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Existence of Angular Boundary Values and Cauchy–Green Formula

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The aim of the paper is to find the conditions on the function $F(z)$, determined in the domain G and having in G continuous partial derivatives, for validity of the following statements:

i) almost everywhere $F(z)$ has the finite angular boundary values $F(t)$ on $\gamma = \partial G$;

ii) the boundary values $F(t)$ are A-integrable on γ;

iii) the analog of the Cauchy–Green formula holds for $F(z)$.

Key words: angular boundary values, A-integral, Cauchy–Green formula, non-tangential maximal function, Cauchy integral.

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Introduction

Let G be a bounded simply-connected domain of the complex plane $\mathbb{C}$ with the boundary $\gamma = \partial G$. A set of functions satisfying the condition

$$\|f\|_{L_p(\overline{G})} \overset{def}{=} \left(\iint\limits_G |f(z)|^p dxdy \right)^{\frac{1}{p}} < \infty,\ z = x + iy,\ p \geq 1,$$

is denoted by $L_p(\overline{G})$.

For a function F, given in the domain G, on γ we determine the function $F^*_\alpha(t)$ and put

$$F^*_\alpha(t) = \sup\{|F(z)| : z \in G_{t,\alpha}\},\ \alpha \in (1, \infty)$$

if the set $G_{t,\alpha}$ is not empty, and $F^*_\alpha(t) = 0$ otherwise, where $\rho(z, \gamma)$ is an Euclidean distance from the point z to the curve γ, and $G_{t,\alpha} = \{z \in G : |z - t| < \alpha\rho(z, \gamma)\}$. The function F^*_α is a natural analog of a non-tangential maximal function (see

[1, Ch. 1, §$\dot{5}$, p. 36]) for the case of functions F determined in the arbitrary domains of the plane $\mathbb{C}$.

Everywhere in the sequel, if there is no other restrictions on G, we will consider γ as a Jordan rectifiable curve.

Let m be a Lebesgue measure on γ. A complex valued function f on γ that is measurable with respect to the measure m is said to be A-integrable on γ if

$$m\{t \in \gamma : |f(t)| > \lambda\} = o\left(\frac{1}{\lambda}\right), \quad \lambda \to +\infty, \tag{0.1}$$

and there exists a finite limit

$$(A)\int_{\gamma} f(t)dt \stackrel{def}{=} \lim_{\lambda\to+\infty} \int_{\{t\in\gamma:|f(t)|\leq\lambda\}} f(t)dt. \tag{0.2}$$

An attempt to determine an integral by means of limit (0.2) can hardly be referred to any author. Without suitable restrictions this attempt meets an obstacle because the integral determined by means of limit (0.2) has no additivity property [2].

Titchmarch [2] showed that, when applied to the theory of trigonometric series conjugate to the Fourier–Lebesgue series, this integration (determined through the integral (0.2)) gives many natural results. Kolmogorov [3] showed that the function conjugate to a summable function possesses the property (0.1). Titchmarch [2] noticed that it is the property (0.1) that guarantees the additivity of A-integral.

The papers by P.L. Ulyanov [4, 6], Yu.S. Ochan [7], T.S. Salimov [8] were devoted to the problem of application of A-integral to the theory of trigonometric series and to the theory of boundary properties of analytic functions.

It is known [9, 10] that if the analytic function $F(z)$ is a Cauchy-type integral of some finite measure, then ν, then $F(z)$ almost everywhere on γ has a finite angular boundary value $F(t)$. P.L. Ulyanov proved the following theorem.

Theorem A [6, Theorem 4]. *Let a finite domain G be bounded by a contour γ that satisfies the conditions of C (see the definition in* [6]*). Thereby, if*

$$F(z) = \frac{1}{2\pi i}\int_{\gamma} \frac{\varphi(t)}{t-z}dt \ \ (z \in G),$$

then

$$F(z) = \frac{1}{2\pi i}(A)\int_{\gamma} \frac{F(t)}{t-z}dt \ \ (z \in G).$$

In other words, the Cauchy-type integral of absolutely continuous measure is a Cauchy A-integral of its own boundary values.

The author of [11] gave an appropriate representation of the analytic function $F(z)$ in its own boundary values $F(t)$ for a circle in the case of arbitrary finite measures.

T.S. Salimov [8] considered a problem on the conditions on the analytic function $F(z)$ determined in the domain G so that $F(z)$ has almost everywhere the finite angular boundary values $F(t)$, the function $F(t)$ is A-integrable, and $F(z)$ admits representation on the domain G by the Cauchy A-integral in its own boundary values.

Theorem B [8, Theorem 2]. *Let the function $F(z)$ be analytic in G and for some $\alpha \in (2, \infty)$ $m\{t \in \gamma : F^*_\alpha(t) > \lambda\} = o\left(\frac{1}{\lambda}\right)$, $\lambda \to +\infty$. Then F has a finite angular boundary value $F(t)$ for almost all $t \in \gamma$ and the following equalities are valid:*

a) $(A)\int\limits_\gamma F(t)dt = 0$;

b) $F(z) = \dfrac{1}{2\pi i}(A)\int\limits_\gamma \dfrac{F(t)}{t-z}dt, \;\; z \in G.$

Theorem B is also proved for the case $\alpha \in (1, 2]$ under some additional conditions set on the domain G (see [8], Theorem 9).

The present paper is devoted to finding the conditions on the arbitrary (not necessarily analytic) functions $F(z)$, determined in the domain G and having in G continuous partial derivatives, for validity of the following statements:

i) almost everywhere $F(z)$ has the angular boundary values $F(t)$ on $\gamma = \partial G$ (see Theorem 1 in Sec. 1);

ii) the boundary values $F(t)$ are A-integrable on γ (see item a) of the statement of Theorem 2 in Sec. 2);

iii) the analog of the Cauchy–Green formula holds for $F(z)$ (see item b) of the statement of Theorem 2 in Sec. 2).

