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Elementary Solutions of the Bernstein Problem on Two Intervals

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First we note that the best polynomial approximation to $|x|$ on the set, which consists of an interval on the positive half-axis and a point on the negative half-axis, can be given by means of the classical Chebyshev polynomials. Then we explore the cases when a solution of the related problem on two intervals can be given in elementary functions.

Key words: Chebyshev polynomials, polynomial approximation of $|x|$, Bernstein problem.

Mathematics Subject Classification 2000: 41A10.

1. Introduction

1.1. Setting of the Problem

In [1, 2] Eremenko and Yuditskii studied asymptotics of the error L_n of the best polynomial approximation of the function $\operatorname{sgn}(x)$ on two intervals $[-A,-1]\cup[1,B]$. It was noted that in the case when one of the intervals degenerates to a point, say $A=1$, the extremal polynomial $P_n(x)$ can be explicitly written by means of the classical Chebyshev polynomials. Moreover, such a relation holds true for a certain sufficiently small extension of the set until it reaches the critical value $A_*>1$ (note that a further extension of the interval leads to a representation of the extremal solution via the Zolotarev polynomials), (see Figure 1).

With these results in mind, we study the best polynomial approximation to the function $|x|$ on the set $[-D,-C]\cup[B,A]$ that consists of two intervals, one on the positive and one on the negative half-axes, so that the problem also admits an elementary solution. Recall that Bernstein [3, 4] found that for the error E_n

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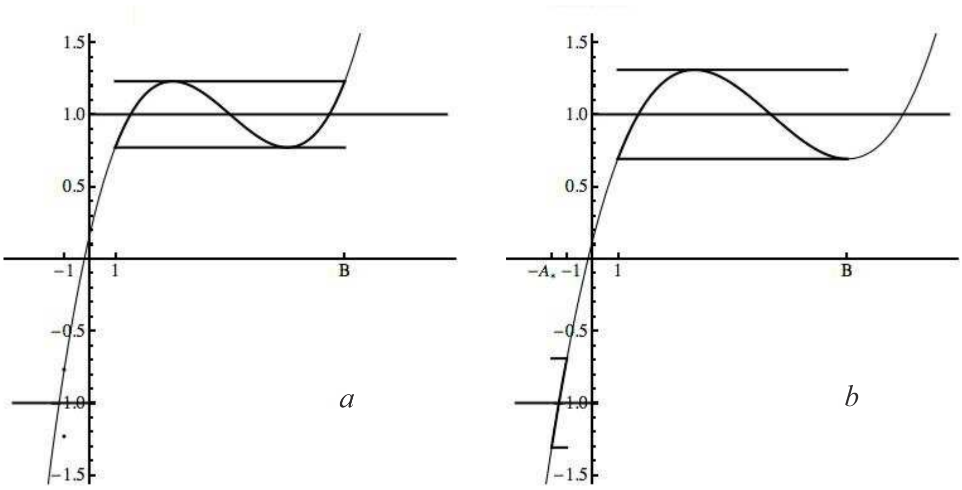

Fig. 1. Best polynomial approximation of sgn(x) with $B = 10$, $n = 3$. Solution on $\{-1\} \cup [1, B]$ (a). Maximal extension with Chebyshev polynomials (b).

of the best uniform approximation of $|x|$ on $[-1, 1]$ by polynomials of degree n the following limit exists:

$$\lim_{n\to\infty} nE_n = \mu > 0.$$

His question, whether μ can be expressed in terms of any known transcendental functions, remains one of the most famous open problems in approximation theory (for recent developments in this area see [5]).

Similarly to the ideas in [1], we start with the set $\{-C\} \cup [B, A]$ that consists of an interval on the positive half-axis and a point on the negative half-axis and show that for arbitrary $A, B, C > 0$ these extremal polynomials can be described in terms of Chebyshev polynomials.

The polynomials of the best approximation of sgn(x) possess the following crucial property: all critical values, barring one possible exception, are on the level $1 \pm L_n$ on the positive half-axis and on the level $-1 \pm L_n$ on the negative half-axis (see Figure 1). It allows in an easy way to relate some of them with the Chebyshev and Zolotarev polynomials. The structure of the extremal polynomials that approximate $|x|$ is much more delicate. Nevertheless, we are able to find the maximal interval bound $\mathfrak{D} = \mathfrak{D}(A, B, C)$ such that for given A, B, C the extremal polynomial on $[-\mathfrak{D}, -C] \cup [B, A]$ can still be described in terms of Chebyshev polynomials.

