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# METHOD FOR STUDYING SPATIAL VIBRATIONS OF A VEHICLE DURING ITS MOVEMENT ALONG THE RAIL TRACK ON SEPARATE SUPPORTS WITH ELASTIC-DISSIPATIVE AND INERTIAL PROPERTIES

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**Abstract.** In the paper, the authors describe a method for studying spatial vibrations of a vehicle with wheels freely rotating on wheelset axles during their movement along the rail track on individual elastic supports with elastic-dissipative properties of points of rail resting on sleepers, as well as with bending of rails between sleepers in the vertical and transverse directions when transport units moves along curved and rectilinear sections of track with inertial properties. The purpose of the work is to improve accuracy of studies of the interaction between components of the transport system by using a new method for studying the mechanics of motion of a vehicle along the rail track with individual elastic-dissipative supports instead of the existing method of studies of motion along the rail track with a solid elastic base. The proposed method of studies is most closely approximated to the actual conditions of operation of the rail track consisting of rail-sleeper grid. A mathematical model of the new method for studying the motion of a vehicle along rail track with separate elastic supports is presented. An important feature of the new method is consideration of: track discreteness, which is caused by the presence of a rail-sleeper grid with variable distance between supports and track laying diagram with different length of spans between the sleepers; inertial, stiffness and dissipative properties of intermediate rail fasteners, sleepers and base in points where rails are installed on sleepers; elastic properties of rails between supports and directly on supports. Moreover, there is a possibility to study loads and forced vibrations of a vehicle in vertical and transverse directions for cases with reinforced concrete, wooden or other sub-rail base. The proposed method and mathematical model are provided for purposeful studies of the influence of the rail track laying diagram with certain or necessary spans between supports or sleepers on the loading and related stress-strain state of all rail track elements, including rails, sleepers and sub-sleeper base. The results of the study can be applied to research conducted on railway, industrial and underground rail transport.

**Keywords:** mechanics of motion, rail track, rail-sleeper grid, vehicles.

## 1. Introduction

Analysis of studies in the field of interaction between the rolling stock and the rail track shows that initially the calculation scheme of the vehicle was taken with the assumption that the rail track is rigid with deterministic irregularities [1]. Such a simplification of the rail track scheme caused the approximation of the evaluation of vibrations and loads of both the rolling stock and the rail-sleeper grid of the rail track. In addition, mainly not spatial, but plane vibrations of vehicles in the vertical and longitudinal planes were investigated [2, 3].

In a number of other studies [4], the inertia and elasticity of the track were taken into account by assuming that the rail-sleeper grid forms a continuous base, instead of a discrete rail-sleeper grid, according to the hypothesis of N.P. Petrov [5]. However, this assumption does not fully reflect the properties of the track structure due to its discreteness, which leads to incorrect research results.

G.M. Shakhunyants [6] did not take into account dynamic loads caused by vehicle oscillation in his calculations of the rail track as a one-dimensional continuous system.

Differential equations of motion of a vehicle along inertial deformed rail track were obtained and solved by V.A. Lazarian and his students [7] under the assumption that the sub-rail base obeys V.Z. Vlasov's hypothesis of a continuous elastic base [8].

Plane vibrations of moving units in the vertical plane with the representation of the rail base by two layers with taking into account the inertial and elastic-dissipative properties of the track are described in the studies of A.I. Zalessky [9].

A special fundamentality is also present in studies of the interaction between the rolling stock and rail track in a great number of works of such scientists as P.S. Anisimov, Yu.M. Bondrenko, M.F. Verigo, S.V. Vertinsky, Yu.D. Voloshko, L.O. Gracheva, V.N. Danilov, A.Y. Kogan, S.S. Krepkogorsky, N.N. Kudryavtsev, B.V. Medel, Yu.S. Romen, I.I. Chelnokov, and others [11-16].

The joint vibrations of moving units and the rail track are the basis of these studies, the track was represented as a continuous system and its vibrations were described in partial derivatives. In [16], a description of the carriage and the rail track is given as a discrete system, however, this model also does not reflect to the required extent the peculiarities of the track structure. It takes into account the inertial properties of the track by adding the reduced track masses to each wheel, while the unevenness of rail deflections due to the discreteness of the rail-sleeper grid was not taken into account in the research.

The above-mentioned studies of the interaction between the moving units and the rail track can be improved by considering that the rail track corresponds to the real design of the rail-sleeper grid with discontinuous trackways resting on individual supports and not to the rails continuous resting on the base

These conditions were confirmed by many years' researches of the non-uniform elasticity of rail track in sleeper cross-sections and spans between these sleepers, as well as between individual sleepers. This is exemplified by the results of research on railway transport, outlined in [17, 18], as well as on industrial and mining rail transport [19, 20]. At the same time, the inelasticity along the rail track in sleeper span sections reaches from 7% to 35%, and the inelasticity along the rail track reaches 200%.

Apart from the above, the method for investigating the interaction between the moving units and the rail track with the assumption of a continuous elastic base does not allow to determine the actual loads on individual sleepers and to establish the influence of different sleeper spacing on the force impact on the sleepers and the influence of the laying diagram on the stress-strain state of the rail track.

The above data confirm the relevance of creating a new method for investigating the interaction between the moving units and the rail track on individual supports with elastic-dissipative and inertial properties.

The purpose of the work is to improve the accuracy of interaction studies of the components of the transport system by means of a new method of studying the mechanics of vehicle motion along the rail track with individual elastic and dissipative supports instead of the existing method of studying motion along the rail track with a continuous elastic base.

## 2. Methods

The proposed methodology for studying vehicle movement along the rail track takes into account the elastic-dissipative properties of rail resting on sleepers or bars as well as the bending of rails between the sleepers in the vertical and transverse directions when transport units move along straight and curved sections of track, which allows to get as close as possible to description of the real operating conditions

Since the disturbances acting on vehicles are predominantly caused by heterogeneities and imperfections in the track structure, the methodology under discussion will make it possible to identify reserves for improving the performance of the rolling stock and the track structure.

The new mathematical model of interaction between the vehicle and the rail track proposed in addition to the previously obtained results [21] is somewhat different from the previous one, as it takes into account the main structural features of the track structure, including switches, to a certain extent more accurately.