1. Existence of Angular Boundary Values

We will need the following known theorems.

Theorem C [12, Theorem 1.16]. *If $f = \dfrac{\partial g}{\partial \overline{z}} \in L_1(\overline{G})$, then*

$$g(z) = \Phi(z) + (T_G f)(z), \; z \in G,$$

where Φ is analytic in G, $(T_G f)(z) = -\dfrac{1}{\pi}\iint\limits_G \dfrac{f(\zeta)}{\zeta - z}d\xi d\eta$, $\zeta = \xi + i\eta$.

Theorem D [12, Theorem 1.19]. *If $f \in L_p(\overline{G})$ for some $p \in (2, \infty)$, then the function $T_G f$ satisfies the condition*

$$|(T_G f)(z_1) - (T_G f)(z_2)| \le M_p \, \|f\|_{L_p(\overline{G})} \, |z_1 - z_2|^\alpha, \;\; \alpha = (p-2)/p,$$

where z_1 *and* z_2 *are arbitrary points of the plane* $\mathbb{C}$, *and* M_p *is a constant dependent only on* p.

Theorem E [8, Lemma 6]. *Let the function* F *be analytic in* G *and for some* $\alpha \in (1, \infty)$ $F^*_\alpha(t)$ *be finite for all* t *from the measurable set* $P \subset \gamma$. *Then* F *has the finite angular boundary value* $F(t)$ *for almost all* $t \in P$.

Theorem 1. *Let the function* F *be determined in the domain* G *and satisfy the following conditions:*

1) F *is absolutely continuous in* G, *moreover,* $\dfrac{\partial F}{\partial \overline{z}} \in L_p(\overline{G})$, $z = x + iy$, *for some* $p \in (2, \infty)$;

2) *for some* $\alpha \in (1, \infty)$,

$$m\{t \in \gamma : F^*_\alpha(t) > \lambda\} = o\left(\frac{1}{\lambda}\right), \quad \lambda \to +\infty.$$

Then, for m*-almost all* $t \in \gamma$ *there exists the finite angular boundary value* $F(t)$.

P r o o f. Assume

$$h(z) = \left(T_G \frac{\partial F}{\partial \overline{\zeta}}\right)(z) = -\frac{1}{\pi} \iint\limits_G \frac{\partial F}{\partial \overline{\zeta}} \frac{d\xi d\eta}{\zeta - z}.$$

It follows from Theorem C that the function $\Psi(z) = F(z) - h(z)$ is analytic in G, and from Theorem D we get that the function $h(z)$ is continuous in $\overline{G}$. Hence and from condition 2) of Theorem 1 it follows that the function $\Psi(z)$ satisfies all the conditions of Theorem E. Thus, $\Psi(z)$ has a finite angular boundary value $\Psi(t)$ for almost all $t \in \gamma$. On the other hand, the function $h(z)$ is continuous in $\overline{G}$, and therefore the function $F(z) = \Psi(z) + h(z)$ also has the finite angular boundary value $F(t)$ for almost all $t \in \gamma$.

Theorem 1 is proved.

Corollary 1. *If the function* $F(z)$ *is bounded and has a bounded derivative* $\dfrac{\partial F}{\partial \overline{z}}$ *in* G, *then there exists the finite angular boundary value* $F(t)$ *for* m*-almost all* $t \in \gamma$.

In fact, the second condition of Theorem 1 follows from the boundedness of $F(z)$, and the first condition follows from the boundedness of the derivative $\dfrac{\partial F}{\partial \overline{z}}$. Therefore, $F(z)$ has the finite angular boundary value $F(t)$ for almost all $t \in \gamma$.

Remark 1. *M.B. Balk* [13] *proved the following statement: if a polyanalytic function* $F(z)$, *i.e. the function of the form* $F(z) = \varphi_0(z) + \overline{z}\varphi_1(z) + ... + \overline{z}^{n-1}\varphi_{n-1}(z)$, *where* $\varphi_i(z)$ *are analytic functions,* $i = \overline{0, n-1}$, *together with its* *derivative* $\dfrac{\partial F}{\partial \overline{z}}$ *is bounded in* G *and there is an analytic arch* Γ *on the boundary* $\gamma = \partial G$*, then there exists the finite angular boundary value* $F(t)$ *almost everywhere on* Γ*. Corollary 1 shows that the existence of almost everywhere finite angular boundary value holds for any bounded functions* $F(z)$ *bounded by the derivative* $\dfrac{\partial F}{\partial \overline{z}}$*. Moreover, neither polyanaliticity nor the analytic arch* Γ *on the curve* γ *is required.*

2. Cauchy–Green Formula

In this section we prove the following

Theorem 2. *Let the function* F *be determined in the domain* G *and satisfy the following conditions:*

1) F *has the continuous partial derivatives* $\dfrac{\partial F}{\partial x}$ *and* $\dfrac{\partial F}{\partial y}$ *in* G*, moreover,* $\dfrac{\partial F}{\partial \overline{z}} \in L_1(\overline{G}),\ z = x + iy$;

2) *for some* $\alpha \in (1, \infty)$,

$$m\{t \in \gamma : F^*_\alpha(t) > \lambda\} = o\left(\frac{1}{\lambda}\right),\ \lambda \to +\infty;$$

