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Asymptotics of Rarefaction Wave Solution to the mKdV Equation

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The modified Korteveg de Vries equation

$$q_t + 6q^2 q_x + q_{xxx} = 0$$

on the line is considered. The initial data is the pure step function, i.e. $q(x,0) = c_r$ for $x \geq 0$ and $q(x,0) = c_l$ for $x < 0$, where $c_l > c_r > 0$ are arbitrary real numbers. The goal of this paper is to study the asymptotic behavior of the solution of initial-value problem as $t \to -\infty$, i.e. to study the long-time dynamics of the rarefaction wave. Using the steepest descent method and the so-called g-function mechanism we deform the original oscillatory matrix Riemann–Hilbert problem to the explicitly solvable model forms and show that the solution of the initial-value problem has different asymptotic behavior in different regions of the xt-plane. In the regions $x < 6c_l^2 t$ and $x > 6c_r^2 t$, the main term of asymptotics of the solution is equal to c_l and c_r, respectively. In the region $6c_l^2 t < x < 6c_r^2 t$, the asymptotics of the solution tends to $\sqrt{\dfrac{x}{6t}}$.

Key words: nonlinear equations, Riemann–Hilbert problem, the steepest descent method, asymptotics.

Mathematics Subject Classification 2000: 35Q15, 35B40.

1. Introduction

The step-like initial value problems for the Korteweg–de Vries equation were firstly studied by Gurevich, Pitaevsky [1] and Khruslov [2]. Using the inverse

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scattering transform in the form of Marchenko integral equations and the so-called asymptotic solitons, Khruslov and Kotlyarov (sf. [3–8]) studied the asymptotic behavior of solutions of the step-like initial value problems in a neighborhood of leading edge. Egorova, Teschl and coauthors (see [9, 10] and the references therein) obtained very deep and rigorous results related to the Toda hierarchy with quasi-periodic background and the Cauchy problem for the Korteweg–de Vries equation with step-like finite-gap initial data. Remarkable results on the problems for integrable PDEs with the different finite-gap boundary conditions as $x \to \pm\infty$ were obtained by Bikbaev [11–13], Novokshenov [14] and others (see the references in [11–14]).

In the short note [11] the initial value problem

$$q_t + 6q^2 q_x + q_{xxx} = 0, \tag{1.1}$$

$$q(x,0) = q_0(x) \to \begin{cases} c_r, & x \to +\infty \\ c_l, & x \to -\infty \end{cases} \tag{1.2}$$

was considered. The initial function was supposed to be an arbitrary step-like function, i.e. $q_0(x)$ was sufficiently smooth and $q_0(x) \to c_l$ as $x \to -\infty$, $q_0(x) \to c_r$ as $x \to \infty$ ($c_l > c_r \geq 0$). The author announced (without any justification) that the solution of the problem tends to $\sqrt{\dfrac{x}{6t}}$ when $6c_l^2 t < x < 6c_r^2 t$ and $t \to -\infty$, and it becomes equal to c_l or c_r when $x < 6c_l^2 t$ and $x > 6c_r^2 t$, respectively. Our goal is to justify this result in a modern rigorous form and to bring new results to the theory of shock problems, especially for the case of non self-adjoint Lax operators, and to develop the so-called g-function mechanism which allows us to deform the original oscillatory matrix Riemann–Hilbert problem to the explicitly solvable model forms. We suppose that the solution $q(x,t)$ of this problem

- exists for $x \in \mathbb{R}$, and $t \in \mathbb{R}_+$;
- is sufficiently smooth;
- tends (rapidly) to the constant c_r as $x \to \infty$ and to the constant c_l as $x \to -\infty$.

2. Jost Solutions of the Lax Equations

To study the initial value problem (1.1)–(1.2), we will use the Lax representation of the mKdV equation given in [15, 16] in the form of the overdetermined system of differential equations

$$\Phi_x + \mathrm{i}k\sigma_3\Phi = Q(x,t)\Phi, \tag{2.1}$$

$$\Phi_t + 4\mathrm{i}k^3\sigma_3\Phi = \hat{Q}(x,t,k)\Phi, \tag{2.2}$$

where $\Phi = \Phi(x,t,k)$ is a 2×2 matrix-valued function,

$$\sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad Q(x,t) := \begin{pmatrix} 0 & q(x,t) \\ -q(x,t) & 0 \end{pmatrix},$$

$$\hat{Q}(x,t,k) = 4k^2Q(x,t,k) - 2ik(Q^2(x,t,k) + Q_x(x,t,k))\sigma_3 + 2Q^3(x,t,k) - Q_{xx}(x,t,k),$$

and $k \in \mathbb{C}$. Equations (2.1) and (2.2) are compatible if and only if the function $q(x,t)$ satisfies mKdV equation (1.1). To apply the inverse scattering transform to problem (1.1)–(1.2), we have to define the matrix valued Jost solutions of the Lax equations. We define them as the solutions of compatible equations (2.1) and (2.2) satisfying the asymptotic conditions

$$\Phi_r(x,t,k) = E_r(x,t,k) + o(1), \quad x \to +\infty, \quad \operatorname{Im} k = 0, \tag{2.3}$$

$$\Phi_l(x,t,k) = E_l(x,t,k) + o(1), \quad x \to -\infty, \quad \operatorname{Im} k = 0. \tag{2.4}$$

Here $E_l(x,t,k)$, $E_r(x,t,k)$ are the solutions of the linear differential equations

$$\begin{aligned} E_{lx} + \mathrm{i}k\sigma_3 E_l &= Q_{c_l} E_l, \\ E_{lt} + 4\mathrm{i}k^3\sigma_3 E_l &= \hat{Q}_{c_l}(k) E_l, \end{aligned} \tag{2.5}$$

$$\begin{aligned} E_{rx} + \mathrm{i}k\sigma_3 E_r &= Q_{c_r} E_r, \\ E_{rt} + 4\mathrm{i}k^3\sigma_3 E_r &= \hat{Q}_{c_r}(k) E_r \end{aligned} \tag{2.6}$$

with the constant matrix coefficients

$$Q_{c_l} := \begin{pmatrix} 0 & c_l \\ -c_l & 0 \end{pmatrix},$$

$$\hat{Q}_{c_l}(k) = 4k^2 Q_{c_l} - 2\mathrm{i}k Q_{c_l}^2 \sigma_3 + 2Q_{c_l}^3,$$

$$Q_{c_r} := \begin{pmatrix} 0 & c_r \\ -c_r & 0 \end{pmatrix},$$

$$\hat{Q}_{c_r}(k) = 4k^2 Q_{c_r} - 2\mathrm{i}k Q_{c_r}^2 \sigma_3 + 2Q_{c_r}^3.$$

We choose the solutions $E_l(x,t,k)$, $E_r(x,t,k)$ in the form

$$E_{l,r}(x,t,k) = \frac{1}{2}\begin{pmatrix} \varkappa_{l,r}(k) + \dfrac{1}{\varkappa_{l,r}(k)} & \varkappa_{l,r}(k) - \dfrac{1}{\varkappa_{l,r}(k)} \\ \varkappa_{l,r}(k) - \dfrac{1}{\varkappa_{l,r}(k)} & \varkappa_{l,r}(k) + \dfrac{1}{\varkappa_{l,r}(k)} \end{pmatrix} \mathrm{e}^{-\mathrm{i}xX_{l,r}(k)\sigma_3 - \mathrm{i}t\Omega_{l,r}(k)\sigma_3},$$

where

$$X_{l,r}(k) = \sqrt{k^2 + c_{l,r}^2}, \quad \Omega_{l,r}(k) = 2(2k^2 - c_{l,r}^2)X_{l,r}(k), \quad \varkappa_{l,r}(k) = \sqrt[4]{\frac{k - \mathrm{i}c_{l,r}}{k + \mathrm{i}c_{l,r}}}. \tag{2.7}$$

The branches of the roots are fixed by the conditions $X_{l,r}(1) > 0$, $\varkappa_{l,r}(\infty) = 1$. Then the functions $X_{l,r}(k)$ and $\varkappa_{l,r}(k)$ are analytic in $\mathbb{C}\backslash[\mathrm{i}c, -\mathrm{i}c]$, where $c = c_l$ and $c = c_r$, respectively.

Solutions (2.3), (2.4) can be represented in the forms

$$\Phi_l(x,t,k) = E_l(x,t,k) + \int_{-\infty}^{x} K_l(x,y,t)E_l(y,t,k)dy, \qquad \operatorname{Im} k = 0, \tag{2.8}$$

$$\Phi_r(x,t,k) = E_r(x,t,k) + \int_{x}^{\infty} K_r(x,y,t)E_r(y,t,k)dy, \qquad \operatorname{Im} k = 0, \tag{2.9}$$

where the kernels $K_{l,r}(x,y,t)$ are sufficiently smooth and decrease to zero rapidly as $x + y \to \pm\infty$. Omitting details of the proof of these representations, we formulate below the properties of the solutions.

The matrices $\Phi_l(x,t,k)$ and $\Phi_r(x,t,k)$ are defined by (2.8), (2.9) and their columns $\Phi_{lj}(x,t,k)$ and $\Phi_{rj}(x,t,k)$, $j = 1,2$ have the following properties:

1. Determinants:
   $\det \Phi_{l,r}(x,t,k) = 1$;

2. Analyticity:
   $\Phi_{r1}(x,t,k)$ is analytic in $k \in \mathbb{D}_{r-} := \mathbb{C}_- \setminus [0, -\mathrm{i}c_r]$,
   $\Phi_{r2}(x,t,k)$ is analytic in $k \in \mathbb{D}_{r+} := \mathbb{C}_+ \setminus [0, \mathrm{i}c_r]$,
   $\Phi_{l1}(x,t,k)$ is analytic in $k \in \mathbb{D}_{l+} := \mathbb{C}_+ \setminus [0, \mathrm{i}c_l]$,
   $\Phi_{l2}(x,t,k)$ is analytic in $k \in \mathbb{D}_{l-} := \mathbb{C}_- \setminus [-\mathrm{i}c_l, 0]$;

3. Continuity:
   $\Phi_{r1}(x,t,k)$ is continuous for $k \in \mathbb{D}_{r-} \cup (-\mathrm{i}c_r, \mathrm{i}c_r)_- \cup (-\mathrm{i}c_r, \mathrm{i}c_r)_+$,
   $\Phi_{r2}(x,t,k)$ is continuous for $k \in \mathbb{D}_{r+} \cup (-\mathrm{i}c_r, \mathrm{i}c_r)_- \cup (-\mathrm{i}c_r, \mathrm{i}c_r)_+$,
   $\Phi_{l1}(x,t,k)$ is continuous for $k \in \mathbb{D}_{l+} \cup (-\mathrm{i}c_l, \mathrm{i}c_l)_- \cup (-\mathrm{i}c_l, \mathrm{i}c_l)_+$,
   $\Phi_{l2}(x,t,k)$ is continuous for $k \in \mathbb{D}_{l-} \cup (-\mathrm{i}c_l, \mathrm{i}c_l)_- \cup (-\mathrm{i}c_l, \mathrm{i}c_l)_+$,
   where $(-\mathrm{i}c_{l,r}, \mathrm{i}c_{l,r})_-$ and $(-\mathrm{i}c_{l,r}, \mathrm{i}c_{l,r})_+$ are the left and the right banks of the interval $(-\mathrm{i}c_{l,r}, \mathrm{i}c_{l,r})$;

