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On the Solution of the Monge–Ampere Equation $Z_{xx}Z_{yy} - Z_{xy}^2 = f(x, y)$ with Quadratic Right Side

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For the Monge–Ampere equation $Z_{xx}Z_{yy} - Z_{xy}^2 = b_{20}x^2 + b_{11}xy + b_{02}y^2 + b_{00}$ we consider the question on the existence of a solution $Z(x, y)$ in the class of polynomials such that $Z = Z(x, y)$ is a graph of a convex surface. If

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Z is a polynomial of odd degree, then the solution does not exist. If Z is a polynomial of 4-th degree and $4b_{20}b_{02} - b_{11}^2 > 0$, then the solution also does not exist. If $4b_{20}b_{02} - b_{11}^2 = 0$, then we have solutions.

Key words: Monge–Ampere equation, polynomial, convex surface.

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1. Introduction

Numerous works are devoted to the study of the Monge–Ampere equation. The well-known Jörgens theorem [1] affirms that the equation

$$Z_{xx}Z_{yy} - Z_{xy}^2 = 1 \tag{1}$$

has a unique solution

$$Z = a_{20}x^2 + a_{11}xy + a_{02}y^2 + a_{10}x + a_{01}y + a_{00}$$

with the condition $4a_{20}a_{02} - a_{11}^2 = 1$ if $Z(x, y)$ is convex and a complete surface is determined on the whole plane x, y.

In the work by Yu. Volkov and S. Vladimirova [2] the Jörgens theorem was applied to the proof of the following remarkable result: *every isometric immersion of the Euclidean plane into Lobachevsky space* $\mathbb{L}^3$ *is either a horosphere or a surface of rotation of some equidistant of a geodesic around this geodesic.*

The Jörgens theorem has been generalized to the n-dimensional case for the equation

$$det\,|Z_{ij}| = 1,$$

where

$$Z = Z(x_1, \ldots, x_n),$$

by Calabi for the case $n = 3, 4$ [3] and by Pogorelov for $n \geq 5$ [4].

Now the methods of construction of solutions for nonlinear differential equations in the form of solitons are well elaborated with the help of inverse scattering problem, but taking into attention the possibility to approximate every continue function by polynomials at x, y, it is natural to apply the straight method to find the solution in the form of polynomials for the equation

$$Z_{xx}Z_{yy} - Z_{xy}^2 = f(x, y), \tag{2}$$

where $f(x, y)$ is a polynomial, in particular, of the second degree.

We will prove the following theorems.

Theorem 1. *The equation*

$$Z_{xx}Z_{yy} - Z_{xy}^2 = b_{20}x^2 + b_{11}xy + b_{02}y^2 + b_{00} \tag{3}$$

with the conditions

$$b_{20} > 0, b_{02} > 0, 4b_{20}b_{02} - b_{11}^2 \geq 0, b_{00} > 0 \tag{4}$$

does not have a solution in the form of a polynomial of odd degree.

Theorem 2. *The equation*

$$Z_{xx}Z_{yy} - Z_{xy}^2 = b_{20}x^2 + b_{11}xy + b_{02}y^2 + b_{00} \tag{5}$$

with the conditions

$$b_{20} > 0, b_{02} > 0, 4b_{20}b_{02} - b_{11}^2 \geq 0, b_{00} > 0 \tag{6}$$

has a solution in the class of polynomials of the 4-th degree if and only if

$$4b_{20}b_{02} - b_{11}^2 = 0.$$

Hence, if $4b_{20}b_{02} - b_{11}^2 > 0$, then equation (5) with condition (6) does not have a solution in the class of polynomial of 4-th degree.

We remark that (4) is the consequence of convexity of the surface $Z = Z(x, y)$.