3) *for* m*-almost all* $t \in \gamma$ *there exists the finite angular boundary value* $F(t)$*. Then*

a)

$$(A)\int\limits_\gamma F(t)dt = 2i \iint\limits_G \frac{\partial F}{\partial \overline{\zeta}} d\xi d\eta,\ \ \zeta = \xi + i\eta, \tag{2.1}$$

where $F(t)$ *is a finite angular boundary value of the function* $F(z)$ *as* $z \to t \in \gamma$*;*

b) for all $z \in G$,

$$F(z) = \frac{1}{2\pi i}(A)\int\limits_\gamma \frac{F(t)}{t-z}dt - \frac{1}{\pi}\iint\limits_G \frac{\partial F}{\partial \overline{\zeta}} \cdot \frac{d\xi d\eta}{\zeta - z}. \tag{2.2}$$

Remark 2. *If* $\dfrac{\partial F}{\partial \overline{z}} \in L_p(\overline{G})$ *for some* $p > 2$*, then the condition* 3) *follows from Theorem 1 and can be eliminated.*

Remark 3. *Since for the function* $F(z)$ *analytic in the domain* G *the integral in the right-hand side of* (2.1) *equals zero, then it follows from Theorems 1 and 2 that the statement of Theorem B holds for* $\alpha \in (1, \infty)$.

To prove Theorem 2 we will need the following known statements proved by A.S. Bezikovich [14] and P.L. Ulyanov [6].

Theorem F ([14], Ch. 1, § 1, p. 13). *Let A be a bounded set in R^n, and for each $x \in A$ there be given a closed Euclidean ball $B(x, r(x))$ centered in x and of radius $r(x)$. Then from the set $\{B(x, r(x)) : x \in A\}$ we can choose at most a denumerable set of balls $\{B_k\}$ satisfying the following conditions:*

i) $A \subset \bigcup\limits_k B_k$;

ii) none of the points from R^n is contained in more than θ_n balls from the set $\{B_k\}$, where θ_n is a number dependent only on n;

iii) a set of the balls $\{B_k\}$ can be divided into the ξ_n families of disjoint balls, where ξ_n is a number dependent only on n.

Theorem G [6, Lemma 2]. *Let the functions $f(x)$ and $\varphi(x)$ be determined on the segment $[a; b]$. Thereby, if $m\{x \in [a, b] : |f(x)| > \lambda = o(1/\lambda)$, $\lambda \to +\infty$, $|\varphi(x)| \le D$ for $x \in [a, b]$, where D is a positive constant, then*

$$\lim_{\lambda \to +\infty} \left\{ \int\limits_{\{x \in [a;b] : |f(x)\varphi(x)| \le \lambda\}} f(x)\varphi(x)dx - \int\limits_{\{x \in [a;b] : |f(x)| \le \lambda\}} f(x)\varphi(x)dx \right\} = 0.$$

2.1. Let E be an open set on $\gamma = \partial G$, $E \ne \emptyset$, $E \ne \gamma$, and $\alpha > 1$ be a given number. Denote $P = \gamma \backslash E$,

$$r = \frac{\alpha - 1}{3\alpha}, \ \beta = \frac{2\alpha + 1}{\alpha + 2}, \ \delta = \frac{1}{2} \arccos \frac{1}{\beta}, \ n = \left[\frac{\pi}{\delta}\right] + 1, \tag{2.3}$$

where $[x]$ is an entire part of a number $x \in R$.

Using Theorem F, from the system $\{B(t, r\rho(t, P))\}_{t \in E}$ we choose at most a denumerable set of the circles $\{B_q\}_{q \in Q}$ ($B_q = B(t_q, r\rho(t_q, P))$, $q \in Q$), such that

$$E \subset \bigcup_{q \in Q} B_q \tag{2.4}$$

and each point from $\mathbb{C}$ is overlapped by at most θ circles from $\{B_q\}$, and the number θ is an absolute constant. As each point from E is overlapped by at most θ circles from $\{B_q\}$, we get the estimation

$$\sum_{q \in Q} m(E \cap B_q) \le \theta m E.$$

Since the point t_q (the center of the circle B_q) belongs to the set E, then the measure of intersection of the set E with the circle B_q is smaller than the diameter of the circle B_q, i.e. $m(E \cap B_q) \ge 2r\rho(t_q, P)$, $q \in Q$, and thus we have

$$\sum_{q \in Q} \rho(t_q, P) \le \frac{\theta}{2r} m E. \tag{2.5}$$

Let $l_k,\ k=\overline{0,n-1}$ be straight lines in the plane $\mathbb{C}$ given by the equations

$$x\cdot\sin\left(\frac{\pi}{n}k\right)-y\cdot\cos\left(\frac{\pi}{n}k\right)=0,\ k=\overline{0,n-1}.$$

For each point $t_q,\ q\in Q$, divide the plane $\mathbb{C}$ into $2n$ sectors by the straight lines parallel to $l_k,\ k=\overline{0,n-1}$, and crossing the point t_q. Denote these sectors by $S_q^{(k)},\ k=\overline{1,2n}$. Let

$$P_q^{(k)}=P\cap\overline{S_q^{(k)}},\ k=\overline{1,2n}.$$

Since the set $P_q^{(k)}$ is closed, we can take a point $t_q^{(k)}\in P_q^{(k)}$, $k=\overline{1,2n}$, such that

$$\rho\left(t_q,P_q^{(k)}\right)=|t_q-t_q^{(k)}|,\ k=\overline{1,2n}.$$

Denoting

$$\tau_q^{(k)}=t_q+\frac{t_q-t_q^{(k)}}{\beta^2-1},\ k=\overline{1,2n},$$

we set

$$K_q=\bigcap_{k=1}^{2n}B\left(\tau_q^{(k)},\frac{\beta}{\beta^2-1}|t_q-t_q^{(k)}|\right),\ q\in Q,\tag{2.6}$$