4. Symmetries:

$$\begin{aligned}
\overline{\Phi_{22}(x,t,\overline{k})} &= \Phi_{11}(x,t,k), & \Phi_{22}(x,t,-k) &= \Phi_{11}(x,t,k), \\
\overline{\Phi_{12}(x,t,\overline{k})} &= -\Phi_{21}(x,t,k), & \Phi_{12}(x,t,-k) &= -\Phi_{21}(x,t,k), \\
\overline{\Phi_{jl}(x,t,-\overline{k})} &= \Phi_{jl}(x,t,k), & j,l &= \overline{1,2},
\end{aligned}$$

where $\Phi(x,t,k)$ denotes $\Phi_l(x,t,k)$ or $\Phi_r(x,t,k)$;

5. Large k asymptotics:

$$\left.\begin{array}{l}\Phi_{r1}(x,t,k)\mathrm{e}^{+\mathrm{i}kx+4\mathrm{i}k^3t}\\ \Phi_{l2}(x,t,k)\mathrm{e}^{-\mathrm{i}kx-4\mathrm{i}k^3t}\end{array}\right\} = 1+O\left(\frac{1}{k}\right), \quad k\to\infty, \quad \operatorname{Im}k\le 0,$$

$$\left.\begin{array}{l}\Phi_{l1}(x,t,k)\mathrm{e}^{+\mathrm{i}kx+4\mathrm{i}k^3t}\\ \Phi_{r2}(x,t,k)\mathrm{e}^{-\mathrm{i}kx-4\mathrm{i}k^3t}\end{array}\right\} = 1+O\left(\frac{1}{k}\right), \quad k\to\infty, \quad \operatorname{Im}k\ge 0;$$

6. Jump:
$\Phi_-(x,t,k)=\Phi_+(x,t,k)\begin{pmatrix}0 & \mathrm{i}\\ \mathrm{i} & 0\end{pmatrix}, \quad k\in(\mathrm{i}c,-\mathrm{i}c),$
where $\Phi(x,t,k)$ and c denotes $\Phi_l(x,t,k)$ and c_l or $\Phi_r(x,t,k)$ and c_r, respectively, and $\Phi_\pm(x,t,k)$ are the non-tangential boundary values of matrix $\Phi(x,t,k)$ from the left $(-)$ and from the right $(+)$ of the downward-oriented interval $(-\mathrm{i}c,\mathrm{i}c)$.

The matrices $\Phi_l(x,t,k)$ and $\Phi_r(x,t,k)$ are the solutions of (2.1) and (2.2). Hence, they are linear dependent, i.e. there exists the independent of x,t matrix

$$T(k)=\Phi_r^{-1}(x,t,k)\Phi_l(x,t,k), \quad k\in\mathbb{R}, \tag{2.10}$$

which is defined for real k. Some elements of this matrix have an extended domain of definition. Indeed, using (2.10), we find

$$\begin{aligned}T_{11}(k)&=\det(\Phi_{l1},\Phi_{r2}),\\ T_{21}(k)&=\det(\Phi_{r1},\Phi_{l1}),\\ T_{12}(k)&=\det(\Phi_{l2},\Phi_{r2}),\\ T_{22}(k)&=\det(\Phi_{r1},\Phi_{l2}).\end{aligned}$$

Then the above properties of the solutions $\Phi_r(x,t,k)$ and $\Phi_l(x,t,k)$ imply:

- $T_{11}(k)$ is analytic in $k\in\mathbb{C}_+\setminus[0,\mathrm{i}c_l]$ and has a continuous extension to $(0,\mathrm{i}c_l)_-\bigcup(0,\mathrm{i}c_l)_+$;
- $T_{22}(k)$ is analytic in $k\in\mathbb{C}_-\setminus[0,\mathrm{i}c_l]$ and has a continuous extension to $(-\mathrm{i}c_l,0)_-\bigcup(-\mathrm{i}c_l,0)_+$;
- $T_{21}(k)$ is continuous in $k\in(-\infty,0)\bigcup(0,-\mathrm{i}c_l)_-\bigcup(-\mathrm{i}c_l,0)_+\bigcup(0,+\infty)$;
- $T_{12}(k)$ is continuous in $k\in(-\infty,0)\bigcup(0,\mathrm{i}c_l)_-\bigcup(\mathrm{i}c_l,0)_+\bigcup(0,+\infty)$,

where, as before, the signs $-$ and $+$ denote the left and the right banks of the interval;

- $\overline{T_{22}(\overline{k})} = T_{11}(k), \quad T_{22}(-k) = T_{11}(k),$
- $\overline{T_{12}(\overline{k})} = -T_{21}(k), \quad T_{12}(-k) = -T_{21}(k),$
- $\overline{T_{jk}(-\overline{k})} = T_{jk}(k), \quad j,k = \overline{1,2}.$

Denote

$$a(k) = T_{11}(k),$$
$$b(k) = T_{21}(k).$$

Define the reflection coefficient

$$r(k) = \frac{b(k)}{a(k)}.$$

It has the following property:

$$\overline{r(-\overline{k})} = r(k).$$

The columns of the matrices Φ_l and Φ_r satisfy the following jump conditions:

7. $\dfrac{(\Phi_{l1})_-(x,t,k)}{a_-(k)} - \dfrac{(\Phi_{l1})_+(x,t,k)}{a_+(k)} = f_1(k)\Phi_{r2}(x,t,k), \quad k \in (\mathrm{i}c_r, \mathrm{i}c_l);$

8. $\dfrac{(\Phi_{l2})_-(x,t,k)}{\overline{a_-(\overline{k})}} - \dfrac{(\Phi_{l2})_+(x,t,k)}{\overline{a_+(\overline{k})}} = f_2(k)\Phi_{r1}(x,t,k), \quad k \in (-\mathrm{i}c_r, -\mathrm{i}c_l),$

where

$$f_1(k) = \frac{\mathrm{i}}{a_-(k)a_+(k)}, \quad k \in (0, \mathrm{i}c_l), \qquad f_2(k) = -\overline{f_1(\overline{k})}, \quad k \in (-\mathrm{i}c_l, 0).$$

3. The Basic Riemann–Hilbert Problem

Scattering relations (2.10) between the matrix-valued functions $\Phi_l(x,t,k)$ and $\Phi_r(x,t,k)$ and jump conditions 6, 7, 8 can be rewritten in terms of the Riemann–Hilbert problem. To do this, define the matrix-valued function

$$M(\xi,t,k) = \begin{cases} \left(\dfrac{\Phi_{l1}(x,t,k)}{a(k)} \mathrm{e}^{\mathrm{i}t\theta(k,\xi)}, \Phi_{r2}(x,t,k)\mathrm{e}^{-\mathrm{i}t\theta(k,\xi)} \right), & k \in \mathbb{C}_+ \backslash [0, \mathrm{i}c_l], \\ \\ \left(\Phi_{r1}(x,t,k)\mathrm{e}^{\mathrm{i}t\theta(k,\xi)}, \dfrac{\Phi_{l2}(x,t,k)}{\overline{a(\overline{k})}} \mathrm{e}^{-\mathrm{i}t\theta(k,\xi)} \right), & k \in \mathbb{C}_- \backslash [-\mathrm{i}c_l, 0], \end{cases} \tag{3.1}$$

where $x = 12\xi t$ and $\theta(k, \xi) = 4k^3 + 12k\xi$ $(\xi = x/12t)$. To make our presentation more transparent, we consider below the simplest shock problem where the initial function is discontinuous and piecewise constant (pure step function)

$$q_0(x) = \begin{cases} c_r, & x \geq 0 \\ c_l, & x < 0 \end{cases}. \tag{3.2}$$

Then

$$a(k) = \frac{1}{2}\left(\varkappa(k) + \frac{1}{\varkappa(k)}\right),\ b(k) = \frac{1}{2}\left(\varkappa(k) - \frac{1}{\varkappa(k)}\right),\ r(k) = \frac{\varkappa^2(k) - 1}{\varkappa^2(k) + 1} \tag{3.3}$$

are analytic in $k \in \mathbb{C}\setminus([-\mathrm{i}c_l, -\mathrm{i}c_r] \cup [\mathrm{i}c_l, \mathrm{i}c_r])$ since the function $\varkappa(k) := \dfrac{\varkappa_l(k)}{\varkappa_r(k)}$ (see (2.7)) is analytic in this domain. The transition coefficient $a^{-1}(k)$ is bounded in $k \in \mathbb{C}_+\setminus[\mathrm{i}c_l, \mathrm{i}c_r]$ because the function $a(k)$ equals zero nowhere and, hence, the set of eigenvalues of the linear problem (2.1) is empty. We also have

$$f(k) := f_1(k) = f_2(k) = r_-(k) - r_+(k), \quad k \in (-\mathrm{i}c, \mathrm{i}c). \tag{3.4}$$

Let us define the oriented contour $\Sigma = \mathbb{R} \cup (\mathrm{i}c_l, -\mathrm{i}c_l)$ as in Figure 1. Then the

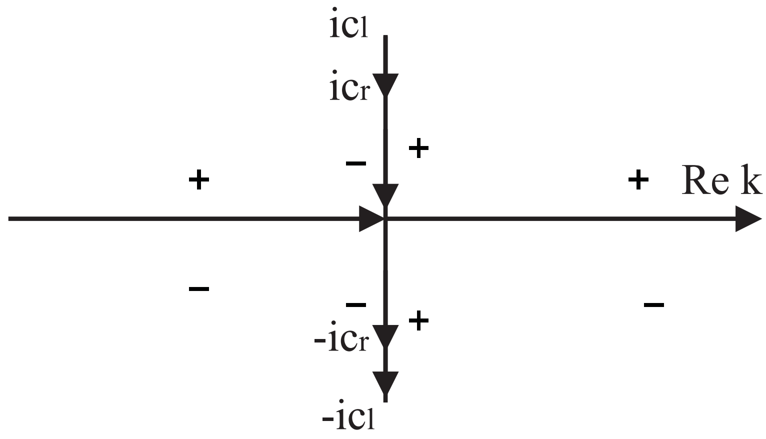


Fig. 1. Oriented contour Σ.

matrix (3.1) solves the next Riemann–Hilbert problem:

- the matrix valued function $M(\xi, t, k)$ is analytic in the domain $\mathbb{C} \setminus \Sigma$;
- $M(\xi, t, k)$ is bounded in the neighborhood of the end points $\mathrm{i}c_l$, $\mathrm{i}c_r$, $-\mathrm{i}c_l$, $-\mathrm{i}c_r$ and at the origin $(k = 0)$;