An important feature of the new methodology is the consideration of: a) track discreteness, which is caused by the rail-sleeper grid; b) inertial, stiffness and dissipative properties of rails, intermediate fasteners, sleepers and base in the points where rails are installed on sleepers; c) elastic properties of rails between supports; d) variable characteristics in the area of individual elements of switches.

The proposed method and mathematical model contain equations of spatial vibrations of a rail vehicle during its uniform motion along an inertial, elastic-dissipative track structure of arbitrary outline in the plan, represented in the form of concentrated masses in points of rail resting on sleepers and elastic trackways.

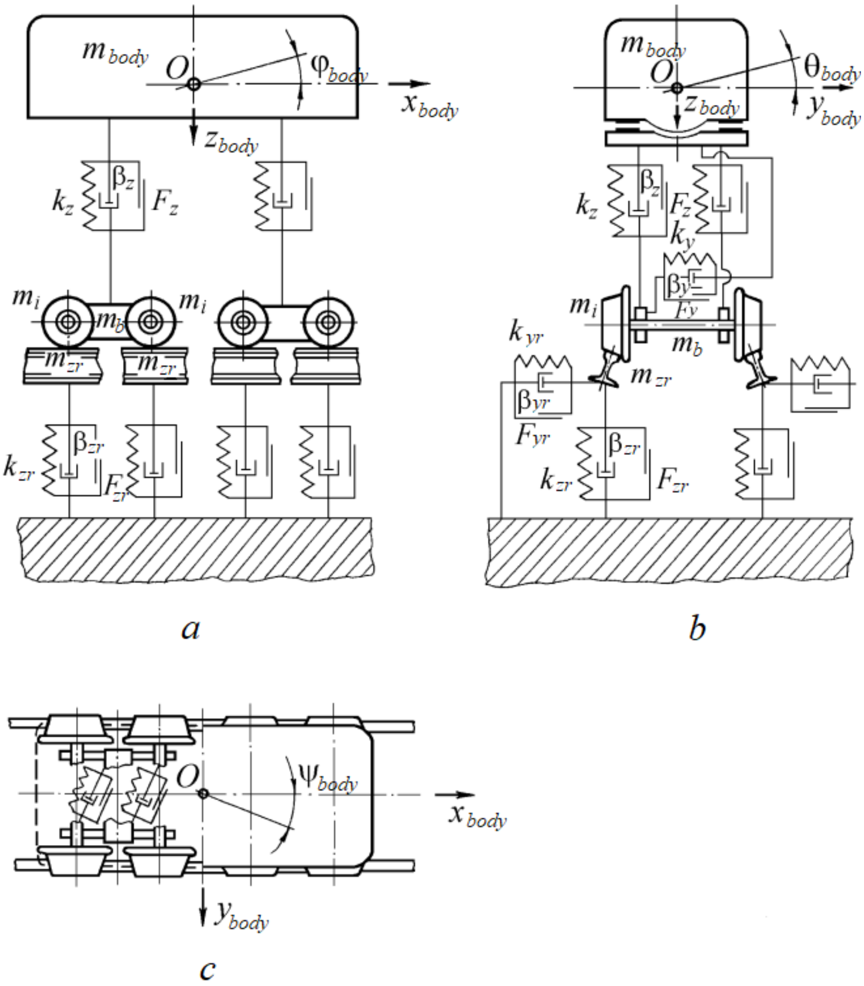
As an object for mathematical description of the considered movement, let's choose, for example, a four-axle car of mine rail transport of VG-28-960 type (Fig. 1) or another model.

The body of this vehicle is supported by two double-axle bogies whose wheels can turn freely on their axles. Unsprung parts of the bogies (side frames and wheelset axles) can take a parallelogram shape in plan within the limits determined by the choice of clearances between the side jaws of the sidewalls covered the wheelset axles, therefore, we take the angles of rotation of wheelset axles and truck bolsters in plan for each bogie to be equal to each other. We take into account the elastic-dissipative relations between the body and bogie side frames (through the sprung beam) in vertical and transverse directions, between the body and the truck bolsters during their mutual turns in vertical and transverse planes, as well as between the bogie side frames and truck bolsters during their mutual turns in plan.

We represent the vehicle as a mechanical system of solid bodies (body, two truck bolsters, four bogie side frames, four wheelset axles and eight wheels) connected with each other by elastic-dissipative rigid or hinged links [21].

The rail track consists of two elastic inseparable beams on elastic-dissipative supports in the vertical and transverse directions, in which the reduced masses, rails, sleepers and base are concentrated (Fig. 2).

In addition, dynamic properties of the carriage are significantly affected by the displacements of the reduced masses and rails (due to their elasticity) only in a certain end section along the axis of the track structure. For this reason, the number of the reduced concentrated masses attributable to one trackway is assumed to be equal to twice the number of sleepers located between the outermost axles of the wheelsets and sleepers outside the first and last axles of the wheelsets.




*a* – side view; *b* – top view; *c* – end view

Figure 1 – Calculation scheme of the interaction between vehicle and track

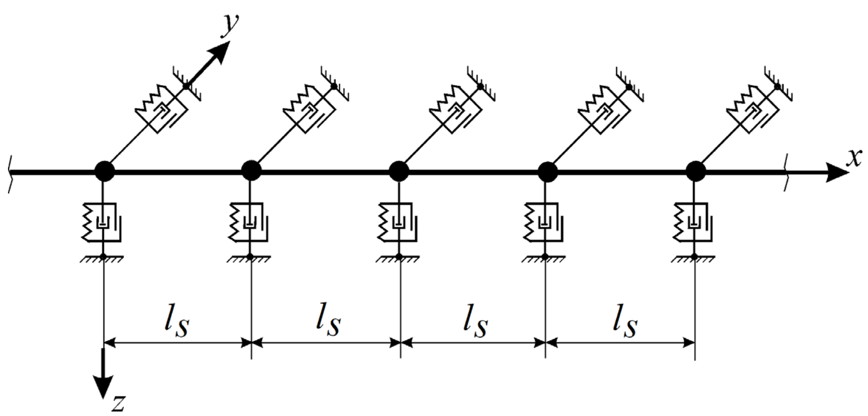



Figure 2 – Calculation scheme for the rail track with elastic-dissipative connections on the supports in the vertical ($z$) and transverse ($y$) directions

If the carriage base and bogie base, respectively, are designated by $2l$ and $2a$, and the distance between sleeper axles – by $l_s$, then the expression for the number of the reduced concentrated masses of the track structure can be written as follows:

$$N_r = \left\{ \left[ \frac{2\ell + 2a}{l_s} \right] + 2n_0 \right\}, \tag{1}$$

where in brackets $[\ \ ]^0$ only the integer part of the number is taken into account; $2n_0$ is the number of considered concentrated track masses outside the first and last wheelsets.