and

$$G(E,\alpha)=G\setminus\bigcup_{q\in Q}K_q.\tag{2.7}$$

Remark 4. *In the paper* [8], *T.S. Salimov considered the set $G(E,\alpha)=G\setminus\cup_{q\in Q}B'_q$, where B'_q is a circle centered at the point t_q and of radius $h\rho(t_q,P)$, $h\in(0;1)$. Unfortunately, this set is not suitable for obtaining the necessary result for $\alpha\in(1;2]$ for the domains with boundaries having external angles smaller than $4\arccos\frac{\alpha}{2}$. Under such structure of the set $G(E,\alpha)$, if E is a sufficiently small vicinity of the point t_0 on $\gamma=\partial G$, with an external angle smaller than $4\arccos\frac{\alpha}{2}$, then for the point z_0 on the internal bisectrix* (*and for the points z in some proximity z_0*) *at the distance $r\rho(t_0,P)$ from t_0, neither the inequality $\rho(z,P)<\alpha\cdot\rho(z,\gamma)$* (*see Lemma 2.1 below*) *that provides inclusion $z\in F^*_\alpha(t_0)$, nor the inequality $|F(z)|\le\lambda$ is fulfilled.*

In the present paper, we consider the set $G(E,\alpha)=G\setminus\cup_{q\in Q}K_q$, where K_q is determined by formula (2.6) *that allows to establish Theorem B for the case of the values $\alpha\in(1;2)$.*

Lemma 2.1. *If $z\in G(E,\alpha)$, then $\rho(z,P)<\alpha\rho(z,\gamma)$.*

P r o o f. Choose a point $t\in\gamma$ such that $\rho(z,\gamma)=|z-t|$. In the case $t\in P$ we get $\rho(z,P)=|z-t|=\rho(z,\gamma)<\alpha\rho(z,\gamma)$. Consider the case $t\in E$. By (2.4), $t\in B_q$ for some $q\in Q$, hence for any $k=\overline{1,2n}$

$$|t_q-t|\le r\rho(t_q,P)\le r\rho(t_q,P_q^{(k)})=r|t_q-t_q^{(k)}|.\tag{2.8}$$

Further, in virtue of $z \notin K_q$ (see (2.7)) there exists $k_0 \in \{1, 2, \ldots, 2n\}$ such that for

$$z \notin B\left(\tau_q^{(k_0)}, \frac{\beta}{\beta^2 - 1}|t_q - t_q^{(k_0)}|\right). \tag{2.9}$$

It is easy to show that the geometric place of the points ξ, satisfying the inequality

$$|\xi - t_q^{(k_0)}| \geq \beta|\xi - t_q|,$$

is the circle $B\left(\tau_q^{(k_0)}, \frac{\beta}{\beta^2 - 1}|t_q - t_q^{(k_0)}|\right)$. Therefore, from (2.9) we get

$$|z - t_q^{(k_0)}| < \beta|z - t_q|. \tag{2.10}$$

By virtue of the triangle inequality, from (2.8) and (2.10) we have

$$|z - t_q| \leq |z - t| + |t_q - t| \leq |z - t| + r|t_q - t_q^{(k_0)}|$$

$$\leq |z - t| + r\left\{|z - t_q| + |z - t_q^{(k_0)}|\right\} < |z - t| + r(1 + \beta)|z - t_q|,$$

whence

$$|z - t_q| < \frac{1}{1 - r(1 + \beta)}|z - t|. \tag{2.11}$$

Taking into account that $|z - t| = \rho(z, \gamma)$ and $\rho(z, P) \leq \rho(z, P_q^{(k_0)}) \leq |z - t_q^{(k_0)}|$, by (2.10), (2.11) and (2.3) we get

$$\rho(z, P) \leq |z - t_q^{(k_0)}| < \beta|z - t_q| < \frac{\beta}{1 - r(1 + \beta)}\rho(z, \gamma) = \alpha\rho(z, \gamma). \qquad \blacksquare$$

Lemma 2.2. *Let $t \in P$ and a tangent to γ exist at the point t. Then there might be found a number $\varepsilon > 0$ such that for all the points z lying on the internal normal to G at the point t and satisfying the inequality $|z - t| < \varepsilon$, the inclusion $z \in G(E, \alpha)$ holds.*

P r o o f. Choose $\varepsilon_1 > 0$ such that if the point z lies on the internal normal to G at the point t and $|z - t| < \varepsilon_1$, then $z \in G$. Choose $\varepsilon_2 > 0$ such that if $\tau \in \gamma$ and $|\tau - t| < \varepsilon_2$, then the angle between the tangent at the point t and the ray $(t\tau)$ is smaller than δ_0.

Assume $\varepsilon = \min\{\varepsilon_1, \frac{\beta - 1}{\beta}\varepsilon_2\}$. Let the point z lie on the internal normal to G at the point t and $|z - t| < \varepsilon$. Then $|z - t| < \varepsilon_1$, and therefore $z \in G$. For proving the lemma, by (2.7) it is enough to verify that $z \notin K_q$ for all $q \in Q$. Since for any $q \in Q$ the plane $\mathbb{C}$ is divided into $2n$ sectors $S_q^{(k)}$, then there exists such a number $k_1 \in \{1, 2, \ldots, 2n\}$ that $t \in S_q^{(k_1)}$. Denote an angle between the tangent at the point t and the chord $[t, t_q]$ by φ_1, and the angle between the straight lines (tt_q) and $(t_q t_q^{(k_1)})$ by φ_2. Two cases are possible.