Our considerations are based on the Chebyshev Alternation Theorem which we would like to recall before stating our main result.

1.2. Cheybshev Polynomials and Alternation Theorem

We follow [6] (see also [7]) in this introductory subsection. Let $K \subset \mathbb{R}$ be an arbitrary compact set consisting of at least $n+2$ points, and let $s(x)$ and $f(x)$ be real continuous functions with $s(x) > 0$ for $x \in K$. Due to Chebyshev, we want to determine a polynomial $P(x)$ of degree at most n such that its deviation from $f(x)$ with the weight $s(x)$, namely

$$L(P) = ||(f - P)/s||_{C(K)},$$

is as small as possible. Denote by

$$M(P) = \{x \in K : |f(x) - P(x)|/s(x) = L(P)\}$$

the set of points of maximal deviation. A set of points of alternation is defined to be a maximal subset $\{x_1, \dots, x_\omega\} \subset M(P)$ such that $(f(x) - P(x))/s(x)$ takes the values $\pm L(P)$ with alternating signs at the points of this set. Note that in general a set of points of alternation is not unique. However, it follows from the definition that all sets of points of alternation have the same cardinality. Now we can state the important theorem which is usually referred to as the Chebyshev Alternation Theorem:

Theorem 1. *There exists a unique polynomial $P(x)$ of degree at most n deviating least from the function $f(x)$ with the weight $s(x)$ on the compact set K. This polynomial is completely characterized by the property that a set of points of alternation consists of at least $n + 2$ points.*

Note that we can consider the polynomial of degree $n-1$ deviating least from the function $f(x) = x^n$ with the weight $s(x) = 1$ on $[-1, 1]$ or, in other words, the polynomial of degree n with leading coefficient 1 deviating least from zero on $[-1, 1]$. This is the classical Chebyshev problem and its solutions are known as Chebyshev polynomials $T_n(x)$ of degree n. The polynomials T_n have always exactly $n + 1$ alternation points in $[-1, 1]$.

1.3. Notations and Main Result

Let $P_n(x)$ be the best approximation of $|x|$ by polynomials of degree at most n on the set $I := [-D, -C] \cup [B, A]$, with $A, B, C, D > 0$ and $A > B$ and $D \geq C$. Our goal is to obtain a representation of the extremal polynomial on I for given A, B, C and different values of D. We will study in which cases this representation can be written in terms of Chebyshev polynomials T_n of degree n. Our main result can be stated as follows:

Theorem 2. *For given $A, B, C > 0$, there are values L, α and β such that there exists $\mathfrak{D}(A, B, C)$ with the property that for all values D with $\mathfrak{D} \geq D \geq C$ the extremal polynomial on $I = [-D, -C] \cup [B, A]$ can be written in the form*

$$P_n(x) = L \cdot (-1)^n T_n(y(x, \alpha, \beta)) + x, \tag{1}$$

where L denotes the approximation error and $y(x, \alpha, \beta) = -1 + \frac{x-\beta}{\alpha-\beta} \cdot 2$.

Our paper is structured as follows. In Section 2, we consider the problem of finding a solution to our problem on $I = \{-Q\} \cup [B, A]$. In this case $\alpha = A$, $\beta = B$ and $L = L_n(A, B, Q)$ can be given by the explicit Formula (5). In Section 3, we apply the results of Section 2 and prove the first version of our main result. In this case $\alpha = A, \beta = B$ and $L = L_n(A, B, Q)$, where Q is an appropriately chosen point in the second interval $[-D, -C]$. In Section 4, we study further extensions of I by certain changes of the parameters α and β. We conclude our considerations with the observation that any further extension of I involves polynomials that are not Chebyshev polynomials any more. Section 5 is finally concerned with the special case $B = 0$, where the set I might degenerate to an interval.

2. Polynomials of Least Deviation from $|x|$ on $\{-Q\} \cup [B, A]$

In this section we will show that there always exists an extremal polynomial of the above form for fixed but arbitrary A, B, Q on $I = \{-Q\} \cup [B, A]$. Let

$$H_n(x, \alpha, \beta, \gamma) = L_n(\alpha, \beta, \gamma) \cdot (-1)^n T_n(y(x, \alpha, \beta)) + x, \tag{2}$$

where $y(x, \alpha, \beta) = -1 + \frac{x-\beta}{\alpha-\beta} \cdot 2$.