### 3. Results and discussion

Thus, we will consider the spatial vibrations of a mechanical system consisting of solid bodies simulating a body, a carriage, two truck beams, four bogie side frames, four wheelset axles and eight wheels connected by elastic-dissipative elements on elastic-dissipative supports. The mechanical system also includes the reduced concentrated masses of the track structure within the limited track section. In the scheme, vibrations of the carriage and the track section both under and outside the car at a certain interval of the track length are considered.

Note that the number of reduced concentrated track masses can be determined from test calculations with using accepted evaluation criteria.

In general, we take into account vertical and horizontal track irregularities, given as deterministic or random variables [21].

We do not take into account the longitudinal vibrations of the system.

In order to compose a mathematical model describing the motion of a vehicle along a track of arbitrary outline in plan, one moving coordinate system was used and for each solid body of the mechanical system under the study - two moving systems: a natural and an invariably connected to the solid body [9, 20].

The linear in the transverse and vertical directions $y$, $z$, and angular displacements $\vartheta$, $\varphi$, $\psi$ of the solids corresponding to the vehicle (body and its running gears) and the linear displacements $y$ and $z$ of the solids corresponding to the track were chosen as generalized coordinates $q_v$.

Mathematical models of motion of the systems under the study were obtained by using Lagrange equations of the II kind. In general, the equations of motion of the systems under the consideration can be represented as:

$$D_v + \Pi_v + \Phi_v = Q_v, \ \left(v = 1, 2, ..., n\right), \tag{2}$$

where $D_v$, $\Pi_v$, $\Phi_v$ are the differential operators corresponding to the Lagrange equations of the II kind:

$$D_v = \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_v} - \frac{\partial T}{\partial q_v}; \quad \Pi_v = \frac{\partial \Pi}{\partial q_v}; \quad \Phi_v = \frac{\partial \Phi}{\partial q_v}, \tag{3}$$

where $Q_v$ are the generalized forces corresponding to the generalized coordinates $q_v$; $n$ is the number of degrees of freedom of the system; $T$, Π, Φ are the kinetic and potential energies and the scattering function.

With the assumptions accepted, the expressions $D_v$, $\Pi_v$, $\Phi_v$ can be represented as:

$$D_\text{v} = f(m, I, q_\text{v}, \ddot{q}_\text{v}),\ \Pi_\text{v} = f_1(k, q_\text{v}) + \Pi_h(q_\text{v}),\ \Phi_\text{v} = f_2(\beta, \dot{q}_\text{v}) + f_3(F, \dot{q}_\text{v}), \tag{4}$$

where $m$, $I$ are inertial characteristics (masses and moments of inertia) of solid bodies of the system; $k_z$, $k_{zr}$, $k_{zs}$, $k_{zf}$ and $\beta_z$, $\beta_{zr}$, $\beta_{zs}$, $\beta_{zf}$ are coefficients of rigidity of elastic and viscous dissipative elements respectively of the carriage, rails, sleepers and foundation (base); $F_z$, $F_{zr}$, $F_{zs}$, $F_{zf}$ are forces of dry friction in the bonds between solid bodies respectively of the carriage, rails, sleepers and foundation (base); $\Pi_h(q_v)$ is the component of potential energy due to changes in height of gravity centers of solid bodies at displacements $q_v$.

The generalized forces $Q_v$ were determined as functions of forces of interaction between wheels and rails $P_{nj}$, $F_{ynj}$, $X_{nj}$ in vertical, transverse and longitudinal directions (here $n$ is the number of wheel set; $j$=1.2 is the number of wheel of the $n^{\text{th}}$ wheel set).

Let's specify distinctive features of determining forces $P_{nj}$, $F_{ynj}$, $X_{nj}$ in case of free and rigid wheel attachment on wheelset axles.

The forces $P_{nj}$ acting on the wheels in the vertical direction were generally calculated in the same way and are the sum of the static and dynamic forces determined through dynamic deflections and their time derivatives and through stiffness and viscosity coefficients (or dry friction forces) of the spring suspension elements.

The forces of interaction between the wheels and rails in transverse direction $F_{ynj}$ were determined as the sum of forces of pseudo-slip in transverse direction $y_{nj}$ and forces of lateral pressure of wheel flanges on rail heads $W_{nj} = -P_{nj}\dfrac{dr_{nj}}{dy_{nj}}$, where $\Delta r_{nj}$ is increments of wheels radii during their transverse movement relative to rails $y_{nj}$.

Tangential forces of interaction between the wheels and the rails (pseudo-slip forces) $F_{nj}$ were found based on the creep hypothesis with taking into account their nonlinear dependence on the dimensionless slip characteristics, from the expression:

$$F_{nj} = -f_{nj}\varepsilon_{nj}\left[\left(\frac{f_{nj}\varepsilon_{nj}}{k\ f\ P_{nj}}\right)^2 + 1\right]^{-1/2}, \tag{5}$$

and the components of the pseudo-slip forces $X_{nj}$, $Y_{nj}$ in the longitudinal and transverse directions are as follows:

$$X_{nj} = F_{nj}\frac{\varepsilon_{xnj}}{\varepsilon_{nj}}; \quad Y_{nj} = F_{nj}\frac{\varepsilon_{ynj}}{\varepsilon_{nj}}, \tag{6}$$

where $\varepsilon_{nj}$, $\varepsilon_{xnj}$, $\varepsilon_{ynj}$ are dimensionless characteristics of wheel slip and its components in the longitudinal and transverse directions; $\varepsilon_{nj} = \left(\varepsilon_{xnj}^2 + \varepsilon_{ynj}^2\right)^{1/2}$, $f_{nj}$ are pseudo-slip coefficients determined in accordance with the works [9, 20]; $k_f$ is coefficient of friction between the wheels and rails.