1) Case $|t - t_q| < \varepsilon_2$. Here, by $t_q \in \gamma$ and $t \in S_q^{(k_1)}$, there follows that $\varphi_1 < \delta$ and $\varphi_2 < \dfrac{2\pi}{2n} < \delta$, respectively (see (2.3)). Let ℓ be a tangent to γ at the point t. Since $\beta = \dfrac{2\alpha+1}{\alpha+2} < 2$ and $\delta = \dfrac{1}{2}\arccos\dfrac{1}{\beta} < \dfrac{\pi}{6}$ for any $\alpha > 1$, we have

$$|z - \tau_q^{(k_1)}| \geq np_\ell\left[z; \tau_q^{(k_1)}\right] = np_\ell\,[t; t_q] + np_\ell\left[t_q; \tau_q^{(k_1)}\right] = |t_q - t| \cdot \cos\varphi_1$$

$$+|\tau_q^{(k_1)} - t_q| \cdot \cos(\varphi_1 + \varphi_2) > \cos(2\delta) \cdot \left[|t_q - t| + |\tau_q^{(k_1)} - t_q|\right],$$

where $np_\ell[z_1; z_2]$ is a length of the projection of the segment $[z_1, z_2]$ on the straight line ℓ.

Taking into consideration that $\cos(2\delta) = 1/\beta$ (see (2.3)), $|t_q - t| \leq \rho(t_q, P_q^{(k_1)})$ $= |t_q - t_q^{(k_1)}|$ by $t \in S_q^{(k_1)}$, and $|\tau_q^{(k_1)} - t_q| = \dfrac{1}{\beta^2 - 1}|t_q - t_q^{(k_1)}|$ by definition of $\tau_q^{(k)}$, we get

$$|z - \tau_q^{(k_1)}| > \frac{1}{\beta}\left[|t_q - \tau_q^{(k_1)}| + \frac{1}{\beta^2 - 1}|t_q - t_q^{(k_1)}|\right] = \frac{\beta}{\beta^2 - 1}|t_q - t_q^{(k_1)}|.$$

Hence it follows that $z \notin B\left(\tau_q^{(k_1)}, \dfrac{\beta}{\beta^2 - 1}|t_q - t_q^{(k_1)}|\right)$ and, consequently, $z \notin K_q$.

2) Case $|t - t_q| \geq \varepsilon_2$. By the triangle inequality and the condition $|z - t| < \varepsilon \leq \dfrac{\beta - 1}{\beta}\varepsilon_2$ we have

$$|z - \tau_q^{(k_1)}| \geq |t - \tau_q^{(k_1)}| - |t - z|$$

$$\geq |t - \tau_q^{(k_1)}| - \frac{\beta - 1}{\beta} \cdot \varepsilon_2 \geq |t - \tau_q^{(k_1)}| - \frac{\beta - 1}{\beta}|t - t_q|. \tag{2.12}$$

Denote by τ' a projection of the point $\tau_q^{(k_1)}$ on the straight line (tt_q). Since $\varphi_2 < \delta < \dfrac{\pi}{6}$, then

$$|t - \tau_q^{(k_1)}| \geq |t - \tau'| = |t - t_q| + |t_q - \tau'| = |t - t_q| + |\tau_q^{(k_1)} - t_q|\cos\varphi_2$$

$$> |t - t_q| + |\tau_q^{(k_1)} - t_q|\cos(2\delta) = |t - t_q| + \frac{1}{\beta}|\tau_q^{(k_1)} - t_q|.$$

From the above and from (2.12) it follows that

$$|z - \tau_q^{(k_1)}| \geq \frac{1}{\beta}\left[|t - t_q| + |\tau_q^{(k_1)} - t_q|\right].$$

Further, taking into account that $|t_q - t| \geq \rho(t_q, P_q^{(k_1)}) = |t_q - t_q^{(k_1)}|$, by $t \in S_q^{(k_1)}$ and $|\tau_q^{(k_1)} - t_q| = \dfrac{1}{\beta^2 - 1}|t_q - t_q^{(k_1)}|$ by the definition of $\tau_q^{(k)}$, we get

$$|z - \tau_q^{(k_1)}| \geq \frac{1}{\beta}\left[|t_q - t_q^{(k_1)}| + \frac{1}{\beta^2-1}|t_q - t_q^{(k_1)}|\right] = \frac{\beta}{\beta^2-1}\left[|t_q - t_q^{(k_1)}|\right]$$

and, consequently, $z \notin K_q$.

Lemma 2.2 is proved.

Lemma 2.3. *$G(E,\alpha)$ is an open subset of the plane $\mathbb{C}$.*

P r o o f. Let $z \in G(E,\alpha)$. By (2.7), $z \in G$ and therefore $\rho(z,\gamma) > 0$. At first prove that for all $q \in Q$ there holds the inclusion

$$K_q \subset B\left(t_q, \frac{1}{\beta-1}\rho(t_q,P)\right). \tag{2.13}$$

In fact, for any point $z \in K_q$, by the definition of the set K_q, we have $z \in B\left(\tau_q^{(k)}, \dfrac{\beta}{\beta^2-1}|t_q - t_q^{(k)}|\right)$ for each $k = \overline{1,2n}$. Then, $|z - \tau_q^{(k)}| < \dfrac{\beta}{\beta^2-1}|t_q - t_q^{(k)}|$ and

$$|z - t_q| = |z - \tau_q^{(k)} + \tau_q^{(k)} - t_q| = \left|z - \tau_q^{(k)} + \frac{t_q - t_q^{(k)}}{\beta^2-1}\right|$$

$$\leq |z - \tau_q^{(k)}| + \frac{1}{\beta^2-1}\cdot|t_q - t_q^{(k)}| < \frac{\beta}{\beta^2-1}\cdot|t_q - t_q^{(k)}| + \frac{1}{\beta^2-1}\cdot|t_q - t_q^{(k)}|$$

$$= \frac{1}{\beta-1}\cdot|t_q - t_q^{(k)}| \leq \frac{1}{\beta-1}\rho(t_q,P),$$

whence inclusion (2.13) follows.