- $M_-(\xi,t,k) = M_+(\xi,t,k)J(\xi,t,k), \quad k\in\Sigma\setminus\{0\}$,
  where

$$J(\xi,t,k) = \begin{pmatrix} 1 & r(k)\mathrm{e}^{-2\mathrm{i}t\theta(k,\xi)} \\ -r(k)\mathrm{e}^{2\mathrm{i}t\theta(k,\xi)} & 1+|r(k)|^2 \end{pmatrix}, \qquad k\in\mathbb{R}\backslash\{0\}, \tag{3.5}$$

$$= \begin{pmatrix} 1 & 0 \\ f(k)\mathrm{e}^{2\mathrm{i}t\theta(k,\xi)} & 1 \end{pmatrix}, \qquad k\in(\mathrm{i}c_r,\mathrm{i}c_l), \tag{3.6}$$

$$= \begin{pmatrix} 1 & f(k)\mathrm{e}^{-2\mathrm{i}t\theta(k,\xi)} \\ 0 & 1 \end{pmatrix}, \qquad k\in(-\mathrm{i}c_r,-\mathrm{i}c_l), \tag{3.7}$$

$$= \begin{pmatrix} \mathrm{i}r(k) & \mathrm{i}\mathrm{e}^{-2\mathrm{i}t\theta(k,\xi)} \\ f(k)\mathrm{e}^{2\mathrm{i}t\theta(k,\xi)} & -\mathrm{i}r(k) \end{pmatrix}, \qquad k\in(0,\mathrm{i}c_r), \tag{3.8}$$

$$= \begin{pmatrix} -\mathrm{i}r(k) & f(k)\mathrm{e}^{-2\mathrm{i}t\theta(k,\xi)} \\ \mathrm{i}\mathrm{e}^{2\mathrm{i}t\theta(k,\xi)} & \mathrm{i}r(k) \end{pmatrix}, \qquad k\in(0,-\mathrm{i}c_r); \tag{3.9}$$

- $M(\xi,t,k) = I + O(k^{-1}), \quad k\to\infty$,

  where $r(k) = -\overline{r(\bar{k})} = -r(-k)$ is given in (3.3), and $f(k)$ in (3.4).

If the initial function is an arbitrary step-like data, then $a(k)$ can have zeroes in the domain of analyticity. In this case the matrix $M(\xi,t,k)$ will be meromorphic and the residue relations between the columns of the matrix $M(\xi,t,k)$ must be added.

In what follows we suppose that the solution $q(x,t)$ of shock problem (1.1)–(1.2) with pure step initial function (3.2) does exist. The above Riemann–Hilbert problem gives $q(x,t)$ in the form

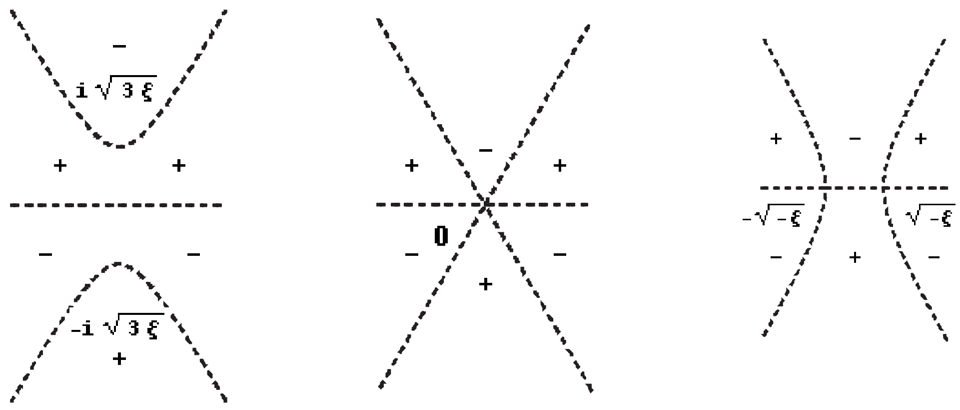
$$q(x,t) = 2\mathrm{i}\lim_{k\to\infty} k[M(x/12t,t,k)]_{12}, \tag{3.10}$$

where $[M(x/12t,t,k)]_{12}$ is the appropriate entry of the matrix $M(x/12t,t,k)$.

4. Long-Time Asymptotic Analysis of the Riemann–Hilbert Problem

4.1. The phase function. The jump matrix $J(\xi,t,k)$ in (3.5)–(3.9) depends on $\exp\{\pm 2\mathrm{i}t\theta(k,\xi)\}$. Hence the table of signs of the imaginary part of $\theta(k,\xi)$ plays a very important role as the phase function. The table of signs of the function

$$\operatorname{Im}\theta(k,\xi) = 4(3\operatorname{Re}^2 k - \operatorname{Im}^2 k + 3\xi)\operatorname{Im} k$$



a) $\xi > 0$ b) $\xi = 0$ c) $\xi < 0$

Fig. 2. The table of signs of the function $\operatorname{Im}\theta(k)$ for different ξ.

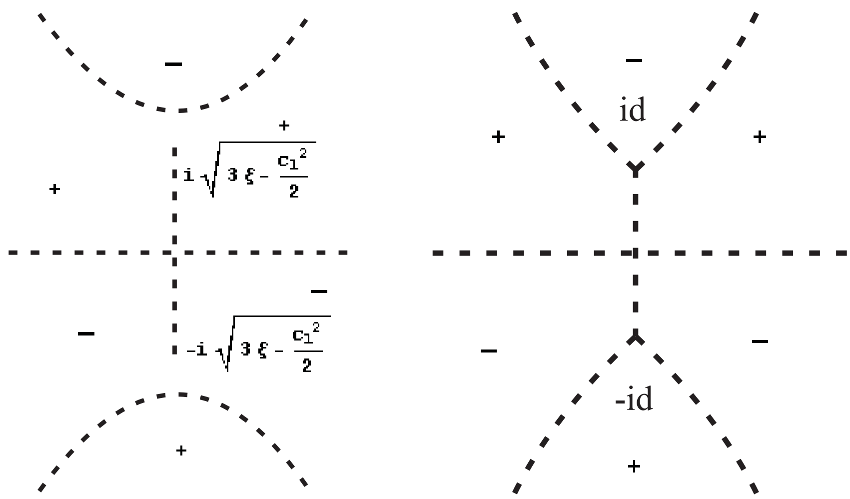
is depicted in Figures 2. Further, we will deform the contour and therefore the initial RH problem into some chain of other RH problems. Some of them admit explicit solutions and give the main contribution to the final asymptotics, while the solutions of others are much more complicated and thus cannot be obtained in an appropriate form. Fortunately, there is a possibility to overcome these difficulties. We choose the deformed contours in such a way that the corresponding jump matrices tend to identity matrix as $t \to -\infty$. Then these RH problems do not contribute to the main terms of asymptotics but they yield appropriate estimates. Thus, we must impose some restrictions on a new phase function on the vertical segment $[\mathrm{i}c_l, -\mathrm{i}c_l]$ preserving the main properties of the initial phase function $\theta(k,\xi)$. More explicitly, we will use the phase function $g(k,\xi)$, which takes different forms for different intervals of variable ξ

$$g(k,\xi) = (4k^2 - 2c_l^2 + 12\xi)X_{c_l}(k), \qquad \xi > \frac{c_l^2}{2}; \tag{4.1}$$

$$= 4X^3_{\sqrt{2\xi}}(k), \qquad \xi \in \left(\frac{c_r^2}{2}, \frac{c_l^2}{2}\right); \tag{4.2}$$

$$= (4k^2 - 2c_r^2 + 12\xi)X_{c_r}(k), \qquad \xi < \frac{c_r^2}{2}. \tag{4.3}$$

Here $X_d(k) = \sqrt{k^2 + d^2}$ is analytic in $\mathbb{C}\backslash[\mathrm{i}d, -\mathrm{i}d]$ and has the following asymptotic behavior near the infinity: $X_d(k) = k + O\left(\frac{1}{k}\right)$.



a) $\xi > \frac{c_l^2}{2}$ b) $\frac{c_r^2}{2} < \xi < \frac{c_l^2}{2}$

Fig. 3. The table of signs of the function $\operatorname{Im} g(k)$ for different ξ

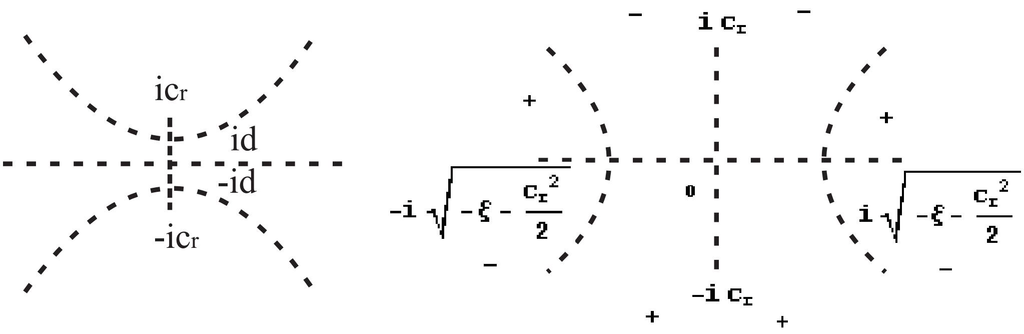
The function $g(k,\xi)$ possesses the following properties:

- for any ξ $g(k,\xi)$ is analytic in k in $\mathbb{C}\backslash[-\mathrm{i}d, \mathrm{i}d]$;
- for any ξ $\exists \lim\limits_{k\to\infty} (g(k,\xi) - \theta(k,\xi)) = 0$;
- $\operatorname{Im}(g(k,\xi))$ is bounded in k in some neighborhood of $[-\mathrm{i}d, \mathrm{i}d]$;
- the table of signs of $\operatorname{Im}(g(k,\xi))$ is depicted in one of Figures 3, 4.

The first three properties are evident. Let us consider the fourth one and pict the table of signs for $\operatorname{Im} g(k,\xi)$. For definiteness we consider $\xi \in \left(\frac{c_r^2}{2}, \frac{c_l^2}{2}\right)$. Other cases are treated in a similar way. First, we note that the function $X_d(k) = \sqrt{k^2+d^2}$, where $d = \sqrt{2\xi}$, makes a conformal map from the complex k-plane with the cut along $(-\mathrm{i}d, \mathrm{i}d)$ to the complex plane of variable X with the cut along $(-d, d)$. Therefore, we can draw the table of signs for $\operatorname{Im} g(k,\xi)$ in the X-plane and then make the inverse conformal map to the k-plane. Denote by X_1 and X_2 the real and imaginary parts of X. Then we have

$$\operatorname{Im} g(k,\xi) = \operatorname{Im}(4X^3) = \operatorname{Im}(4(X_1 + \mathrm{i}X_2)^3) = 4X_2(3X_1^2 - X_2^2).$$

We get the lines, where the imaginary part of $g(k,\xi)$ is zero, which are real axis of the X-plane and two lines which intersect the real axis at the origin. Finally, after inverse conformal mapping, we get Figure 3, b.



a) $-\dfrac{c_r^2}{2} < \xi < \dfrac{c_r^2}{2}$ b) $\xi < -\dfrac{c_r^2}{2}$

Fig. 4. The table of signs of the function $\operatorname{Im} g(k)$ for different ξ.