The components of the dimensionless slip characteristics $\varepsilon_{ynj}$ were determined for rigid and free wheel attachments using the formula:

$$\varepsilon_{ynj} = \frac{1}{V}\dot{y}_{nj} - \psi_{nj}, \tag{7}$$

and the dimensionless characteristic component $\varepsilon_{xnj}$ were determined from the expressions:

– for rigid wheel attachment

$$\varepsilon_{xnj} = (-1)^{j+1}\left[\frac{d_1}{V}\left(\dot{\psi}_{nj} + \dot{\chi}_{nj}\right) + (-1)^j \frac{\Delta r_{nj}}{r}\right], \tag{8}$$

and

– for free wheel attachment

$$\varepsilon_{xnj} = (-1)^{j+1}\left[\frac{d_1}{V}\left(\dot{\psi}_{nj} + \dot{\chi}_{nj}\right) + (-1)^j \frac{\Delta r_{nj}}{r} + \left(-1^j \frac{r\dot{\varphi}_{nj}}{V}\right)\right], \tag{9}$$

where $V$ is the speed of uniform motion of the vehicle; $\dot{y}_{nj}$ is relative speed of movement of points of wheel and rail head contact; $2a$ is the bogie base (for carriage with no bogie – the vehicle base); $k_n$ is track curvature under the $n^{\text{th}}$ wheelset; $2d_1$ is distance between the middle wheelset rolling circles; $r$ is radius of the middle wheelset rolling circle; $\dot{\chi}_{nj}$ is angular speed due to track curvature under the $j^{\text{th}}$ wheel of the $n^{\text{th}}$ wheelset.

The track irregularities in the plan were given as a sine or cosine, and the junction vertical irregularities were given as a cosine of limited extent within its period [7, 20].

We evaluate the calculation schemes of transport systems under the study according to indicators characterizing loading of running parts, spring sets of cars and track in vertical and transverse directions, and according to traffic safety indicators, in particular, characterizing stability against shifting of rail-sleeper grid, unloading of wheels, rolling of wheels on rails. Therefore, the following parameters are chosen as criteria for estimating dynamic characteristics of the investigated types of mine rolling stock: the maximum values of forces acting on wheels, spring systems and trackways in vertical and transverse directions (respectively $P_{nj}$, $Q_{ynj}$, $F_{znj}$, $F_{ynj}$ and $Q_{znjp}$, $Q_{ynjp}$), transverse forces $Q_{yn}$ acting on wheelsets, stability coefficients at unloading of wheels $k_{unload}$, shift of rail-sleeper grid $k_{shift}$ and rolling of wheels on rails $k_{roll}$.

Here is a mathematical description of the movement of the part of the track structure whose vibrations are considered together with the vehicle.

For both trackways represented by continuous beams on elastic dissipative supports, the equations describing their vibrations in the vertical and transverse directions will have the following form:

$$\begin{gathered}
m_{za}\ddot{z}_{a\lambda} + \beta_{z\lambda\lambda}\dot{z}_{a\lambda} + F_{z\lambda\lambda}\operatorname{sgn}\dot{z}_{a\lambda} + C_{z\lambda 1}z_{a1} + C_{z\lambda 2}z_{a2} + \ldots + C_{z\lambda k}z_{ak} = Q_{za\lambda};\\
m_{zb}\ddot{z}_{b\lambda} + \beta_{z\lambda\lambda}\dot{z}_{b\lambda} + F_{z\lambda\lambda}\operatorname{sgn}\dot{z}_{b\lambda} + C_{z\lambda 1}z_{b1} + C_{z\lambda 2}z_{b2} + \ldots + C_{z\lambda k}z_{bk} = Q_{za\lambda};\\
m_{ya}\ddot{y}_{a\lambda} + \beta_{y\lambda\lambda}\dot{y}_{a\lambda} + F_{y\lambda\lambda}\operatorname{sgn}\dot{y}_{a\lambda} + C_{y\lambda 1}y_{a1} + C_{y\lambda 2}y_{a2} + \ldots + C_{y\lambda k}y_{ak} = Q_{ya\lambda};\\
m_{yb}\ddot{y}_{b\lambda} + \beta_{y\lambda\lambda}\dot{y}_{b\lambda} + F_{y\lambda\lambda}\operatorname{sgn}\dot{y}_{b\lambda} + C_{y\lambda 1}y_{b1} + C_{y\lambda 2}y_{b2} + \ldots + C_{y\lambda k}y_{bk} + Q_{yb\lambda};\\
\lambda = 1,\ 2,\ 3, \ldots k;\ \left(k = N_r/2\right),
\end{gathered} \tag{10}$$

where $m_{za}$, $m_{zb}$, $m_{ya}$, $m_{yb}$ are the reduced track masses referred to the left and right trackways, in vertical and transverse directions; $\beta_{z\lambda\lambda}$, $\beta_{y\lambda\lambda}$ and $F_{z\lambda\lambda}$, $F_{y\lambda\lambda}$ are the coefficients of viscous resistance and dry friction forces at the rail resting on sleepers in vertical and transverse directions; $C_{z\lambda 1}$, $C_{z\lambda 2}$, …, $C_{z\lambda k}$ and $C_{y\lambda 1}$, $C_{y\lambda 2}$, …, $C_{y\lambda k}$ are quasi-elastic coefficients of the rail as a continuous beam on elastic supports in vertical and transverse directions; $z_{a\lambda}$, $z_{b\lambda}$ and $y_{a\lambda}$, $y_{b\lambda}$ are displacements of λ elastic dissipative supports of the left and right trackways in vertical and transverse directions (index *a* corresponds to the left, in the direction of the car, trackway, and *b* to the right trackway); $Q_{za\lambda}$, $Q_{zb\lambda}$ and $Q_{ya\lambda}$, $Q_{yb\lambda}$ are generalized forces corresponding to the loads transmitted to λ elastic dissipative supports of the left and right trackways in vertical and transverse directions.

Quasi-elastic coefficients are determined from the partial system for each beam (rigid hinged supports are installed at the points of mass concentration) as the reactions in the supports with numbers 1, 2, ...*k* from the unit displacement of the λ support [21]. The stiffness of the λ elastic supports in vertical and transverse directions is added for the coefficients $C_{z\lambda\lambda}$ and $C_{y\lambda\lambda}$ obtained in this way.

Below are the mathematical transformations required to determine the generalized forces $Q_{za\lambda}$, $Q_{zb\lambda}$, $Q_{ya\lambda}$, $Q_{yb\lambda}$.