If some circle $B\left(t_q, \dfrac{1}{\beta-1}\rho(t_q,P)\right)$ intersects a circle $B(z,\rho(z,\gamma)/2)$, then by $t_q \in \gamma$ it follows that

$$\frac{1}{\beta-1}\rho(t_q,P) + \rho(z,\gamma)/2 \geq \rho(z,\gamma)$$

and thus $\rho(t_q,P) \geq \dfrac{\beta-1}{2}\rho(z,\gamma)$. The above and inclusion (2.4) gives that the circle $B(z,\rho(z,\gamma)/2)$ can intersect only a finite number of circles from $\left\{B\left(t_q, \dfrac{1}{\beta-1}\rho(t_q,P)\right)\right\}_{q\in Q}$. Taking also into account inequality (2.11), we get that the circle $B(z,\rho(z,\gamma)/2)$ can intersect only a finite number of sets from the family $\{K_q\}_{q\in Q}$. Since the sets K_q, $q \in Q$ are closed, then for some $\delta > 0$ the circle $B(z,\delta)$ intersects none of the sets $K_q,\ q \in Q$. Then, by $z \in G$ and (2.7) it follows that some vicinity of the point z is contained in $G(E,\alpha)$.

Lemma 2.3. is proved.

Lemma 2.4. *The following relations hold:*

$$\partial G(E,\alpha) \cap E = \emptyset, \quad \partial G(E,\alpha) \subset P \cup \left(\underset{q\in Q}{\cup} \partial K_q\right),$$

where $P = \gamma \backslash E$.

P r o o f. It follows from the definition of the set $G(E,\alpha)$ (2.7) and Lemma 2.3 that

$$\partial G(E,\alpha) \subset \gamma \cup \left(\underset{q\in Q}{\cup} \partial K_q\right).$$

Therefore, it is enough to prove that $\partial G(E,\alpha) \cap E = \emptyset$. If $t \in E$, then by (2.4), $t \in B_q$ for some $q \in Q$. Subsequently, $|t - t_q| \le r\rho(t_q, P)$. Since for any $k \in \{1, 2, \ldots, 2n\}$ the inequality $|t_q - t_q^{(k)}| \ge \rho(t_q, P)$ is fulfilled, and by (2.3), $r < (\beta + 1)^{-1}$, we have

$$|t - \tau_q^{(k)}| \le |t - t_q| + |t_q - \tau_q^{(k)}| \le r\rho(t_q, P)$$

$$+\frac{1}{\beta^2 - 1}|t_q - t_q^{(k)}| \le \left[r + \frac{1}{\beta^2 - 1}\right]|t_q - t_q^{(k)}| < \frac{\beta}{\beta^2 - 1}|t_q - t_q^{(k)}|.$$

From whence it follows that t is an internal point of the set K_q, and therefore $t \notin \partial G(E,\alpha)$.

Lemma 2.4 is proved.

Let $\{G_j\}_{j\in J}$ be the connected components of the set $G(E,\alpha)$. From the construction of $G(E,\alpha)$ it follows that ∂G_j is a Jordan curve for any $j \in J$. Further, by Lemma 2.4 we have

$$\partial G_j \backslash \gamma \subset \underset{q\in Q}{\cup} \partial K_q, \tag{2.14}$$

$$\partial G_j \cap \gamma \subset P,\ j \in J. \tag{2.15}$$

The set K_q is an intersection of the $2n$ circles and therefore K_q is a convex set. Since one of these circles is of radius $\dfrac{\beta}{\beta^2 - 1}\rho(t_q, P)$ and K_q is a subset of this circle, then the length of the boundary K_q is not greater than the boundary of the indicated circle, i.e. not greater than $\dfrac{2\pi\beta}{\beta^2 - 1}\rho(t_q, P)$. Therefore, from (2.14) and (2.5) we get that ∂G_j is rectifiable, $j \in J$.

If at the point $t \in P$ there exists a tangent γ, then by Lemma 2.2, $t \in \partial G_j$ for some $j \in J$ and, consequently,

$$m\left(P \backslash \underset{j\in J}{\cup} \partial G_j\right) = 0. \tag{2.16}$$

Lemma 2.5. *If $j_1, j_2 \in J$ and $j_1 \neq j_2$, then the intersection $\partial G_{j_1} \cap \partial G_{j_2}$ contains at most one point.*

P r o o f. By the construction of $G(E, \alpha)$, $\partial G_{j_1} \cap \partial G_{j_2}$ does not contain an arch. Therefore, if this set contains at least two points, then the set $\mathbb{C}\backslash(G_{j_1} \cup G_{j_2})$ contains a bounded connected component D. Since $\partial D \subset (\partial G_{j_1} \cup \partial G_{j_2}) \subset \overline{G}$, then from the coherence of G we get $D \subset G$. Furthermore, $D \cap G(E, \alpha) \neq D$, otherwise the set $G_{j_1} \cup G_{j_2} \cup D$ would be contained in some connected component of the set $G(E, \alpha)$, that is impossible. Further, by $D \subset G$ and $D \cap G(E, \alpha) \neq D$, we have $D \cap K_q \neq D$ for some $q \in Q$. Thus, taking into account the inclusion $D \subset G$, we get that the intersection of ∂D with the interior of K_q is not empty. The last statement contradicts the inclusion $\partial D \subset (\partial G_{j_1} \cup \partial G_{j_2})$ and the inequality (2.5), and this proves the lemma.