Let us represent $Z(x, y)$ in the form

$$Z = \sum_{r=2}^{4} \sum_{i+j=r} a_{ij}x^i y^j. \tag{7}$$

We prove that in the case $4b_{20}b_{02} - b_{11}^2 = 0$ the solution has the form

$$Z = t^2 \left(\sqrt{b_{20}}x + \varepsilon\sqrt{b_{02}}y\right)^4 + a_{20}x^2 + a_{11}xy + a_{02}y^2 \tag{8}$$

if

$$24t^2 \left(a_{20}b_{20} - \varepsilon\sqrt{b_{20}b_{02}}a_{11} + a_{02}b_{02}\right) = 1, \tag{9}$$

and

$$4a_{20}a_{02} - a_{11}^2 = b_{00}, \tag{10}$$

where $t \neq 0$ is an independent parameter, and $\varepsilon = \pm 1$.

2. Proof of Theorem 1

If Z is a polynomial of odd order $2r+1$,

$$Z = a_{2r+1,0}x^{2r+1} + a_{2r,1}x^{2r}y + \dots , \tag{11}$$

then, without loss of generality, we can suppose that

$$a_{2r+1,0} > 0,$$

because this condition can be obtained by rotation in the plane x, y.

So we have the second derivative on the axis $y = 0$,

$$Z_{xx} = (2r+1)(2r)a_{2r+1,0}x^{2r-1} + \dots . \tag{12}$$

If $x \to \infty$, then $Z_{xx} > 0$, and similarly, if $x \to -\infty$, then $Z_{xx} < 0$. Thus we can deduce that $Z_{xx} = 0$ at some point. Consequently, at this point

$$Z_{xx}Z_{yy} - Z_{xy}^2 = -Z_{xy}^2 \le 0. \tag{13}$$

But this contradicts our condition that $Z_{xx}Z_{yy} - Z_{xy}^2 = -Z_{xy}^2 > 0$. So our theorem is proved.

3. Proof of Theorem 2

Verify at first that (8) is a solution if $b_{11} = 2\varepsilon\sqrt{b_{02}b_{20}}$ with $\varepsilon = \pm 1$.

Let us denote

$$\sqrt{b_{20}}x + \varepsilon\sqrt{b_{02}}y = u. \tag{14}$$

We have

$$\begin{aligned} Z_x &= 4t^4u^3\sqrt{b_{20}} + 2a_{20}x + a_{11}y, \\ Z_y &= 4\varepsilon t^2u^3\sqrt{b_{02}} + a_{11}x + 2a_{02}y, \\ Z_{xx} &= 12t^2u^2b_{20} + 2a_{20}, \\ Z_{xy} &= 12\varepsilon t^2u^2\sqrt{b_{20}b_{02}} + a_{11}, \\ Z_{yy} &= 12t^2u^2b_{02} + 2a_{02}. \end{aligned} \tag{15}$$

Hence,

$$\begin{aligned} Z_{xx}Z_{yy} - Z_{xy}^2 &= 24t^2u^2(a_{20}b_{02} - \varepsilon\sqrt{b_{02}b_{20}}a_{11} + a_{02}b_{20}) + 4a_{20}a_{02} - a_{11}^2 \\ &= (b_{20}x^2 + b_{11}xy + b_{02}y^2) + b_{00}. \end{aligned} \tag{16}$$

Thus, the function in (8) is a solution of (5).

Write the right-hand side of (7) in a more detailed form

$$\begin{aligned} Z &= a_{40}x^4 + a_{31}x^3y + a_{22}x^2y^2 + a_{13}xy^3 + a_{04}y^4 + a_{30}x^3 \\ &+ a_{21}x^2y + a_{12}xy^2 + a_{03}y^3 + a_{20}x^2 + a_{11}xy + a_{02}y^2. \end{aligned} \tag{17}$$

We divide the system of equations in the coefficients a_{ij} into 5 systems (0), (I), (II), (III), (IV) in accordance with the degree of members, which we obtain in the expression $Z_{xx}Z_{yy} - Z_{xy}^2$ after calculation of derivatives of Z. For readers' comfort we write the expression of the second derivatives as follows:

$$\begin{aligned} Z_{xx} &= 12a_{40}x^2 + 6a_{31}xy + 2a_{22}y^2 + 6a_{30}x + 2a_{21}y + 2a_{20}, \\ Z_{yy} &= 12a_{04}y^2 + 6a_{13}xy + 2a_{22}x^2 + 6a_{03}y + 2a_{12}x + 2a_{02}, \\ Z_{xy} &= 3a_{31}x^2 + 4a_{22}xy + 3a_{13}y^2 + 2a_{21}x + 2a_{12}y + a_{11}. \end{aligned}$$

We use the theorem on the equality coefficients of two polynomials and obtain the following Lemma.