The values of λ, which correspond to each wheel of the car (assuming that $inj^{\text{th}}$ wheel is to the right of the $\lambda^{\text{th}}$ support) are determined from the relation:

$$\lambda_{inj} = \left[\frac{S_{inj}}{\ell ш}\right]^{0} + 1 + n_0; \tag{11}$$

where $S_{inj}$ is the distance travelled by $inj^{th}$ wheel, is measured from the centre of gravity of the body and is determined from the ratios:

$$S_{11_j} = \upsilon t + \ell + a; \quad S_{12_j} = \upsilon t + \ell - a\,;$$
$$S_{21_j} = \upsilon t - \ell + a; \quad S_{22_j} = \upsilon t - \ell - a, \quad j = 1, 2\,. \tag{12}$$

Brackets $[\quad]^0$ mean that only the integer part of the number is counted.

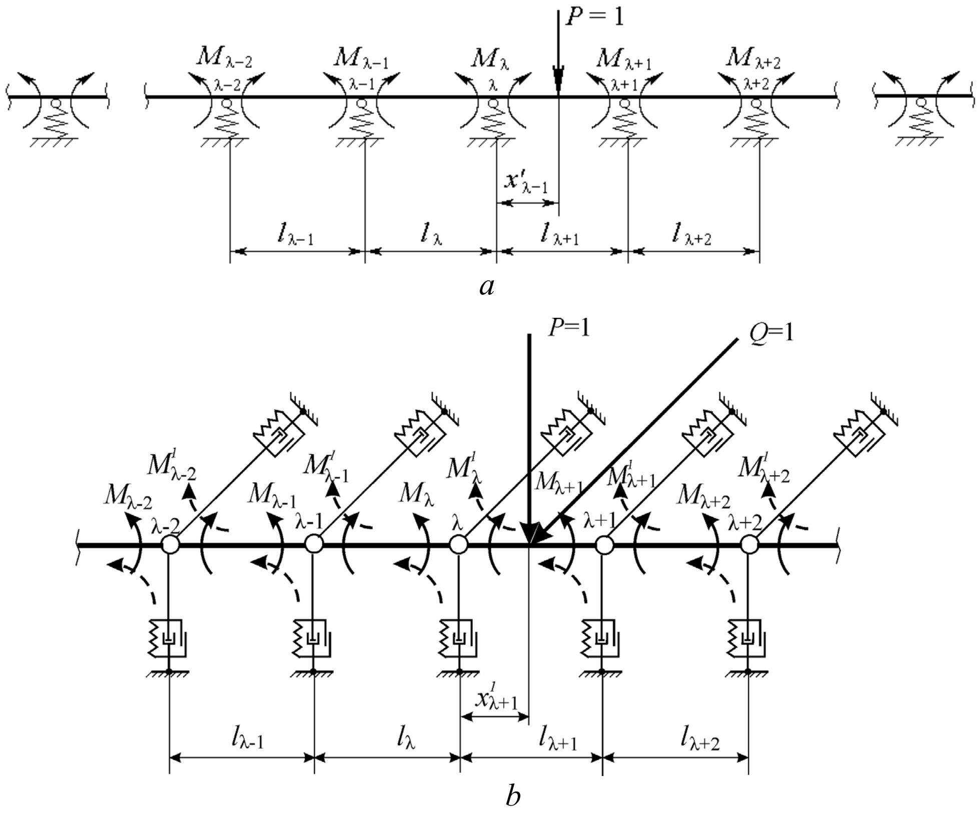
The distance from the $\lambda^{th}$ support to the $in^{th}$ wheelset $x_{inj}$ ($j$=1, 2) can be determined as the ratio:

$$x'_{inj} = S_{inj} - \left(\lambda_{inj} - 1 - n_0\right)\ell_s\,. \tag{13}$$

Thus, expression (13) allows to determine where the $inj^{th}$ wheel is at any given moment in relation to the $\lambda^{th}$ support.

To simplify the description, we will further consider the motion along the continuous beam on elastic-dissipative supports as the unit force $P$.

In order to determine the support reactions of a continuous beam due to action of force $P$, let's consider this beam as a hinge-supported system with two moments applied in opposite directions (Fig. 3).




*a* – beam on individual elastic supports; *b* – beam on individual supports with elastic-dissipative links in vertical and transverse directions

Figure 3 – Calculation diagrams of railways on individual hinged supports

We will find the support moments by using the canonical equations of the five moments [22].

For the $\lambda^{th}$ elastic support, this equation is:

$$\delta_{\lambda,\lambda-2}M_{\lambda-2}+\delta_{\lambda,\lambda-1}M_{\lambda-1}+\delta_{\lambda,\lambda}M_{\lambda}+\delta_{\lambda,\lambda+1}M_{\lambda+1}+ \\ +\delta_{\lambda,\lambda+2}M_{\lambda+2}+\Delta_{\lambda p}=0, \qquad (14)$$

where

$$\delta_{\lambda,\lambda-2}=\frac{C_{\lambda-1}}{\ell_{\lambda-1}\ell_{\lambda}}; \quad \delta_{\lambda,\lambda-1}=\frac{\ell_{\lambda}}{6EI_{\lambda}}-\frac{C_{\lambda-1}}{\ell_{\lambda}}\left(\frac{1}{\ell_{\lambda-1}}+\frac{1}{\ell_{\lambda}}\right)-\frac{C_{\lambda}}{\ell_{\lambda}}\left(\frac{1}{\ell_{\lambda}}+\frac{1}{\ell_{\lambda+1}}\right);$$

$$\delta_{\lambda,\lambda}=\frac{\ell_{\lambda}}{3EI_{\lambda}}+\frac{\ell_{\lambda+1}}{3EI_{\lambda+1}}+\frac{C_{\lambda-1}}{\ell_{\lambda}^{2}}+C_{\lambda}\left(\frac{1}{\ell_{\lambda}}+\frac{1}{\ell_{\lambda+1}}\right)^{2}+\frac{C_{\lambda+1}}{\ell_{\lambda+1}^{2}};$$

$$\delta_{\lambda,\lambda+1}=\frac{\ell_{\lambda}}{6EI_{\lambda}}+\frac{C_{\lambda+1}}{\ell_{\lambda}}\left(\frac{1}{\ell_{\lambda+1}}+\frac{1}{\ell_{\lambda}}\right)-\frac{C_{\lambda}}{\ell_{\lambda}}\left(\frac{1}{\ell_{\lambda}}+\frac{1}{\ell_{\lambda+1}}\right); \quad \delta_{\lambda,\lambda+2}=\frac{C_{\lambda+1}}{\ell_{\lambda+1}\ell_{\lambda}};$$

$$\lambda_{\lambda p}=\frac{B_{\lambda}^{\Phi}}{EI_{\lambda}}+\frac{A_{\lambda+1}^{\Phi}}{EI_{\lambda+1}}+\frac{C_{\lambda-1}R_{\lambda-1}}{\ell_{\lambda-1}}-C_{\lambda}R_{\lambda}\left(\frac{1}{\ell_{\lambda}}+\frac{1}{\ell_{\lambda+1}}\right)+\frac{C_{\lambda+1}R_{\lambda+1}}{\ell_{\lambda+1}}. \qquad (15)$$