2.2. Proof of item a) of Theorem 2.

Denote

$$E_\lambda = \{t \in \gamma; F^*_\alpha > \lambda\},\ P_\lambda = \gamma \backslash E_\lambda.$$

If $E_\lambda \neq \emptyset$ for some $\lambda > 0$, then $|F(z)| \leq \lambda$ for all $z \in G$ and the validity of items a) and b) of Theorem 2 is established (see [12, pp. 28 and 42]).

Now, let $E_\lambda \neq \emptyset$ for all $\lambda > 0$. It follows from the definition of the function F^*_α that E_λ is an open subset of γ. Let $G(E_\lambda, \alpha)$ be a set constructed in subsection 2.1, $\{G_j\}_{j\in J}$ be the connected components of $G(E_\lambda, \alpha)$. On ∂G_j choose the positive orientation (with respect to G_j). It follows from Lemma 2.1 and the definition of the function F^*_α that $|F(z)| \leq \lambda$ for $z \in G(E_\lambda, \alpha)$. Therefore, $|F(t)| \leq \lambda$ for almost all $t \in \partial G_j$, and the following equality is valid:

$$\int\limits_{\partial G_j} F(t)dt = 2i \iint\limits_{G_j} \frac{\partial F}{\partial \bar{\xi}} dxdy,\ \xi = x + iy,$$

and

$$\left| \int\limits_{\partial G_j \cap \gamma} F(t)dt - 2i \iint\limits_{\partial G_j} \frac{\partial F}{\partial \bar{\xi}} dxdy \right| = \left| \int\limits_{\partial G_j \backslash \gamma} F(t)dt \right| \leq \lambda \int\limits_{\partial G_j \backslash \gamma} |dt|.$$

By summing up these inequalities and taking into account (2.14) and Lemma 2.6, we get

$$\sum_{j\in J}\left|\int\limits_{\partial G_j\cap\gamma} F(t)dt - 2i\iint\limits_{G_j}\frac{\partial F}{\partial\bar{\xi}}dxdy\right| \le \lambda\sum_{q\in Q}\int\limits_{\partial K_q}|dt|.$$

As the length of the boundary of the set K_q is not greater than $2\pi\beta(\beta^2-1)^{-1}$ $\times\rho(t_q,P)$, by using (2.5) and taking into account (2.3), we obtain

$$\sum_{j\in J}\left|\int\limits_{\partial G_j\cap\gamma} F(t)dt - 2i\iint\limits_{G_j}\frac{\partial F}{\partial\bar{\xi}}dxdy\right| \le \theta\frac{3\pi\beta\alpha}{(\alpha-1)(\beta^2-1)}\lambda m E_\lambda.$$

Further, by (2.15), (2.16) and Lemma 2.5, we have

$$\int\limits_{P_\lambda} F(t)dt = \sum_{j\in J}\int\limits_{\partial G_j\cap\gamma} F(t)dt.$$

From the last two relations we get the estimation

$$\left|\int\limits_{P_\lambda} F(t)dt - 2i\iint\limits_{G(E_\lambda,\alpha)}\frac{\partial F}{\partial\bar{\xi}}dxdy\right| \le \theta\frac{3\pi\beta\alpha}{(\alpha-1)(\beta^2-1)}\lambda m E_\lambda. \tag{2.17}$$

Denote by P'_λ only a set of the points $t\in\gamma$ having the angular boundary value $F(t)$, and $|F(t)|\le\lambda$. The conditions of the theorem imply that $m(P_\lambda\backslash P'_\lambda)=0$ and, therefore,

$$\int\limits_{P'_\lambda} F(t)dt = \int\limits_{P_\lambda} F(t)dt + \int\limits_{P'_\lambda\backslash P_\lambda} F(t)dt.$$

On the other hand, we have

$$\left|\int\limits_{P'_\lambda\backslash P_\lambda} F(t)dt\right| \le \lambda m(P'_\lambda\backslash P_\lambda) \le \lambda m(\gamma\backslash P_\lambda) = \lambda m E_\lambda. \tag{2.18}$$

By estimations (2.17) and (2.18), we get

$$\left|\int\limits_{P'_\lambda} F(t)dt - 2i\iint\limits_{G}\frac{\partial F}{\partial\bar{\xi}}dxdy\right|$$

$$\leq \left[1+\theta\frac{3\pi\beta\alpha}{(\alpha-1)(\beta^2-1)}\right]\lambda mE_\lambda + \iint\limits_{G\backslash G(E_\lambda,\alpha)} \left|\frac{\partial F}{\partial\bar\xi}\right| dxdy.$$

Since the area of the domain $G\backslash G(E_\lambda,\alpha)$ tends to zero as $\lambda\to+\infty$, then from conditions 1) and 2) of Theorem 2 there follows the equality from item a) of Theorem 2.

2.3. Proof of item b) of Theorem 2.

Fixing $z\in G$ and using condition 2) of Theorem 2 choose $\lambda_0>0$ such that

$$mE_{\lambda_0} < \frac{1}{2}\min\{(\beta^2-1)\rho(z,\gamma), m\gamma\}. \tag{2.19}$$

Now, let an arbitrary $\lambda>\lambda_0$. The set E_λ is open, $E_\lambda\neq\emptyset$, and by (2.19), $E_\lambda\neq\gamma$. Using the construction from subsection 2.1 (for $E=E_\lambda$), we get a system of the sets $\{K_q\}_{q\in Q}$ and the set $G(E_\lambda,\alpha)$. Let $\{G_j\}_{j\in J}$ be connected components of $G(E_\lambda,\alpha)$. Choose the positive orientation on ∂G_j (with respect to G_j), $j\in J$.