Lemma 3. *If $Z(x, y)$ in the form (17) is a solution of (5), then the following 5 systems have place:*

$$4a_{20}a_{02} - a_{11}^2 = b_{00}, \tag{0}$$

$$\begin{aligned} 3a_{30}a_{02} + a_{12}a_{20} - a_{21}a_{11} = 0, \\ 3a_{03}a_{20} + a_{21}a_{02} - a_{12}a_{11} = 0, \end{aligned} \tag{I}$$

$$\begin{aligned} 4a_{22}a_{20} - 6a_{31}a_{11} + 24a_{40}a_{02} + 4(3a_{30}a_{12} - a_{21}^2) = b_{20}, \\ 12a_{13}a_{20} - 8a_{22}a_{11} + 12a_{31}a_{02} + 4(9a_{30}a_{03} - a_{12}a_{21}) = b_{11}, \\ 24a_{04}a_{20} - 6a_{13}a_{11} + 4a_{22}a_{02} + 4(3a_{03}a_{21} - a_{12}^2) = b_{02}, \end{aligned} \tag{II}$$

$$\begin{aligned} a_{22}a_{30} - a_{31}a_{21} + 2a_{40}a_{12} + 0 = 0, \\ 3a_{13}a_{30} - a_{22}a_{21} + 0 + 6a_{40}a_{03} = 0, \\ 6a_{04}a_{30} + 0 - a_{22}a_{12} + 3a_{31}a_{03} = 0, \\ 0 + 2a_{04}a_{21} - a_{13}a_{12} + a_{22}a_{03} = 0, \end{aligned} \tag{III}$$

$$\begin{aligned} 8a_{40}a_{22} - 3a_{31}^2 = 0, \\ 6a_{40}a_{13} - a_{31}a_{22} = 0, \\ 24a_{40}a_{04} + 3a_{31}a_{13} - 2a_{22}^2 = 0, \\ 6a_{31}a_{04} - a_{22}a_{13} = 0, \\ 8a_{04}a_{22} - 3a_{13}^2 = 0. \end{aligned} \tag{IV}$$

We call a_{ij} with $i + j = m$ the coefficient of the degree m.

Lemma 4. *If $a_{40}a_{04} \neq 0$, then all coefficients of the 4-th degree have expressions in terms of a_{40}, a_{04}:*

$$\begin{aligned} a_{31}^2 &= 16a_{40}\sqrt{a_{40}a_{04}}, \\ a_{22} &= 6\sqrt{a_{40}a_{04}}, \\ a_{13}^2 &= 16a_{04}\sqrt{a_{40}a_{04}}, \end{aligned} \tag{18}$$

that is the consequence of (IV).

If the coefficient $a_{04} = 0$, from the system (IV) we obtain that $a_{31} = a_{22} = a_{13} = 0$. If $a_{40} = 0$ also, then Z is the polynomial of degree of 3-rd order. Subsequently we can suppose that $a_{40} \neq 0$. Then, from (III) we obtain $a_{12} = a_{03} = 0$. From the third equation of (II) we obtain $b_{02} = 0$ that contradicts (6). So we can put $a_{40}a_{04} \neq 0$. Then $a_{22} \neq 0, a_{31} \neq 0, a_{13} \neq 0$. From (I) we obtain

$$(3a_{30}a_{12} - a_{21}^2)a_{02} = (3a_{03}a_{21} - a_{12}^2)a_{20}. \tag{19}$$

Consequently, there exists some number λ such that

$$\begin{aligned} 3a_{30}a_{12} - a_{21}^2 &= \lambda a_{20}, \\ 3a_{03}a_{21} - a_{12}^2 &= \lambda a_{02}. \end{aligned} \tag{20}$$