In relation (15) $C_{\lambda}$, $C_{\lambda-1}$, $C_{\lambda-2}$, $C_{\lambda+1}$, $C_{\lambda+2}$ are the flexibility of the respective supports; $EI_{\lambda}$, $EI_{\lambda+1}$, $EI_{\lambda-1}$ are the bending stiffness values of the beam in the spans $\ell_{\lambda}, \ell_{\lambda+1}, \ell_{\lambda-1}$;

$$A_{\lambda+1}^{\Phi}=P\ell_{\lambda+1}^{2}u_{\lambda+1}v_{\lambda+1}\left(1+\tilde{u}_{\lambda+1}\right); \quad B^{\Phi}+0; \\ R_{\lambda-1}=0; \quad R_{\lambda}=v_{\lambda+1}P; \quad R_{\lambda+1}=u_{\lambda+1}P, \qquad (16)$$

where $u_{\lambda+1}=\dfrac{x'_{\lambda+1}}{\ell_{\lambda+1}}$; $v_{\lambda+1}=1-u_{\lambda+1}$; $x'_{\lambda+1}$ are the distance from $\lambda^{th}$ support to the force application point $P$.

In equation (14), the coefficient is following:

$$\Delta_{\lambda p}=\frac{P\ell_{\lambda+1}^{2}u_{\lambda+1}v_{\lambda+1}\left(1+u_{\lambda+1}\right)}{EI_{\lambda+1}}-C_{\lambda}v_{\lambda+1}P\left(\frac{1}{\ell_{\lambda}}+\frac{1}{\ell_{\lambda+1}}\right)+\frac{C_{\lambda}u_{\lambda+1}}{\ell_{\lambda+1}}P. \qquad (17)$$

Equations of five moments of the type (14) for each of the beam supports form a system of linear algebraic equations whose order depends on the number of supports (sleepers) taken into account when investigating joint vibrations of the "movable vehicle-rail track" system. Having solved this system of equations, we obtain the values of support moments $M_\lambda$, with the help of which one can define the deflections of beam in any cross-section depending on the position of cargo. For this purpose, let's find beam reactions $R_\lambda$ and $R_{\lambda+1}$, by using calculation schemes of statically determined beams loaded by supporting moments and force $P$ or by single support moments:

$$R_\lambda = R_\lambda^0 + \frac{M_{\lambda+1} - M_\lambda}{\ell_{\lambda+1}} + \frac{M_{\lambda-1} - M_\lambda}{\ell_\lambda}, \tag{18}$$

where $R_\lambda^0$ is the reaction of $\lambda^{th}$ support from the external load $P$ in the basic system (without considering the support moments) for a single span beam $\ell_{\lambda+1}$:

$$R_\lambda^0 = P v_{\lambda+1}. \tag{19}$$

We define the displacement of the $\lambda^{th}$ support from the equality:

$$z_\lambda^0 = C_\lambda R_\lambda = C_\lambda \left[ \frac{M_{\lambda-1}}{\ell_\lambda} - \left( \frac{1}{\ell_\lambda} + \frac{1}{\ell_{\lambda+1}} \right) M_\lambda + \frac{M_{\lambda+1}}{\ell_{\lambda+1}} \right] + C_\lambda R_\lambda^0. \tag{20}$$

Displacement of any beam cross-section in span $\lambda+1$ (Fig. 1) from support subsidence and beam bending is determined by the following equation:

$$z_{\lambda+1} = z_\lambda^0 + \left( z_{\lambda+1}^0 - z_\lambda^0 \right) u_{\lambda+1} + z'_{\lambda+1}, \tag{21}$$

where $z'_{\lambda+1}$ is the span from deflection of the beam in the cross-section next from the left support at a distance $u_{\lambda+1} \cdot \ell_{\lambda+1}$, which is determined for the calculation scheme of the single-span beam loaded with support moments $M_\lambda$, $M_{\lambda+1}$ and force $P$ or only with support moments, if $P$=0 in the considered span.

Let's analyze both cases of beams loading.

Let the span of the beam with the length $\ell_\lambda$ is loaded only by support moments $M_{\lambda-1}$, $M_\lambda$ (Fig. 3, a). For this span the deflection due to bending can be determined by Vereshchagin's method [22]:

$$z'_\lambda = -\frac{\ell_\lambda (1+u_\lambda)(2-u_\lambda)}{36EJ} \left[ M_{\lambda-1}(2-u_\lambda) + M_\lambda (1+u_\lambda) \right]. \tag{22}$$

For the beam span of length $\ell_{\lambda+1}$ (Fig. 3) the deflection is found in the same way:

$$z'_{\lambda+1} = -\frac{\ell_{\lambda+1}(1+u_{\lambda+1})(2-u_{\lambda+1})}{36EJ}\left[M_{\lambda}(2-u_{\lambda+1})+M_{\lambda+1}(1+u_{\lambda+1})\right]+ \\ +\frac{P\ell^3_{\lambda+1}u^2_{\lambda+1}v^2_{\lambda+1}}{3EJ}(u_{\lambda+1}+v_{\lambda+1}). \quad (23)$$