When $\xi \in \left(\dfrac{c_r^2}{2}, \dfrac{c_l^2}{2}\right)$, then $d = d(\xi)$ varies over (c_r, c_l). Thus, $g(k, \xi)$ has the following properties on the vertical cut:

- $\xi > \dfrac{c_r^2}{2}$. Then $g(., \xi)$ is analytic in $\mathbb{C}\backslash[\mathrm{i}d, -\mathrm{i}d]$ and

$$g_+(k, \xi) + g_-(k, \xi) = 0, \quad k \in (-\mathrm{i}d, \mathrm{i}d),$$

where we define $d = d(\xi)$ as follows:

$$\begin{aligned} d = d(\xi) &= \sqrt{\xi + \frac{c_r^2}{2}}, \quad \text{for } \xi \in \left(-\frac{c_r^2}{2}, \frac{c_r^2}{2}\right), && (4.4)\\ &= \sqrt{2\xi}, \quad \text{for } \xi \in \left(\frac{c_r^2}{2}, \frac{c_l^2}{2}\right), && (4.5)\\ &= c_l, \quad \text{for } \xi > \frac{c_l^2}{2}; && (4.6)\end{aligned}$$

- $\xi < \dfrac{c_r^2}{2}$. Then $g(., \xi)$ is analytic in $\mathbb{C}\backslash[\mathrm{i}c_r, -\mathrm{i}c_r]$ and

$$g_+(k, \xi) + g_-(k, \xi) = 0, \quad k \in (-\mathrm{i}c_r, \mathrm{i}c_r).$$

The functions $g_+(k, \xi)$ and $g_-(k, \xi)$ are boundary values of the function $g(k, \xi)$ for k on vertical cuts. Also, for all $g(k, \xi)$ we have

$$\lim_{k\to\infty} (g(k, \xi) - \theta(k, \xi)) = 0.$$

4.2. Changing of phase function. The Riemann–Hilbert problem for matrix $M(\xi,t,k)$ with the jump contour $\Sigma = \mathbb{R}\cup(\mathrm{i}c_l,-\mathrm{i}c_l)$ has to be considered now with a new phase function $g(k,\xi)$, where g is defined in (4.1)–(4.3). Let us define a new matrix

$$M^{(1)}(\xi,t,k) = M(\xi,t,k)G^{(1)}(\xi,t,k),$$

where $G^{(1)}(\xi,t,k) = \mathrm{e}^{\mathrm{i}t(g(k,\xi)-\theta(k,\xi))\sigma_3}$. Then the function $M^{(1)}(\xi,t,k)$ solves the following RH problem:

$$M_-^{(1)}(\xi,t,k) = M_+^{(1)}(\xi,t,k)J^{(1)}(\xi,t,k), \qquad M^{(1)}(\xi,t,k)\to I, \quad k\to\infty,$$

where

$$J^{(1)}(\xi,t,k) = \begin{pmatrix} 1 & r(k)\mathrm{e}^{-2\mathrm{i}tg(k,\xi)} \\ -\overline{r(k)}\mathrm{e}^{2\mathrm{i}tg(k,\xi)} & 1+|r(k)|^2 \end{pmatrix}, \quad k\in\mathbb{R}\backslash\{0\}, \tag{4.7}$$

$$= \begin{pmatrix} \mathrm{e}^{\mathrm{i}t(g_-(k,\xi)-g_+(k,\xi))} & 0 \\ f(k)\mathrm{e}^{\mathrm{i}t(g_-(k,\xi)+g_+(k,\xi))} & \mathrm{e}^{-\mathrm{i}t(g_-(k,\xi)-g_+(k,\xi))} \end{pmatrix}, \quad k\in(\mathrm{i}c_r,\mathrm{i}c_l), \tag{4.8}$$

$$= \begin{pmatrix} \mathrm{e}^{\mathrm{i}t(g_-(k,\xi)-g_+(k,\xi))} & \overline{f(\bar k)}\mathrm{e}^{-\mathrm{i}t(g_-(k,\xi)+g_+(k,\xi))} \\ 0 & \mathrm{e}^{-\mathrm{i}t(g_-(k,\xi)-g_+(k,\xi))} \end{pmatrix}, \quad k\in(-\mathrm{i}c_r,-\mathrm{i}c_l), \tag{4.9}$$

$$= \begin{pmatrix} \mathrm{i}r(k)\mathrm{e}^{\mathrm{i}t(g_-(k,\xi)-g_+(k,\xi))} & \mathrm{i}\mathrm{e}^{-\mathrm{i}t(g_-(k,\xi)+g_+(k,\xi))} \\ f(k)\mathrm{e}^{\mathrm{i}t(g_-(k,\xi)+g_+(k,\xi))} & -\mathrm{i}r(k)\mathrm{e}^{-\mathrm{i}t(g_-(k,\xi)-g_+(k,\xi))} \end{pmatrix}, \quad k\in(0,\mathrm{i}c_r), \tag{4.10}$$

$$= \begin{pmatrix} -\mathrm{i}\overline{r(\bar k)}\mathrm{e}^{\mathrm{i}t(g_-(k,\xi)-g_+(k,\xi))} & \overline{f(\bar k)}\mathrm{e}^{-\mathrm{i}t(g_-(k,\xi)+g_+(k,\xi))} \\ \mathrm{i}\mathrm{e}^{\mathrm{i}t(g_-(k,\xi)+g_+(k,\xi))} & \mathrm{i}\overline{r(\bar k)}\mathrm{e}^{-\mathrm{i}t(g_-(k,\xi)-g_+(k,\xi))} \end{pmatrix}, \quad k\in(0,-\mathrm{i}c_r). \tag{4.11}$$

4.3. δ-transformation. To transfer the jump contour from the real axis, we use the following factorizations of the jump matrix on the real axis:

$$J^{(1)}(\xi,t,k) = \begin{pmatrix} 1 & 0 \\ -\overline{r(k)}\mathrm{e}^{2\mathrm{i}tg(k,\xi)} & 1 \end{pmatrix}\begin{pmatrix} 1 & r(k)\mathrm{e}^{-2\mathrm{i}tg(k,\xi)} \\ 0 & 1 \end{pmatrix}, \tag{4.12}$$

$$J^{(1)}(\xi,t,k) =$$

$$\begin{pmatrix} 1 & \dfrac{r(k)\mathrm{e}^{-2\mathrm{i}tg(k,\xi)}}{1+|r(k)|^2} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \dfrac{1}{1+|r(k)|^2} & 0 \\ 0 & 1+|r(k)|^2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \dfrac{-\overline{r(k)}\mathrm{e}^{2\mathrm{i}tg(k,\xi)}}{1+|r(k)|^2} & 1 \end{pmatrix}. \tag{4.13}$$

It is easy to see that the first (second) factor in product (4.12) tends to the identity matrix as $t \to -\infty$ in the domains where $\mathrm{Im}\, g(k,\xi) < 0$ ($\mathrm{Im}\, g(k,\xi) > 0$). In product (4.13) the first (third) factor tends to the identity matrix as $t \to -\infty$ in the domains where $\mathrm{Im}\, g(k,\xi) > 0$ ($\mathrm{Im}\, g(k,\xi) < 0$). To remove the diagonal matrix in the second product, we use a transformation

$$M^{(2)}(\xi,t,k) = M^{(1)}(\xi,t,k)\delta^{-\sigma_3}(k,\xi), \qquad \delta^{-\sigma_3}(k,\xi) = \begin{pmatrix} \delta^{-1}(k,\xi) & 0 \\ 0 & \delta(k,\xi) \end{pmatrix},$$

where an analytic in $\mathbb{C}\backslash\mathbb{R}$ function $\delta(k,\xi)$ must be defined. Then the jump matrix $J^{(2)}(\xi,t,k)$ for $k \in \mathbb{R}$ admits the factorizations

$$J^{(2)}(\xi,t,k) = \begin{pmatrix} 1 & 0 \\ -r(k)\delta^{-2}(k,\xi)\mathrm{e}^{2\mathrm{i}tg(k,\xi)} & 1 \end{pmatrix} \begin{pmatrix} 1 & r(k)\delta^{2}(k,\xi)\mathrm{e}^{-2\mathrm{i}tg(k,\xi)} \\ 0 & 1 \end{pmatrix},$$

$$J^{(2)}(\xi,t,k) =$$

$$\begin{pmatrix} 1 & \dfrac{r\delta_+^2\mathrm{e}^{-2\mathrm{i}tg(k,\xi)}}{1+|r|^2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \dfrac{\delta_+}{\delta_-}(1+|r|^2)^{-1} & 0 \\ 0 & \dfrac{\delta_-}{\delta_+}\left(1+|r|^2\right) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \dfrac{-r\mathrm{e}^{2\mathrm{i}tg(k,\xi)}}{(1+|r|^2)\delta_-^2} & 1 \end{pmatrix}.$$

To make the middle matrix factor in the second formula equal to the unit matrix, on $\delta(k,\xi)$ we impose the following conditions:

$$\delta_+(k,\xi) = \delta_-(k,\xi)(1+|r(k)|^2)$$

for $\xi > -\dfrac{c_r^2}{2}$ and $k \in \mathbb{R}$ or $\xi < -\dfrac{c_r^2}{2}$ and $k \in \mathbb{R}\backslash[-\lambda(\xi),\lambda(\xi)]$
and

$$\delta_+(k,\xi) = \delta_-(k,\xi)$$

for $\xi < -\dfrac{c_r^2}{2}$ and $k \in [-\lambda(\xi),\lambda(\xi)]$. Here $\lambda(\xi) = \sqrt{-\xi - \dfrac{c_r^2}{2}}$ for $\xi < -\dfrac{c_r^2}{2}$.
The function $\delta(k,\xi)$ can be found in the form

$$\begin{aligned} \delta(k,\xi) &= \left(\frac{\lambda(\xi)+k}{\lambda(\xi)-k}\right)^{-\mathrm{i}\nu}\chi(k,\xi), && \xi < -\frac{c_r^2}{2}, \\ &= \exp\left(\frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{\infty}\frac{\ln\left(1+|r(s)|^2\right)ds}{s-k}\right), && \xi > -\frac{c_r^2}{2}, \end{aligned} \tag{4.14}$$

where

$$\chi(k,\xi) = \exp\left(\frac{1}{2\pi\mathrm{i}}\int_{\mathbb{R}\backslash[-\lambda(\xi),\lambda(\xi)]}\frac{\ln\left(\dfrac{1+|r(s)|^2}{1+|r(\lambda(\xi))|^2}\right)ds}{s-k}\right),$$

and

$$\nu = \nu(\xi) = \frac{1}{2\pi}\ln(1+|r(\lambda(\xi))|^2), \qquad \lambda(\xi) = \sqrt{-\xi - \frac{c_r^2}{2}}.$$