At first prove that

$$\rho(z,K_q) > \frac{1}{2}\rho(z,\gamma),\ q\in Q. \tag{2.20}$$

Taking into account that $K_q=\bigcap\limits_{k=1}^{2n} B\left(\tau_q^{(k)}, \frac{\beta}{\beta^2-1}|t_q-t_q^{(k)}|\right)$, for any $k\in\{1,2,\dots,2n\}$ we have

$$\rho(z,K_q)\geq\rho\left(z, B\left(\tau_q^{(k)},\frac{\beta}{\beta^2-1}|t_q-t_q^{(k)}|\right)\right)$$

$$\geq|z-\tau_q^{(k)}|\geq|z-t_q|-|t_q-\tau_q^{(k)}|=|z-t_q|-\frac{1}{\beta^2-1}|t_q-t_q^{(k)}|.$$

Let $\rho(t_q,P_\lambda)=|t_q-t_q^{(k_1)}|$. Since $|z-t_q|\geq\rho(z,\gamma)$ and $\rho(z,P_\lambda)\leq mE_\lambda$, then we get

$$\rho(z,K_q)\geq\rho(z,\gamma)-\frac{1}{\beta^2-1}mE_\lambda.$$

Noticing also that by (2.19) $mE_\lambda>\dfrac{\beta^2-1}{2}\rho(z,\gamma)$, we prove the validity of the estimation (2.20).

From (2.7) and (2.20) we have $z\in G(E_\lambda,\alpha)$. Furthermore, it follows from the definition of the set P_λ and Lemma 2.2 that $|F(z')|\leq\lambda$ for $z'\in G(E_\lambda,\alpha)$. Consequently, in $G(E_\lambda,\alpha)$ the Cauchy–Green integral formula is applicable to the function F. Thus, we have

$$F(z)=\frac{1}{2\pi i}\sum_{j\in J}\int\limits_{\partial G_j}\frac{F(t)}{t-z}dt-\frac{1}{\pi}\iint\limits_{G(E_\lambda,\alpha)}\frac{\partial F}{\partial\bar\xi}\frac{dxdy}{\xi-z},\ \xi=x+iy,$$

and

$$\left| F(z) - \frac{1}{2\pi i} \sum_{j\in J} \int_{\partial G_j \cap \gamma} \frac{F(t)}{t-z} dt + \frac{1}{\pi} \iint_G \frac{\partial F}{\partial \bar{\xi}} \frac{dxdy}{\xi - z} \right|$$

$$\leq \frac{1}{2\pi} \sum_{j\in J} \int_{\partial G_j \setminus \gamma} \frac{|F(t)|}{|t-z|} |dt| + \frac{1}{\pi} \iint_{G\setminus G(E_\lambda,\alpha)} \left| \frac{\partial F}{\partial \bar{\xi}} \right| \frac{dxdy}{|\xi - z|}$$

$$\leq \frac{\lambda}{2\pi} \sum_{j\in J} \int_{\partial G_j \setminus \gamma} \frac{|dt|}{|t-z|} + \frac{1}{\pi} \iint_{G\setminus G(E_\lambda,\alpha)} \left| \frac{\partial F}{\partial \bar{\xi}} \right| \frac{dxdy}{|\xi - z|} = \triangle_1 + \triangle_2.$$

Estimate $\triangle_1$ and $\triangle_2$. Taking into account (2.14), Lemma 2.5 and inequality (2.20), for $\triangle_1$ we have

$$\triangle_1 \leq \frac{\lambda}{\pi \rho(z,\gamma)} \sum_{q\in Q} \int_{\partial K_q} |dt|.$$

Since the length of the boundary K_q is not greater than $2\pi\beta(\beta^2-1)^{-1}\rho(t_q, P)$, by inequality (2.5) we get

$$\triangle_1 \leq \theta \frac{3\beta\alpha}{(\alpha-1)(\beta^2-1)\rho(z,\gamma)} \lambda m E_\lambda.$$

From the above and from condition 2) of Theorem 2 it follows that $\triangle_1$ tends to zero as $\lambda \to +\infty$.

Further, by inequality (2.20), for $\triangle_2$ we have

$$\triangle_2 \leq \frac{2}{\pi \rho(z,\gamma)} \iint_{G\setminus G(E_\lambda,\alpha)} \left| \frac{\partial F}{\partial \bar{\xi}} \right| dxdy.$$

Since the area of the domain $G\setminus G(E_\lambda,\alpha)$ tends to zero as $\lambda \to +\infty$, it follows from condition 1) of Theorem 2 that $\triangle_2$ also tends to zero as $\lambda \to +\infty$.

Taking also into account that by (2.15), (2.16) and Lemma 2.5

$$\sum_{j\in J} \int_{\partial G_j \cap \gamma} \frac{F(t)}{t-z} dt = \int_{P_\lambda} \frac{F(t)}{t-z} dt,$$

we get

$$F(z) = \lim_{\lambda\to+\infty} \frac{1}{2\pi i} \int_{P_\lambda} \frac{F(t)}{t-z} dt - \frac{1}{\pi} \iint_G \frac{\partial F}{\partial \bar{\xi}} \frac{dxdy}{\xi - z}. \tag{2.21}$$

Since $m(P_\lambda \backslash P'_\lambda) = 0$, then

$$\left| \int\limits_{P'_\lambda} \frac{F(t)}{t-z} dt - \int\limits_{P_\lambda} \frac{F(t)}{t-z} dt \right| = \left| \int\limits_{P'_\lambda \backslash P_\lambda} \frac{F(t)}{t-z} dt \right|$$

$$\leq \frac{2\lambda}{\rho(z,\gamma)} m(P'_\lambda \backslash P_\lambda) \leq \frac{2\lambda}{\rho(z,\gamma)} m E_\lambda .$$

The last expression tends to zero as $\lambda \to +\infty$ by condition 2) of Theorem 2. Hence, in virtue of (2.21) and Theorem G there follows equality (2.2). The theorem is proved.

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