Consider the system (I) as the system for determining of a_{02}, a_{20}

$$\begin{aligned} 3a_{30}a_{02} + a_{12}a_{20} &= a_{21}a_{11}, \\ a_{21}a_{02} + 3a_{03}a_{20} &= a_{12}a_{11}. \end{aligned} \tag{21}$$

From the system (21) by eliminating a_{20}, we obtain

$$(9a_{30}a_{03} - a_{12}a_{21})a_{02} = a_{11}(3a_{12}a_{03} - a_{12}^2) = \lambda a_{11}a_{02}. \tag{22}$$

Since $a_{02} \neq 0$, we get

$$9a_{30}a_{03} - a_{12}a_{21} = \lambda a_{11}.$$

We have the system of equations

$$\begin{aligned} 3a_{30}a_{12} - a_{21}^2 &= \lambda a_{20}, \\ 9a_{30}a_{03} - a_{12}a_{21} &= \lambda a_{11}, \\ 3a_{03}a_{21} - a_{12}^2 &= \lambda a_{02}. \end{aligned} \tag{23}$$

We show that $\lambda = 0$.

From the system (III) we obtain the expressions of a_{30} and a_{03}

$$\begin{aligned} a_{30} &= \frac{a_{31}a_{21} - 2a_{40}a_{12}}{a_{22}}, \\ a_{03} &= \frac{a_{13}a_{12} - 2a_{04}a_{21}}{a_{22}}, \end{aligned} \tag{24}$$

and by substituting (24) into (23), we obtain

$$\begin{aligned} -6a_{40}a_{12}^2 + 3a_{31}a_{12}a_{21} - a_{22}a_{21}^2 &= \lambda a_{22}a_{20}, \\ 18(-a_{40}a_{13}a_{12}^2 + 8a_{04}a_{40}a_{12}a_{21} - a_{04}a_{31}a_{21}^2) &= \lambda a_{22}^2 a_{11}, \\ -a_{22}a_{12}^2 + 3a_{13}a_{12}a_{21} - 6a_{04}a_{21}^2 &= \lambda a_{22}a_{02}. \end{aligned} \tag{25}$$

In the above we can replace a_{31}, a_{13}, a_{22} with the expressions from Lemma 4. Let $a_{31} > 0$, then also $a_{13} > 0$. Denote

$$T = (a_{40})^{\frac{1}{4}} a_{12} - (a_{04})^{\frac{1}{4}} a_{21}. \tag{26}$$

Then we obtain

$$\begin{aligned} -6\sqrt{a_{40}}T^2 &= \lambda a_{22}a_{20}, \\ -12(a_{40}a_{04})^{\frac{1}{4}}T^2 &= \lambda a_{22}a_{11}, \\ -6\sqrt{a_{04}}T^2 &= \lambda a_{22}a_{02}. \end{aligned} \tag{27}$$

If $\lambda \neq 0$, then

$$\begin{aligned} a_{20} &= \frac{a_{11}}{2}\left(\frac{a_{40}}{a_{04}}\right)^{\frac{1}{4}}, \\ a_{02} &= \frac{a_{11}}{2}\left(\frac{a_{04}}{a_{40}}\right)^{\frac{1}{4}}. \end{aligned} \tag{28}$$

It gives us

$$4a_{20}a_{02} - a_{11}^2 = 0,$$

that is impossible, because $4a_{20}a_{02} - a_{11}^2 = b_{00} > 0$. Similarly is considered the case $a_{31} < 0$. Therefore, $\lambda = 0$. In this case the system (II) is as follows:

$$\begin{aligned} 4a_{22}a_{20} - 6a_{31}a_{11} + 24a_{40}a_{02} &= b_{20}, \\ 12a_{13}a_{20} - 8a_{22}a_{11} + 12a_{31}a_{02} &= b_{11}, \\ 24a_{04}a_{20} - 6a_{13}a_{11} + 4a_{22}a_{02} &= b_{02}. \end{aligned} \tag{29}$$