Knowing the forces (18) acting on supports of the left and right trackway in the vertical and transverse directions, let's determine the generalized forces for the track movements:

$$\begin{aligned} Q_{za\lambda} &= R_{\lambda z111}P_{111}+R_{\lambda z121}P_{121}+R_{\lambda z211}P_{211}+R_{\lambda z221}P_{221}; \\ Q_{zb\lambda} &= R_{\lambda z112}P_{112}+R_{\lambda z122}P_{122}+R_{\lambda z212}P_{212}+R_{\lambda z222}P_{222}; \\ Q_{ya\lambda} &= R_{\lambda y111}Q_{y111p}+R_{\lambda y121}Q_{y121p}+R_{\lambda y211}Q_{y211p}+R_{\lambda y221}Q_{y221p}; \\ Q_{yb\lambda} &= R_{\lambda y112}Q_{y112p}+R_{\lambda y122}Q_{y122p}+R_{\lambda y212}Q_{y212p}+R_{\lambda y222}Q_{y222p}. \end{aligned} \quad (24)$$

where $R_{\lambda zinj}$, $R_{\lambda yinj}$ are the forces transmitted to the $\lambda^{th}$ support due to the unit loads $P$=1 applied respectively in the vertical and transverse directions at the locations of $inj^{th}$ wheels (are determined in accordance with expression (18)).

By using formulas (21-23), it is possible to find deflections not only due to one force but also due to the vertical and transverse loads that are transmitted to each wheel (rail). By applying the superposition principle, it is possible to find deflections in each cross-section of both continuous beams (rails) in vertical and horizontal directions caused by all forces.

Let's write down the expressions for the wheel movements $z^0_{inj}$ under the left and right rail tracks, due to sleeper and rail suppleness. Each wheel corresponds to a certain span number $\mu=\lambda+1$. The following relations apply for $inj$ wheels ($i$=1, 2; $n$=1, 2; $j$=1):

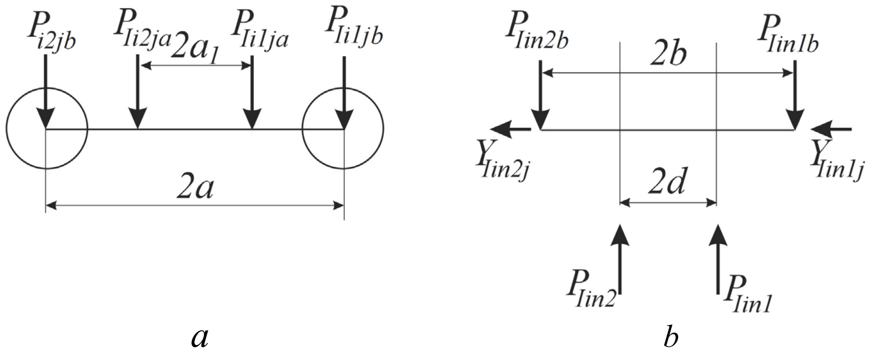
$$\begin{aligned} z^0_{111} &= \bar{z}_{111,111}P_{111}+\bar{z}_{111,121}P_{121}+\bar{z}_{111,211}P_{211}+\bar{z}_{111,221}P_{221}; \\ z^0_{121} &= \bar{z}_{121,111}P_{111}+\bar{z}_{121,121}P_{121}+\bar{z}_{121,211}P_{211}+\bar{z}_{121,221}P_{221}; \\ z^0_{211} &= \bar{z}_{211,111}P_{111}+\bar{z}_{211,121}P_{121}+\bar{z}_{211,211}P_{211}+\bar{z}_{211,221}P_{221}; \\ z^0_{221} &= \bar{z}_{221,111}P_{111}+\bar{z}_{221,121}P_{121}+\bar{z}_{221,211}P_{211}+\bar{z}_{221,221}P_{221}, \end{aligned} \quad (25)$$

where $\bar{z}_{inj,inj}$ is the vertical displacements under the action of force $P$=1; in the designations, the first three indexes before the decimal point correspond to the area of the deflection (under the $inj^{th}$ force), and the second index after the decimal point corresponds to the force $P_{inj}$ under the action of which the deflection occurs. These variables are determined by formula (30) under the action of force $P_{inj}$, applied to the $\mu$ span, in the area of the force action under which the deflection occurs (the number of this force is denoted by three indices before the decimal point). According to the reciprocity of displacements theorem μ:

$$\bar{z}_{111,121}=\bar{z}_{111,111};\quad \bar{z}_{111,211}=\bar{z}_{211,111};\quad \bar{z}_{111,211}=\bar{z}_{211,111};$$
$$\bar{z}_{121,211}=\bar{z}_{211,121};\quad \bar{z}_{121,221}=\bar{z}_{221,121};\quad \bar{z}_{211,221}=\bar{z}_{221,211}\,. \qquad (26)$$

Similarly to (25), we can write the dependencies for vertical displacements ($z^0_{112}, z^0_{122}, z^0_{212}, z^0_{222}$), i.e. for displacements that correspond to $j$=2, and for horizontal displacements $y^0_{in1p}, y^0_{in2p}$ ($i$=1, 2; $n$=1,2), which are functions of unit displacements $\bar{y}_{in1p}, \bar{y}_{in2p}$ and forces $Q_{yinjp}$.

Now let's find the vertical forces $P_{inj}$ acting on the wheels (Fig. 4).




*a* – side view; *b* – front view

Figure 4 – Schematic diagram of the wheelset load distribution

The elastic-dissipative forces $P_{inja}$ and $Y_{inja}$ acting from the springs on the bogie side frames in the vertical and transverse directions can be determined as follows:

$$P_{inja}=k_z\Delta z_{inj}+\beta_z\Delta\dot{z}_{inj}+F_z sgn\,\Delta\dot{z}_{inj};$$
$$Y_{inja}=k_y\Delta y_{inj}+\beta_y\Delta\dot{y}_{inj}+F_y sgn\Delta\dot{y}_{inj}. \qquad (27)$$

The forces $P_{injb}$ and $Y_{injb}$, transmitted from the springs to the wheelset axles in the vertical and transverse directions can be found as follows:

$$P_{i1jb}=\frac{P_{i1ja}\left(a+a_1\right)}{2a}+P_{i2ja}\frac{\left(a-a_1\right)}{2a};\quad P_{i2jb}=\frac{P_{i1ja}\left(a-a_1\right)}{2a}+P_{i2ja}\frac{\left(a+a_1\right)}{2a};$$
$$Y_{i1jb}=\frac{Y_{i1ja}\left(a+a_1\right)}{2a}+Y_{i2ja}\frac{\left(a-a_1\right)}{2a};\quad Y_{i2jb}=\frac{Y_{i1ja}\left(a-a_1\right)}{2a}+Y_{i2ja}\frac{\left(a+a_1\right)}{2a}. \qquad (28)$$