In the following, we set $\alpha = A$, $\beta = B$ and $\gamma = Q$, and show that for the given set I, $P_n = H_n(x, A, B, Q)$, moreover, L_n is the approximation error (5).

Proposition 1. *For given $A, B, Q > 0$, $A, B, Q \in \mathbb{R}$, there exists $L_n > 0$ such that $P_n = H_n(x, A, B, Q)$ with $y(x, A, B) = -1 + \frac{x-B}{A-B} \cdot 2$.*

P r o o f. Note that for $x \in [B, A]$ we have $y(x, A, B) \in [-1, 1]$. Hence we have $n + 1$ alternation points in the interval $[B, A]$. In order to have an extremal polynomial of degree n on I we need $n+2$ alternation points due to the Chebyshev theorem. Therefore we introduce an additional relation, which already suffices to compute $L_n(A, B, Q)$ in a unique way, and solve the following set of equations for L_n:

$$H_n(x, A, B, Q) = L_n(A, B, Q) \cdot (-1)^n T_n(y(x, A, B)) + x \tag{3}$$

$$H_n(-Q, A, B, Q) = Q - L_n(A, B, Q). \tag{4}$$

We get that

$$L_n(A,B,Q) = \frac{2Q}{1 + (-1)^n T_n(y(-Q,A,B))}. \tag{5}$$

Note that $L_n > 0$, and that (4) gives us the required $(n+2)$-th alternation point such that the polynomial H_n is indeed extremal on I. ■

These extremal polynomials have the following crucial properties:

Proposition 2. *For given $A, B, Q > 0$, the first derivative $H_n'(x, A, B, Q)$ of the extremal polynomial P_n on I is monotonously increasing in $(-\infty, B]$ and $H_n'(B, A, B, Q) > -1$.*

P r o o f. Note that $T_n(y(x, A, B))$ has $n+1$ points of alternation in $[B, A]$. Hence n zeros of T_n lie in $[B, A]$, which means that all zeros of T_n lie in $[B, A]$. Consequently, $[B, A]$ contains $n-1$ critical points of T_n, which are exactly the $n-1$ zeros of T_n'. Therefore T_n' is monotonous outside of the interval $[B, A]$. Since we always have by definition that $H_n(B, A, B, Q) = B + L_n$, it follows that T_n' is monotonously increasing in $(-\infty, B]$.

Now suppose $H_n'(B, A, B, Q) \leq -1$. Since $H_n(B, A, B, Q) = B + L_n$ and since the derivative is monotonic, we have $H_n(x, A, B, Q) > |x|$ for all $x \in (-\infty, B)$. However, this is a contradiction to $H_n(-Q, A, B, Q) = Q - L_n$. Hence $H_n'(B, A, B, Q) > -1$. ■

Proposition 3. *For given A, B and for all $Q \in (0, \infty)$, there exists $x \in (Q, \infty)$ such that $H_n(-x, A, B, Q) > x + L_n(A, B, Q)$.*

P r o o f. Recall that

$$H_n(x, A, B, Q) = L_n(A, B, Q)(-1)^n T_n(y(x, A, B)) + x$$

and that $T_n(x)$ behaves asymptotically like x^n for $x \to -\infty$. Therefore, for sufficiently large $x \in (Q, \infty)$, we get

$$H_n(-x, A, B, Q) = L_n(A, B, Q)(-1)^n T_n(y(-x, A, B)) - x > x + L_n(A, B, Q)$$

since

$$\frac{1}{x}((-1)^n T_n(y(-x, A, B)) - 1) > \frac{2}{L_n(A, B, Q)},$$

where the right-hand side is constant. ■

As a consequence of these three propositions we can state the following:

- If for a given Q, $H_n'(-Q, A, B, Q) > -1$, then there is $\tilde{Q} \in (Q, \infty)$ such that $H_n(-\tilde{Q}, A, B, Q) = \tilde{Q} - L_n(A, B, Q)$. Indeed, there exists a point $\xi_1 > Q$ such that $H_n(-\xi_1, A, B, Q) < \xi_1 - L_n(A, B, Q)$ and due to Proposition 3 there also exists a point $\xi_2 > \xi_1$ with $P_n(-\xi_2, A, B, Q) > \xi_2 + L_n(A, B, Q)$. Hence there is $\tilde{Q}$ with $\xi_2 > \tilde{Q} > \xi_1$ for which the claim holds true.