It is easy to see that $\delta(k,\xi)$ is separated from zero and infinity, and

$$\lim_{k\to\infty} \delta(k,\xi) = 1.$$

Thus, the jump matrix $J^{(2)}(\xi,t,k)$ has the lower/upper factorization for $\xi < -\frac{c_r^2}{2}$, $k \in [-\lambda(\xi), \lambda(\xi)]$:

$$J^{(2)}(\xi,t,k) = \begin{pmatrix} 1 & 0 \\ -r(k)\delta^{-2}(k,\xi)\mathrm{e}^{2\mathrm{i}tg(k,\xi)} & 1 \end{pmatrix} \begin{pmatrix} 1 & r(k)\delta^2(k)\mathrm{e}^{-2\mathrm{i}tg(k,\xi)} \\ 0 & 1 \end{pmatrix},$$

and the upper/lower factorization for $\xi < -\frac{c_r^2}{2}$, $k \in \mathbb{R} \setminus [-\lambda(\xi), \lambda(\xi)]$ and for $\xi > -\frac{c_r^2}{2}$, $k \in \mathbb{R}$:

$$J^{(2)}(\xi,t,k) = \begin{pmatrix} 1 & a(k)b(k)\delta_+^2(k,\xi)\mathrm{e}^{-2\mathrm{i}tg(k,\xi)} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -a(k)b(k)\delta_-^{-2}(k,\xi)\mathrm{e}^{2\mathrm{i}tg(k,\xi)} & 1 \end{pmatrix},$$

where we use the identity

$$\frac{r(k)}{1+|r(k)|^2} = \frac{r(k)}{1-r^2(k)} = a(k)b(k).$$

For $k \in (-\mathrm{i}c_l, \mathrm{i}c_l)$, the jump matrix $J^{(2)}(\xi,t,k)$ takes the form

$$J^{(2)}(\xi,t,k)$$

$$= \begin{pmatrix} \mathrm{e}^{\mathrm{i}t(g_-(k,\xi)-g_+(k,\xi))} & 0 \\ f(k)\delta^{-2}(k,\xi)\mathrm{e}^{\mathrm{i}t(g_-(k,\xi)+g_+(k,\xi))} & \mathrm{e}^{-\mathrm{i}t(g_-(k,\xi)-g_+(k,\xi))} \end{pmatrix}, k \in (\mathrm{i}c_r, \mathrm{i}c_l), \tag{4.15}$$

$$= \begin{pmatrix} \mathrm{e}^{\mathrm{i}t(g_-(k,\xi)-g_+(k,\xi))} & f(k)\delta^2(k,\xi)\mathrm{e}^{-\mathrm{i}t(g_-(k,\xi)+g_+(k,\xi))} \\ 0 & \mathrm{e}^{-\mathrm{i}t(g_-(k,\xi)-g_+(k,\xi))} \end{pmatrix}, k \in (-\mathrm{i}c_r, -\mathrm{i}c_l), \tag{4.16}$$

$$= \begin{pmatrix} \mathrm{i}r(k)\mathrm{e}^{\mathrm{i}t(g_-(k,\xi)-g_+(k,\xi))} & \mathrm{i}\delta^2(k,\xi)\mathrm{e}^{-\mathrm{i}t(g_-(k,\xi)+g_+(k,\xi))} \\ f(k)\delta^{-2}(k,\xi)\mathrm{e}^{\mathrm{i}t(g_-(k,\xi)+g_+(k,\xi))} & -\mathrm{i}r(k)\mathrm{e}^{-\mathrm{i}t(g_-(k,\xi)-g_+(k,\xi))} \end{pmatrix}, k \in (0, \mathrm{i}c_r), \tag{4.17}$$

$$= \begin{pmatrix} -\mathrm{i}r(k)\mathrm{e}^{\mathrm{i}t(g_-(k,\xi)-g_+(k,\xi))} & f(k)\delta^2(k,\xi)\mathrm{e}^{-\mathrm{i}t(g_-(k,\xi)+g_+(k,\xi))} \\ \mathrm{i}\delta^{-2}(k,\xi)\mathrm{e}^{\mathrm{i}t(g_-(k,\xi)+g_+(k,\xi))} & \mathrm{i}r(k)\mathrm{e}^{-\mathrm{i}t(g_-(k,\xi)-g_+(k,\xi))} \end{pmatrix}, k \in (0, -\mathrm{i}c_r). \tag{4.18}$$

The jump contour $\Sigma^{(2)}$ for $M^{(2)}(\xi,t,k)$-problem is the initial one, i.e. $\Sigma^{(2)} = \Sigma$.

4.4. Transferring of the jump contour from the real line. Let us define a decomposition of the complex k-plane into domains $\Omega_1, \Omega_2, \ldots$, separated by their common boundary $\Sigma^{(3)}$ as shown in Figures 5, 6, and 7.

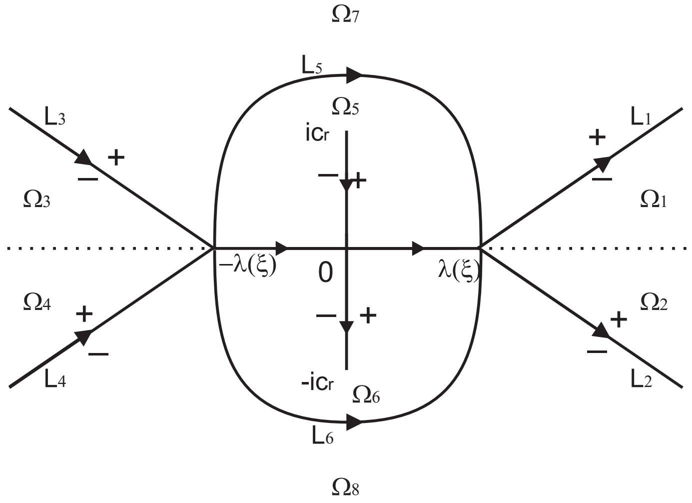


Fig. 5. Contour $\Sigma^{(3)}$ for $M^{(3)}(\xi,t,k)$-problem for $\xi < -\dfrac{c_r^2}{2}$.

Then the next transformation depends on whether ξ lies in $\left(-\infty, -\dfrac{c_r^2}{2}\right)$ or in $\left(-\dfrac{c_r^2}{2}, \infty\right)$:

$$M^{(3)}(\xi,t,k) = M^{(2)}(\xi,t,k)G^{(3)}(\xi,t,k),$$

where for $\xi < -\dfrac{c_r^2}{2}$

$$G^{(3)}(\xi,t,k) = \begin{pmatrix} 1 & 0 \\ -r(k)\delta^{-2}(k,\xi)\mathrm{e}^{2itg(k,\xi)} & 1 \end{pmatrix}, \qquad k \in \Omega_5, \tag{4.19}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad k \in \Omega_7 \cup \Omega_8, \tag{4.20}$$

$$= \begin{pmatrix} 1 & -r(k)\delta^{2}(k,\xi)\mathrm{e}^{-2itg(k,\xi)} \\ 0 & 1 \end{pmatrix}, \qquad k \in \Omega_6, \tag{4.21}$$

$$G^{(3)}(\xi,t,k) = \begin{pmatrix} 1 & a(k)b(k)\delta^{2}(k,\xi)\mathrm{e}^{-2itg(k,\xi)} \\ 0 & 1 \end{pmatrix}, \qquad k \in \Omega_1 \cup \Omega_3, \tag{4.22}$$

$$= \begin{pmatrix} 1 & 0 \\ a(k)b(k)\delta^{-2}(k,\xi)\mathrm{e}^{2itg(k,\xi)} & 1 \end{pmatrix}, \qquad k \in \Omega_2 \cup \Omega_4, \tag{4.23}$$

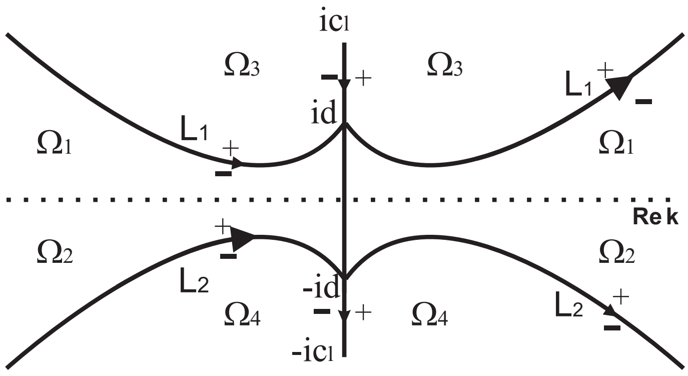


Fig. 6. Contour $\Sigma^{(3)}$ for $M^{(3)}(\xi,t,k)$-problem for $-\frac{c_r^2}{2} < \xi < \frac{c_l^2}{2}$.

and for $\xi > -\frac{c_r^2}{2}$

$$G^{(3)}(\xi,t,k) = \begin{pmatrix} 1 & a(k)b(k)\delta^2(k,\xi)\mathrm{e}^{-2\mathrm{i}tg(k,\xi)} \\ 0 & 1 \end{pmatrix}, \qquad k \in \Omega_1, \tag{4.24}$$

$$= \begin{pmatrix} 1 & 0 \\ a(k)b(k)\delta^{-2}(k,\xi)\mathrm{e}^{2\mathrm{i}tg(k,\xi)} & 1 \end{pmatrix}, \qquad k \in \Omega_2, \tag{4.25}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad k \in \Omega_3 \cup \Omega_4. \tag{4.26}$$

$G^{(3)}$-transformation leads to the following RH problem:

$$M_-^{(3)}(\xi,t,k) = M_+^{(3)}(\xi,t,k)J^{(3)}(\xi,t,k), \qquad M_-^{(3)}(\xi,t,k) = I + O(k^{-1}), \quad k \to \infty,$$

on the contour $\Sigma^{(3)}$ depicted in Figure 5. The jump matrix $J^{(3)}(\xi,t,k)$ is equal to the identity matrix on the real axis and it coincides with the matrix $G^{(3)}(k)$ or $\left(G^{(3)}\right)^{-1}(k)$ from (4.19)–(4.26) written for the contours $k \in L_j$ ($j = 1, 2, \ldots, 6$). It is easy to see that $J^{(3)}(\xi,t,k) = I + O(\mathrm{e}^{-\epsilon t})$ as $t \to \infty$ and $k \in L_j$ with the exception of some neighborhoods of the stationary points $\pm\lambda(\xi)$ or $\pm d$. Therefore, the main contribution to the asymptotics comes from the jump matrix on the interval $(\mathrm{i}c_l, -\mathrm{i}c_l)$, where it takes the form $J^{(3)}(\xi,t,k) = \left(G_+^{(2)}(k)\right)^{-1} J^{(2)}(\xi,t,k)G_-^{(2)}(k)$, i.e.