From the equilibrium of the $in^{th}$ wheelset loaded with forces $P_{in1b}$, $P_{in2b}$, $Y_{in1b}$, $Y_{in2b}$, let's find the forces acting on the wheels:

$$P_{in1}=\frac{P_{in1}\left(b+d\right)-P_{in2}\left(b-d\right)}{2d}-\frac{\left(Y_{in1b}+Y_{in2b}\right)r}{2d};$$
$$P_{in2}=\frac{P_{in2}\left(b+d\right)-P_{in1}\left(b-d\right)}{2d}+\frac{\left(Y_{in1b}+Y_{in2b}\right)r}{2d}. \tag{29}$$

We note that the forces $P_{inja}, P_{injb}, P_{inja}, Q_{za\lambda}, Q_{zb\lambda}, Q_{ya\lambda}, Q_{yb\lambda}$ depend on the movements not only of the sprung parts but also from the rails $z_{inj}^{0}$, $y_{inj}^{0}$ under the wheels in the vertical and transverse directions. This indicates the relationship between equations (2-9) described the vehicle vibrations and equations (10) described the motion of the track structure.

The above mathematical model of vehicle movements and track structure along the straight track sections is also valid for curved (constant and variable curvature) sections.

During the movement along curved sections, the generalized forces corresponding to the transverse attitude, the lateral sway, the wobble of unsprung parts of vehicle bogies and the transverse movements of the trackways are defined as follows:

$$Q_{yTi}=\sum_{n=1}^{2}\sum_{j=1}^{2}\left(Y_{inj}+W_{inj}+\Delta W_{inj}+W_{ainj}\right);$$
$$Q_{\theta Ti}=-H_{T}\sum_{n=1}^{2}\sum_{j=1}^{2}\left(Y_{inj}+W_{inj}+\Delta W_{inj}+W_{ainj}\right)-\left(-1\right)^{j}\sum_{n=1}^{2}\sum_{j=1}^{2}P_{inja1}\cdot d_{1};$$
$$Q_{\Psi in}=\sum_{j=1}^{2}\left(Y_{i1j}+W_{i1j}+\Delta W_{i1j}+W_{ai1j}-Y_{i2j}-W_{i2j}-\Delta W_{i2j}-W_{ai2j}\right)\cdot a\ ;$$
$$Q_{yinjp}=-\left(Y_{inj}+W_{inj}+\Delta W_{inj}+W_{ainj}\right), \tag{30}$$

where $Y_{inj}$, $W_{inj}$ are the transverse forces of interaction between wheels and rails due to elastic slip of wheels (pseudo-slip) and lateral pressure of wheel flanges on rails; $H_T$ is the height of gravity center of unsprung parts of bogies over the rail heads.

Thus, the movement of the vehicle through curved sections of track can be described by equations (2-9), which take into account features reflecting the nature of movement through curves of different radii of curvature.

## 4. Conclusions

1. The method and the mathematical model of the researches of spatial vibrations of a vehicle with free rotation of wheels on wheelset axles during its uniform motion along an inertial, elastic-dissipative rail track with an arbitrary outline in plan presented as concentrated masses in points of rail resting on sleepers as applied to a straight and curvilinear track were developed.

2. The proposed method and mathematical model makes it possible to conduct purposeful studies of the influence of the rail track laying diagram with certain or necessary length of spans between supports or sleepers on the loading and the corre-

sponding stress-strain state of all rail track elements, including rails, sleepers and sub-tie base.

3. A new direction in researching of loading and stress-strain state of rail track under conditions of non-uniform elasticity of individual sleepers resting on the base and non-uniform rigidity of track in sleeper spans and sleepers is proposed.

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## МЕТОД ДОСЛІДЖЕНЬ ПРОСТОРОВИХ КОЛИВАНЬ ТРАНСПОРТНОГО ЗАСОБУ ПРИ РУСІ ПО РЕЙКОВІЙ КОЛІЇ НА ОКРЕМИХ ОПОРАХ З ПРУЖНОДИСИПАТИВНИМИ ТА ІНЕРЦІЙНИМИ ВЛАСТИВОСТЯМИ

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**Анотація.** У статті викладено метод досліджень просторових коливань транспортного засобу з колесами, які вільно обертаються на осях колісних пар при русі по рейковій колії на окремих пружних опорах з пружнодисипативними властивостями місць опирання рейок на шпали, а також з вигином рейок між шпалами в вертикальному і поперечному напрямках при проходженні по криволінійних і прямолінійних ділянках колії з інерційними властивостями. Метою роботи є підвищення точності досліджень взаємодії складових виробів транспортної системи за допомогою нового методу досліджень механіки руху транспортного засобу по рейкововій колії з окремими пружнодисипативними опорами на заміну існуючого методу досліджень руху по рейковій колії із суцільною пружною підшпальною основою. Викладений метод досліджень максимально наближений до реальних умов експлуатації колії з рейкошпальної решітки. Представлено математичну модель нового методу досліджень руху транспортного засобу по рейковій колії з окремими пружними опорами. Важливою особливістю нового методу є: облік дискретності рейкової колії, яка обумовлена наявністю рейкошпальної решітки з змінною відстанню між опорами і епюрою укладання колії з різними величинами прольотів між шпалами; облік інерційних, жорсткістних і дисипативних властивостей проміжних рейкових скріплень, шпал, підшпальної основи в місцях установки рейок на шпалах, а також облік пружних властивостей рейок між опорами і безпосередньо на опорах. При цьому є можливість досліджень навантаженості і вимушених коливань транспортного засобу в вертикальному і поперечному напрямках для випадків із залізобетонною, дерев'яною або іншою підрейковою основою. Пропонований метод і математична модель передбачені для цілеспрямованих досліджень впливу епюри укладання рейкового колії з певними або необхідними величинами прольотів між опорами або шпалами на навантаженість і пов'язаний з цим напружено-деформований стан всіх елементів рейкової колії, включаючи рейки, шпали та підшпальну основу. Результати роботи можуть бути застосовані для досліджень на залізничному, промисловому і підземному рейковому транспорті.

**Ключові слова:** механіка руху, рейкова колія, рейкошпальна решітка, транспортні засоби.

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