Consider (29) as the system for determining of a_{20}, a_{11}, a_{22}. All the coefficients of the system have expressions in terms of a_{40}, a_{04}. Denote $\gamma = (a_{04}a_{40})^{\frac{1}{4}}$, then the determinant of the matrix of coefficients is

$$\begin{vmatrix} 4a_{22} & -6a_{31} & 24a_{40} \\ 12a_{13} & -8a_{22} & 12a_{31} \\ 24a_{04} & -6a_{13} & 4a_{22} \end{vmatrix} = -32 \cdot 36 \cdot 12\gamma\sqrt{a_{04}a_{40}} \begin{vmatrix} \sqrt{a_{40}} & \sqrt{a_{40}} & \sqrt{a_{40}} \\ 2\gamma & 2\gamma & 2\gamma \\ \sqrt{a_{04}} & \sqrt{a_{04}} & \sqrt{a_{04}} \end{vmatrix} = 0$$

if $a_{31} > 0$. When $a_{31} < 0$, we obtain the same statement.

Hence, it is not difficult to verify that the determinant of the coefficient matrix is equal to zero as well as all minors of the second order. In fact, it means that every two vectors from the system

$$(4a_{22}, -6a_{31}, 24a_{40}),$$
$$(12a_{13}, -8a_{22}, 12a_{31}),$$
$$(24a_{04}, -6a_{13}, 4a_{22})$$

are linearly dependent. Therefore, if the system (29) has a solution, then it must be

$$\begin{aligned} \frac{4}{3}\frac{a_{22}}{a_{31}} &= \frac{b_{11}}{b_{20}}, \\ \frac{4}{3}\frac{a_{22}}{a_{13}} &= \frac{b_{11}}{b_{02}}. \end{aligned} \tag{30}$$

Taking into attention Lemma 4, we obtain

$$\frac{b_{11}^2}{b_{02}b_{20}} = \frac{16a_{22}^2}{9a_{31}a_{13}} = 4.$$

So, Theorem 2 is proved.

Now we suppose that $4b_{20}b_{02} - b_{11}^2 = 0$. Let us find the view of polynomial (8). From (30) we obtain

$$\frac{\sqrt{a_{40}}}{\sqrt{a_{04}}} = \frac{a_{31}}{a_{13}} = \frac{b_{20}}{b_{02}}. \tag{31}$$

Introduce some positive number t such that

$$\sqrt{a_{40}} = tb_{20}, \sqrt{a_{04}} = tb_{02}. \tag{32}$$

Lemma 5. *All coefficients of the 3-rd degree are equal to zero, i.e.,* $a_{30} = a_{21} = a_{12} = a_{03} = 0$.

Suppose that $a_{12} \neq 0$, then $a_{21} \neq 0$. From (23) we have

$$\begin{aligned} 3a_{30} &= \frac{a_{21}^2}{a_{12}}, \\ 3a_{03} &= \frac{a_{12}^2}{a_{21}}. \end{aligned} \tag{33}$$

Further, by substituting (33) into the first equation of (I), we get

$$a_{21}a_{02} + \frac{a_{12}^2}{a_{21}}a_{20} - a_{11}a_{12} = 0.$$

We have the equation

$$a_{02}a_{21}^2 - a_{12}a_{21}a_{11} + a_{12}^2a_{20} = 0.$$

This equation does not have a solution, except zero, because $a_{11}^2 - 4a_{20}a_{02} < 0$. So, $a_{12} = a_{21} = 0$. Further, from (II) we obtain $a_{30} = a_{03} = 0$. From Lemmas 4 and 5 there follows the view (8).

We remark that equation (2) with polynomial $f(x, y) > 0$ sometimes has a solution in the polynomial form. Besides (12), we can give the following example. For the polynomial

$$f(x, y) = (3u^2 + 1)^2 + 3u^2(u^2 + u + x^2y^2)$$

of degree eight, where

$$u = \frac{1}{2}(x^2 + y^2),$$

the solution of (2) is the polynomial of degree six, $Z = u(u^2 + 1)$. The surface Z is complete and convex. Obviously $-Z$ is also the solution. Will these two functions be unique solutions on the whole plane x, y ?

Our question in consideration is a part of the more general problem: to construct the solutions of (2) as a polynomial, when f is also a polynomial, but without condition $f > 0$.

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