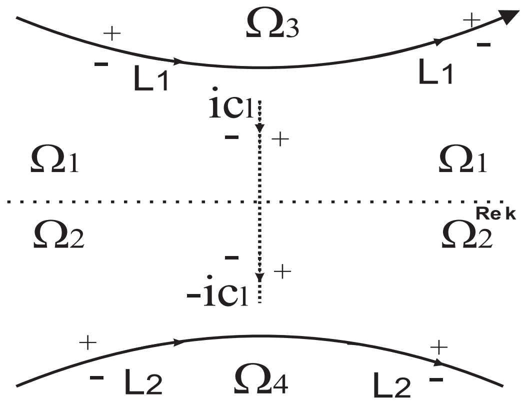


Fig. 7. Contour $\Sigma^{(3)}$ for $M^{(3)}(\xi,t,k)$-problem for $\xi > \dfrac{c_l^2}{2}$.

$$\begin{aligned}
J^{(3)}(\xi,t,k) &= \begin{pmatrix} 1 & -a_+(k)b_+(k)\delta^2(k,\xi)\mathrm{e}^{-2\mathrm{i}tg_c^+(k,\xi)} \\ 0 & 1 \end{pmatrix} J^{(2)}(\xi,t,k) \\
&\times \begin{pmatrix} 1 & a_-(k)b_-(k)\delta^2(k,\xi)\mathrm{e}^{2\mathrm{i}tg_c^+(k,\xi)} \\ 0 & 1 \end{pmatrix}, \quad k \in (0,\mathrm{i}d), \\
&= \begin{pmatrix} 1 & 0 \\ -a_+(k)b_+(k)\delta^{-2}(k,\xi)\mathrm{e}^{2\mathrm{i}tg_c^+(k,\xi)} & 1 \end{pmatrix} J^{(2)}(\xi,t,k) \\
&\times \begin{pmatrix} 1 & 0 \\ a_-(k)b_-(k)\delta^{-2}(k,\xi)\mathrm{e}^{-2\mathrm{i}tg_c^+(k,\xi)} & 1 \end{pmatrix}, \quad k \in (0,-\mathrm{i}d), \\
&= J^{(2)}(\xi,t,k), \quad k \in (\mathrm{i}c_l,\mathrm{i}d) \cup (-\mathrm{i}d,-\mathrm{i}c_l)
\end{aligned}$$

for $\xi > -\dfrac{c_r^2}{2}$. For the semi-line $\xi < -\dfrac{c_r^2}{2}$, we have

$$\begin{aligned}
J^{(3)}(\xi,t,k) &= \begin{pmatrix} 1 & 0 \\ r_+(k)\delta^{-2}(k,\xi)\mathrm{e}^{2\mathrm{i}tg_+(k,\xi)} & 1 \end{pmatrix} J^{(2)}(\xi,t,k) \\
&\times \begin{pmatrix} 1 & 0 \\ -r_-(k)\delta^{-2}(k,\xi)\mathrm{e}^{2\mathrm{i}tg_-(k,\xi)} & 1 \end{pmatrix}, \quad k \in (0,\mathrm{i}c_l), \\
&= \begin{pmatrix} 1 & r_+(k)\delta^2(k,\xi)\mathrm{e}^{-2\mathrm{i}tg_+(k,\xi)} \\ 0 & 1 \end{pmatrix} J^{(2)}(\xi,t,k) \\
&\times \begin{pmatrix} 1 & -r_-(k)\delta^2(k,\xi)\mathrm{e}^{-2\mathrm{i}tg_-(k,\xi)} \\ 0 & 1 \end{pmatrix}, \quad k \in (0,-\mathrm{i}c_l).
\end{aligned}$$

where $d = d(\xi)$ is defined in (4.4)–(4.6). Using the equalities $1 + b_\pm(k)^2 = a_\pm(k)^2$

and

$$a_-(k) = \mathrm{i}b_+(k), \qquad b_-(k) = \mathrm{i}a_+(k), \qquad k \in \pm(\mathrm{i}c_l, \mathrm{i}c_r),$$
$$a_-(k) = a_+(k), \qquad b_-(k) = b_+(k), \qquad k \in (\mathrm{i}c_r, -\mathrm{i}c_r),$$

we obtain

$$J^{(3)}(\xi,t,k)$$

$$= \begin{pmatrix} 0 & \mathrm{i}a_-(k)a_+(k)\delta^2(k,\xi)\mathrm{e}^{-\mathrm{i}t(g_-(k,\xi)+g_+(k,\xi))} \\ \dfrac{\mathrm{i}\mathrm{e}^{\mathrm{i}t(g_-(k,\xi)+g_+(k,\xi))}}{a_-(k)a_+(k)\delta^2(k,\xi)} & 0 \end{pmatrix},$$
$$k \in (0, \mathrm{i}d), \tag{4.27}$$
$$= \begin{pmatrix} 0 & \dfrac{\mathrm{i}\delta^2(k,\xi)\mathrm{e}^{\mathrm{i}t(g_-(k,\xi)+g_+(k,\xi))}}{a_-(k)a_+(k)} \\ \mathrm{i}a_-(k)a_+(k)\delta^{-2}(k,\xi)\mathrm{e}^{-\mathrm{i}t(g_-(k,\xi)+g_+(k,\xi))} & 0 \end{pmatrix},$$
$$k \in (0, -\mathrm{i}d), \tag{4.28}$$
$$= J^{(2)}(\xi,t,k), \qquad k \in \pm(\mathrm{i}c_l, \mathrm{i}d), \tag{4.29}$$

for $\xi > -\dfrac{c_r^2}{2}$, and

$$J^{(3)}(\xi,t,k)$$

$$= \begin{pmatrix} \mathrm{e}^{\mathrm{i}t(g_-(k,\xi)-g_+(k,\xi))} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}t(g_-(k,\xi)-g_+(k,\xi))} \end{pmatrix}, \quad k \in \pm(\mathrm{i}c_l, \mathrm{i}c_r), \tag{4.30}$$
$$= \begin{pmatrix} 0 & \mathrm{i}\delta^2(k,\xi)\mathrm{e}^{-\mathrm{i}t(g_-(k,\xi)+g_+(k,\xi))} \\ \mathrm{i}\delta^{-2}(k,\xi)\mathrm{e}^{\mathrm{i}t(g_-(k,\xi)+g_+(k,\xi))} & 0 \end{pmatrix}, \quad k \in (\mathrm{i}c_r, -\mathrm{i}c_r), \tag{4.31}$$

for $\xi < -\dfrac{c_r^2}{2}$.

4.5. Constant model problem. To obtain the RH problem with a constant in k jump matrix on the vertical cut, close to the identity matrix on other lines, we use different factorizations for different ξ.

4.5.1. Region $\xi > \dfrac{c_l^2}{2}$

In this region and for $k \in [\mathrm{i}c_l, -\mathrm{i}c_l]$ we use the following factorization of the matrix $J^{(3)}(\xi,t,k)$:

$$J^{(3)}(\xi,t,k) = \begin{pmatrix} F_+^{-1}(k,\xi) & 0 \\ 0 & F_+(k,\xi) \end{pmatrix} \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \begin{pmatrix} F_-(k,\xi) & 0 \\ 0 & F_-^{-1}(k,\xi) \end{pmatrix}. \tag{4.32}$$

Direct calculations show that it is possible if

- $F(k,\xi)$ is analytic outside the downward-oriented contour $[\mathrm{i}c_l, -\mathrm{i}c_l]$;
- $F(k,\xi)$ does not vanish;
- $F(k,\xi)$ satisfies the jump relations:

$$F_-(k,\xi)F_+(k,\xi) = \tilde{h}(k)$$

$$= \begin{cases} -\mathrm{i}f(k)\delta^{-2}(k) = (a_-(k)a_+(k))^{-1}\delta^{-2}(k), & k \in (\mathrm{i}c_l, 0), \\ \dfrac{\mathrm{i}}{f(k)\delta^2(k)} = a_-(k)a_+(k)\delta^{-2}(k), & k \in (-\mathrm{i}c_l, 0); \end{cases}$$

- $F(k,\xi)$ is bounded at the infinity;
- $a(k)F(k,\xi)$ is bounded in a small neighborhood of the point $\mathrm{i}c_l$;
- $(a(k))^{-1}F(k,\xi)$ is bounded in a small neighborhood of the point $-\mathrm{i}c_l$.

To solve this scalar RH problem, we use the function $X_{c_l}(k) = \sqrt{k^2 + c_l^2}$:

$$\left[\frac{\log F(k,\xi)}{X_{c_l}(k)}\right]_+ - \left[\frac{\log F(k,\xi)}{X_{c_l}(k)}\right]_- = \frac{\log \tilde{h}(s)}{X_{c_l+}(k)}, \qquad k \in (\mathrm{i}c_l, -\mathrm{i}c_l).$$

The function

$$F(k,\xi) = \exp\left\{\frac{X_{c_l}(k)}{2\pi\mathrm{i}}\int_{\mathrm{i}c_l}^{0}\frac{-\log(a_+(s)a_-(s)\delta^2(k))\mathrm{d}s}{(s-k)X_{c_l+}(s)}\right\}$$
$$\times \exp\left\{\frac{X_{c_l}(k)}{2\pi\mathrm{i}}\int_{0}^{-\mathrm{i}c_l}\frac{\log(a_+(s)a_-(s)\delta^{-2}(k))\mathrm{d}s}{(s-k)X_{c_l+}(s)}\right\} \tag{4.33}$$

satisfies the first, second and third properties. Let us show that $F(\infty,\xi) \equiv 1$. Indeed,

$$F(\infty,\xi) = \exp\left\{\frac{1}{2\pi\mathrm{i}}\int_{\mathrm{i}c_l}^{0}\frac{\log(a_+(s)a_-(s)\delta^2(k))ds}{X_{c_l+}(s)}\right\}$$
$$\times \exp\left\{\frac{1}{2\pi\mathrm{i}}\int_{0}^{-\mathrm{i}c_l}\frac{-\log(a_+(s)a_-(s)\delta^{-2}(k))ds}{X_{c_l+}(s)}\right\}.$$

Taking into account the following properties of the functions $a(s)$, $\delta(s,\xi)$ and $X_{c_l}(s)$:

- for $s \in (-\mathrm{i}c_l, \mathrm{i}c_l)$ $\quad a_+(s) = a_+(-s) \quad \& \quad a_-(s) = a_-(-s)$;
- for $s \in (-\mathrm{i}c_l, \mathrm{i}c_l)$ $\quad \delta(s,\xi) > 0 \quad \& \quad \delta(-s,\xi) = 1/\delta(s,\xi)$;
- for $s \in (-\mathrm{i}c_l, \mathrm{i}c_l)$ $\quad X_{c_l+}(-s) = X_{c_l+}(s)$,

one can find that $F(\infty,\xi) \equiv 1$. Finally, the boundedness of $a^{\pm 1}(k)F(k,\xi)$ near the points $\mathrm{i}c_l$ and $-\mathrm{i}c_l$ is the consequence of the equality

$$\log a(k) = \frac{X_{c_l}(k)}{2\pi\mathrm{i}} \int\limits_{\mathrm{i}c_l}^{-\mathrm{i}c_l} \frac{\log(a_+(s)a_-(s))}{s-k} \frac{ds}{X_{c_l+}(s)},$$

which can be obtained by the Cauchy theorem applied to the function $[\log a(k)]/X_{c_l}(k)$ analytic in $k \in \mathbb{C}\setminus[\mathrm{i}c_l, -\mathrm{i}c_l]$. Thus, (4.33) gives the solution of the above scalar RH problem. As a result, the matrix $J^{(3)}(\xi,t,k)$ takes the form

$$J^{(3)}(\xi,t,k) = F_+^{-\sigma_3}(k,\xi) J^{(mod)} F_-^{\sigma_3}(k,\xi), \qquad k \in (\mathrm{i}c_l, -\mathrm{i}c_l),$$

where

$$J^{(mod)} = \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}.$$

This factorization prompts the next step:

$$M^{(4)}(\xi,t,k) = M^{(3)}(\xi,t,k)F^{-\sigma_3}(k,\xi), \tag{4.34}$$

which gives the following RH problem:

$$\begin{aligned} &M_-^{(4)}(\xi,t,k) = M_+^{(4)}(\xi,t,k)J^{(4)}(\xi,t,k), \quad k \in \Sigma_4, \\ &M^{(4)}(\xi,t,k) = I + O(k^{-1}), \quad k \to \infty, \end{aligned} \tag{4.35}$$

where $\Sigma^{(4)} = \Sigma^{(3)} = \mathbb{R} \cup (\mathrm{i}c_l, -\mathrm{i}c_l) \cup \bigcup\limits_{j=1}^{2} L_j$, and

$$J^{(4)}(\xi,t,k) = \begin{cases} I, & k \in \mathbb{R}, \\ J^{(mod)}, & k \in (\mathrm{i}c_l, -\mathrm{i}c_l), \\ I + O(\mathrm{e}^{-\varepsilon t}), & k \in L_j, \quad j = 1,2. \end{cases}$$

4.5.2. Region $\xi \in \left(\dfrac{c_l^2}{2}, \dfrac{c_r^2}{2}\right)$

In this case the segments $[\mathrm{i}c_l, \mathrm{i}d]$ and $[0, -\mathrm{i}d]$ lay in the domains where $\operatorname{Im} g(k,\xi) > 0$, and the segments $[\mathrm{i}d, 0]$ and $[-\mathrm{i}d, -\mathrm{i}c_l]$ lay in the domains where $\operatorname{Im} g(k,\xi) < 0$. Taking into account that $g_-(k,\xi) = g_+(k,\xi)$ for $k \in [\mathrm{i}c_l, \mathrm{i}d] \cup [-\mathrm{i}c_l, -\mathrm{i}d]$, we find

$$J^{(3)}(\xi,t,k) = I + O\left(\mathrm{e}^{-\varepsilon t}\right), \quad k \in [\mathrm{i}c_l, \mathrm{i}d] \cup [-\mathrm{i}c_l, -\mathrm{i}d].$$

Therefore, the main contribution to the asymptotics is given by the segment $[\mathrm{i}d, -\mathrm{i}d]$. Here we use the same factorization (4.32) as in the case $\xi > \dfrac{c_l^2}{2}$ with the function $F(k,\xi)$, defined by (4.33), where the parameter c_l is substituted with $d = \sqrt{2\xi}$. Therefore, we define $M^{(4)}(\xi,t,k)$ as in (4.34). It solves the RH problem (4.35), where

$$J^{(4)}(\xi,t,k) = \begin{cases} I, & k \in \mathbb{R} \cup (\mathrm{i}c_l, \mathrm{i}d) \cup (-\mathrm{i}c_l, -\mathrm{i}d), \\ J^{(mod)}, & k \in (\mathrm{i}d, -\mathrm{i}d), \\ I + O(\mathrm{e}^{-\varepsilon t}), & k \in L_j, \quad j = 1, 2. \end{cases}.$$

4.5.3. Region $\xi \in \left(\dfrac{c_r^2}{2}, -\dfrac{c_r^2}{2}\right)$

In this case the line $\operatorname{Im} g(k,\xi) = 0$ intersects the segment $[\mathrm{i}c_l, -\mathrm{i}c_l]$ at two points $\pm d$, where $d = \sqrt{\xi + \dfrac{c_r^2}{2}} \in (0, \mathrm{i}c_r)$. Therefore, we need another factorizations on the intervals $(\mathrm{i}c_r, \mathrm{i}d) \cup (-\mathrm{i}c_r, -\mathrm{i}d)$:

$$\begin{aligned} J^{(3)}(\xi,t,k) =& F_+^{-\sigma_3}(k,\xi) \begin{pmatrix} 1 & 0 \\ -\dfrac{r(k)\mathrm{e}^{2\mathrm{i}tg_+(k,\xi)}}{\delta^2(k,\xi)F_+^2(k,\xi)} & 1 \end{pmatrix} \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \\ &\times \begin{pmatrix} 1 & 0 \\ \dfrac{r(k)\mathrm{e}^{2\mathrm{i}tg_-(k,\xi)}}{\delta^2(k,\xi)F_-^2(k,\xi)} & 1 \end{pmatrix} F_-^{\sigma_3}(k,\xi), \quad k \in (\mathrm{i}c_r, \mathrm{i}d), \\ =& F_+^{-\sigma_3}(k,\xi) \begin{pmatrix} 1 & -\dfrac{r(k)\delta^2(k,\xi)F_+^2(k,\xi)}{\mathrm{e}^{2\mathrm{i}tg_+(k,\xi)}} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \\ &\times \begin{pmatrix} 1 & \dfrac{r(k)\delta^2(k,\xi)F_-^2(k,\xi)}{\mathrm{e}^{2\mathrm{i}tg_-(k,\xi)}} \\ 0 & 1 \end{pmatrix} F_-^{\sigma_3}(k,\xi),\ k \in (-\mathrm{i}c_r, -\mathrm{i}d), \end{aligned}$$

and the previous factorization on the interval $(\mathrm{i}d, -\mathrm{i}d)$:

$$J^{(3)}(\xi,t,k) = F_+^{-\sigma_3}(k,\xi) \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} F_-^{\sigma_3}(k,\xi), \quad k \in (\mathrm{i}d, -\mathrm{i}d), \tag{4.36}$$

where

- $F(k,\xi)$ is analytic outside the downward-oriented contour $[\mathrm{i}c_r, -\mathrm{i}c_r]$;
- $F(k,\xi)$ does not vanish;
- $F(k,\xi)$ satisfies the jump relation

$$F_-(k,\xi)F_+(k,\xi) = \tilde{h}(k,\xi)\mathrm{e}^{\mathrm{i}t(g_-(k,\xi)+g_+(k,\xi))},$$

where

$$\tilde{h}(k) = \begin{cases} -\mathrm{i}f(k)\delta^{-2}(k) = (a_-(k)a_+(k))^{-1}\delta^{-2}(k), \quad k \in (\mathrm{i}d, 0), \\ \dfrac{\mathrm{i}}{f(k)\delta^2(k)} = a_-(k)a_+(k)\delta^{-2}(k), \quad k \in (-\mathrm{i}d, 0), \\ \delta^{-2}(k), \quad k \in (\mathrm{i}c_r, \mathrm{i}d) \cup (-\mathrm{i}d, -\mathrm{i}c_r); \end{cases} \tag{4.37}$$

- $\lim\limits_{k\to\infty} F(k,\xi) = 1$;
- $a(k)F(k,\xi)$ is bounded in a small neighborhood of the point $\mathrm{i}c_l$;
- $(a(k))^{-1}F(k,\xi)$ is bounded in a small neighborhood of the point $-\mathrm{i}c_l$.

This scalar RH problem is solved like in the case of $\xi > \dfrac{c_l^2}{2}$, and the solution is given by the formula

$$F(k,\xi) = \exp\left\{ \frac{X_{c_r}(k)}{2\pi\mathrm{i}} \int\limits_{\mathrm{i}c_r}^{-\mathrm{i}c_r} \frac{\log \tilde{h}(s,\xi)}{s-k} \frac{ds}{X_{c_r+}(s)} \right\}. \tag{4.38}$$

Due to the properties of $g(k,\xi)$ on the vertical cut $[\mathrm{i}c_l, -\mathrm{i}c_l]$, we have

$$J^{(3)}(\xi,t,k) = I + O\left(\mathrm{e}^{-\varepsilon t}\right), \quad k \in (\mathrm{i}c_l, \mathrm{i}c_r) \cup (-\mathrm{i}c_r, -\mathrm{i}c_l), \text{ when } t \to -\infty.$$

In opposite to the previous cases now we add some lenses to our contour

$$\Sigma^{(4)} = \Sigma^{(3)} \cup \bigcup_{j=5}^{8} L_j,$$

as depicted in Figure 8. We define the new matrix-function $M^{(4)}(\xi,t,k)$ as follows:

$$M^{(4)}(\xi,t,k) = M^{(3)}(\xi,t,k)F^{-\sigma_3}(k,\xi), \qquad k \notin \bigcup_{j=5}^{8} \Omega_j, \tag{4.39}$$

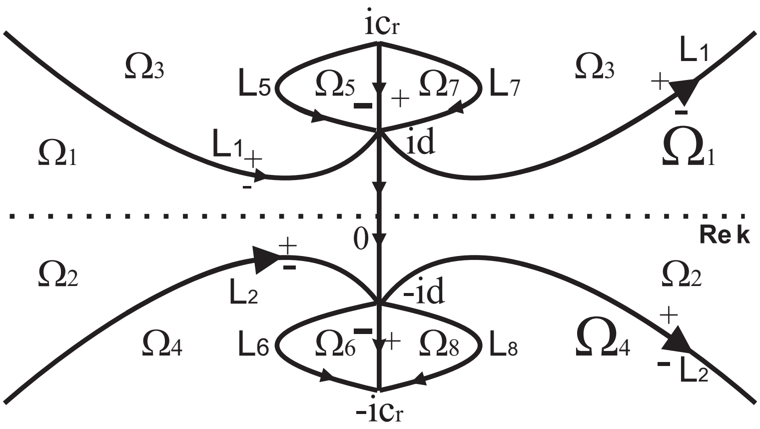


Fig. 8. Contour $\Sigma^{(4)}$ for $M^{(3)}(\xi,t,k)$-problem for $-\dfrac{c_r^2}{2}<\xi<\dfrac{c_r^2}{2}$.

$$M^{(4)}(\xi,t,k)=M^{(3)}(\xi,t,k)F^{-\sigma_3}(k,\xi)\begin{pmatrix}1 & 0\\ \dfrac{-r(k)\mathrm{e}^{2itg(k,\xi)}}{\delta^2(k,\xi)F^2(k,\xi)} & 1\end{pmatrix},\qquad k\in\Omega_7, \tag{4.40}$$

$$M^{(4)}(\xi,t,k)=M^{(3)}(\xi,t,k)F^{-\sigma_3}(k,\xi)\begin{pmatrix}1 & 0\\ \dfrac{r(k)\mathrm{e}^{2itg(k,\xi)}}{\delta^2(k,\xi)F^2(k,\xi)} & 1\end{pmatrix},\qquad k\in\Omega_5, \tag{4.41}$$

$$M^{(4)}(\xi,t,k)=M^{(3)}(\xi,t,k)F^{-\sigma_3}(k,\xi)\begin{pmatrix}1 & \dfrac{-r(k)\delta^2(k,\xi)F^2(k,\xi)}{\mathrm{e}^{2itg(k,\xi)}}\\ 0 & 1\end{pmatrix},\qquad k\in\Omega_8, \tag{4.42}$$

$$M^{(4)}(\xi,t,k)=M^{(3)}(\xi,t,k)F^{-\sigma_3}(k,\xi)\begin{pmatrix}1 & \dfrac{r(k)\delta^2(k,\xi)F^2(k,\xi)}{\mathrm{e}^{2itg(k,\xi)}}\\ 0 & 1\end{pmatrix},\qquad k\in\Omega_6, \tag{4.43}$$

which gives the following RH problem:

$$\begin{aligned}&M_-^{(4)}(\xi,t,k)=M_+^{(4)}(\xi,t,k)J^{(4)}(\xi,t,k),\quad k\in\Sigma^{(4)},\\ &M^{(4)}(\xi,t,k)=I+O(k^{-1}),\quad k\to\infty,\end{aligned} \tag{4.44}$$

where $\Sigma^{(4)}=\mathbb{R}\cup\bigcup\limits_{j=1}^{8}L_j$, and

$$J^{(4)}(\xi,t,k)=\begin{cases} I, & k\in\mathbb{R},\\ J^{(mod)}, & k\in(\mathrm{i}c_r,-\mathrm{i}c_r),\\ I+O(\mathrm{e}^{-\varepsilon t}), & k\in L_j\cup(\mathrm{i}c_l,\mathrm{i}c_r)\cup(-\mathrm{i}c_r,-\mathrm{i}c_l), \quad j=1,2,\dots,8.\end{cases}$$

4.5.4. Region $\xi<-\dfrac{c_r^2}{2}$

The jump matrix $J^{(3)}(\xi,t,k)$ is supported on the line $[\mathrm{i}c_r,-\mathrm{i}c_r]$ and has the following factorization:

$$J^{(3)}(\xi,t,k)=F_+^{-\sigma_3}(k,\xi)\begin{pmatrix}0 & \mathrm{i}\\ \mathrm{i} & 0\end{pmatrix}F_-^{\sigma_3}(k,\xi),\quad k\in(\mathrm{i}c_r,-\mathrm{i}c_r), \tag{4.45}$$

where

$$F(k,\xi)=\exp\left\{\frac{X_{c_r}(k)}{2\pi\mathrm{i}}\int_{\mathrm{i}c_r}^{-\mathrm{i}c_r}\frac{-\log\delta^2(k,\xi)}{(s-k)X_{c_r+}(s)}\right\}.$$

This factorization leads to the matrix-function

$$M^{(4)}(\xi,t,k)=M^{(3)}(\xi,t,k)F^{-\sigma_3}(k,\xi),\quad k\in\mathbb{C}\backslash\Sigma^{(4)}.$$

$M^{(4)}(\xi,t,k)$ solves the following RH problem:

$$\begin{aligned} &M_-^{(4)}(\xi,t,k)=M_+^{(4)}(\xi,t,k)J^{(4)}(\xi,t,k), \qquad k\in\Sigma^{(4)},\\ &M^{(4)}(\xi,t,k)=I+O(k^{-1}),\quad k\to\infty,\end{aligned} \tag{4.46}$$

where $\Sigma^{(4)}=[\mathrm{i}c_l,-\mathrm{i}c_l]\bigcup\limits_{j=1}^{6}L_j$, and

$$J^{(4)}(\xi,t,k)=\begin{cases} I, \quad k\in(\mathrm{i}c_l,\mathrm{i}c_r)\cup(-\mathrm{i}c_r,-\mathrm{i}c_l),\\ J^{(mod)}, \quad k\in(\mathrm{i}c_r,-\mathrm{i}c_r),\\ I+O(\mathrm{e}^{-\varepsilon t}), \quad k\in L_j, \quad j=1,2,\dots,6.\end{cases}$$

4.6. Solving of model problem and asymptotics for $q(x,t)$. The asymptotic analysis of the parametrix solutions near the end points $\mathrm{i}c_r$, $-\mathrm{i}c_r$, $\mathrm{i}d$, $-\mathrm{i}d$ and the stationary points $\pm\lambda(\xi)$ are similar to those done in [18] and [17]. In the first case, since the local representation of $g(k,\xi)$ at the points $\mathrm{i}c_r$ and $-\mathrm{i}c_r$ is characterized by a square root type behavior

$$g(k,\xi)\sim g_0(\mathrm{i}c,\xi)\sqrt{k-\mathrm{i}c_r},\ k\to\mathrm{i}c_r,\quad g(k,\xi)\sim\bar{g}_0(-\mathrm{i}c,\xi)\sqrt{k+\mathrm{i}c_r},\ k\to-\mathrm{i}c_r,$$

the relevant Riemann–Hilbert problems are solvable in terms of the Bessel functions, while in the second case of the real stationary points $\pm\lambda(\xi)$ they are solvable in terms of the parabolic cylinder functions. The main contribution to the asymptotics is given by the stationary points, and this contribution has the order $O(t^{-1/2})$. Therefore, we have

$$M^{(4)}(\xi,t,k)=\left(I+O\left(\frac{1}{t^{1/2}}\right)\right)M^{(mod)}(k),$$

where $M^{(mod)}(k)$ solves the zero-gap model problem

$$\begin{aligned}&M_-^{(mod)}(k)=M_+^{(mod)}(k)J^{(mod)},\quad k\in(\mathrm{i}d_1(\xi),-\mathrm{i}d_1(\xi)),\\&M^{(mod)}(k)=I+O(k^{-1}),\quad k\to\infty\end{aligned}$$

with the constant jump matrix

$$J^{(mod)}=\begin{pmatrix}0&\mathrm{i}\\\mathrm{i}&0\end{pmatrix},$$

where

$$d_1(\xi)=\begin{cases}c_l, & \xi>\dfrac{c_l^2}{2},\\ \sqrt{2\xi}, & \xi\in\left(\dfrac{c_l^2}{2},\dfrac{c_r^2}{2}\right),\\ c_r, & \xi<-\dfrac{c_r^2}{2}.\end{cases}\tag{4.47}$$

To solve the model problem let us use the function

$$\varkappa_{d_1}(k)=\sqrt[4]{\frac{k-\mathrm{i}d_1}{k+\mathrm{i}d_1}},$$

where we fix the branch of the square root in such a way that $\varkappa_{d_1}(\infty)=1$. Since $\varkappa_{d_1-}(k)=\mathrm{i}\varkappa_{d_1+}(k)$ on the cut $(\mathrm{i}d_1,-\mathrm{i}d_1)$, the explicit solution of the model problem takes the form

$$M^{(mod)}(k)=\frac{1}{2}\begin{pmatrix}\varkappa_{d_1}(k)+\dfrac{1}{\varkappa_{d_1}(k)} & \varkappa_{d_1}(k)-\dfrac{1}{\varkappa_{d_1}(k)}\\ \varkappa_{d_1}(k)-\dfrac{1}{\varkappa_{d_1}(k)} & \varkappa_{d_1}(k)+\dfrac{1}{\varkappa_{d_1}(k)}\end{pmatrix}.$$

Finally, we have the following chain of transformations of the RH problem:

$$\begin{aligned}
&M(\xi,t,k)=M^{(1)}(\xi,t,k)\mathrm{e}^{\mathrm{i}t[\theta(k)-g(k,\xi)]\sigma_3},\\
&M^{(1)}(\xi,t,k)=M^{(2)}(\xi,t,k)\delta^{\sigma_3}(k,\xi),\\
&M^{(2)}(\xi,t,k)=M^{(3)}(\xi,t,k)[G^{(3)}(\xi,t,k)]^{-1},\\
&M^{(3)}(\xi,t,k)=M^{(4)}(\xi,t,k)F^{\sigma_3}(k,\xi),\\
&M^{(4)}(\xi,t,k)=M^{(mod)}(k)(I+O(t^{-1/2})).
\end{aligned}$$

Let us put

$$\lim_{k\to\infty}[kM^{(j)}(x/12t,t,k)]_{12}=m_{12}^{(j)}(x,t).$$

Then, taking into account the chain of our transformations and the equation (3.10), we find

$$\begin{aligned}
q(x,t)&=2\mathrm{i}\lim_{k\to\infty}[kM(x/12t,t,k)]_{12}\\
&=2\mathrm{i}m_{12}(x,t)=2\mathrm{i}m_{12}^{(1)}(x,t)=2\mathrm{i}m_{12}^{(2)}(x,t)\\
&=2\mathrm{i}m_{12}^{(3)}(x,t)+O(t^{-1/2})\\
&=2\mathrm{i}m_{12}^{(4)}(x,t)+O(t^{-1/2})\\
&=2\mathrm{i}m_{12}^{(mod)}(x,t)+O(t^{-1/2})\\
&=d_1(\xi)+O(t^{-1/2}),
\end{aligned}$$

where we use the equalities $m_{12}^{(mod)}(x,t)=d_1(\xi)/2\mathrm{i}$, $F(\infty,\xi)\equiv 1$. Thus we arrive to the main result.

Theorem 4.1. *For $t\to-\infty$ the solution of the IBV problem (1.1)–(1.2) takes the form*

$$\begin{aligned}
q(x,t) &= c_l+O(\mathrm{e}^{-Ct}), \quad -\infty<x<6c_l^2t,\\
&= \sqrt{\frac{x}{6t}}+O(t^{-1/2}), \quad 6c_l^2t<x<6c_r^2t,\\
&= c_r+O(t^{-1/2}), \quad x>6c_r^2t,
\end{aligned}$$

where $C>0$ is some positive number.

R e m a r k 4.1. The above result is proved for the pure step function (3.2), but our approach can be extended for an arbitrary sufficiently smooth and fast decreasing to its limits initial step-like function $q_0(x)$.

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