- If for a given Q, $H_n'(-Q, A, B, Q) < -1$, then there is $\tilde{Q} \in (0, Q)$ with $H_n(-\tilde{Q}, A, B, Q) = \tilde{Q} - L_n(A, B, Q)$. Indeed, there exists $\xi < Q$ with $H_n(-\xi, A, B, Q) < \xi - L_n(A, B, Q)$. Due to the monotonicity of the derivative and since $H_n(B, A, B, Q) = B + L_n(A, B, Q)$, there is $\tilde{Q} \in (0, Q)$ for which the claim holds true.

Note that since there is a unique extremal polynomial for a given Q, the corresponding points $\tilde{Q}$ are also unique (see Figure 2).

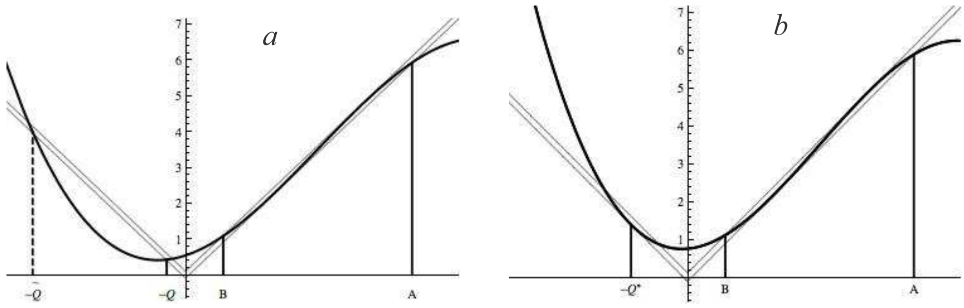

Fig. 2. Solutions of the point-interval problem for $n = 3$ and $A = 6$, $B = 1$ resp. $Q = 0.5$ in (a) and $Q = Q^*(6, 1) = 1.5$ in (b).

Moreover, for different values of Q the corresponding polynomials H_n only differ in the factor L_n with which T_n is multiplied. The polynomial T_n has all its zeros in $[B, A]$, which means that the only points of intersection of the graphs of two extremal polynomials for different values of Q can lie in $[B, A]$. Therefore we can conclude that for the given values $Q_1 > Q_2$ with $H_n'(-Q_1, A, B, Q_1) < -1$ and $H_n'(-Q_2, A, B, Q_2) < -1$ we always have $\tilde{Q}_2 > \tilde{Q}_1$.

Consequently, since we already know that there is a unique solution to our problem for every $Q \in (0, \infty)$, it is easy to see that there exists a unique point $Q^* = Q^*(A, B)$ which is the fixed point of the above described involution $Q \mapsto \tilde{Q}$. For this point it holds that

$$H_n(-Q^*, A, B, Q^*) = Q^* - L_n(A, B, Q^*), \tag{6}$$

$$H_n'(-Q^*, A, B, Q^*) = -1. \tag{7}$$

Furthermore, we can define the values $C^*(A,B) < D^*(A,B)$

$$H_n(-D^*, A, B, Q^*) = D^* + L_n(A, B, Q^*), \tag{8}$$
$$H_n(-C^*, A, B, Q^*) = C^* + L_n(A, B, Q^*). \tag{9}$$

The value $D^*(A,B)$ always exists due to Proposition 3, whereas $C^*(A,B)$ exists because

$$\begin{aligned} H_n(0, A, B, Q^*(A,B)) &= L_n(A, B, Q^*)(-1)^n T_n\left(-1 + 2\frac{0-B}{A-B}\right) + 0 \\ &= L_n(A, B, Q^*)(-1)^n T_n(-1-\epsilon) > L_n(A, B, Q^*). \end{aligned} \tag{10}$$

The values C^*, Q^* and D^* play a crucial role in the first extension of the set (see next section).

3. First Extension of I

We have seen that we can always solve our problem for arbitrary A, B, Q on $I = \{-Q\} \cup [B, A]$. The main idea of our first extension is to move, if possible, the leftmost alternation point (which initially is at $C = Q$). Thus we consider a new point-interval problem with the additional constraint $H_n(-C) \leq C + L_n$. Formally, we change the parameter γ in the solution of the initial point-interval problem.

We will see that if $C \geq Q^*$, then no non-trivial extension of the interval is possible. Note that in the following we always study the maximal possible extension of I to the left, in other words, we always fix A, B, C and compute the biggest possible interval bound $\mathfrak{D} = D(A, B, C) > C$ such that there exists H_n of the form (2) on $[-\mathfrak{D}, -C] \cup [B, A]$.

In order to simplify the proof of Theorem 3, it is useful to have the following property:

Proposition 4. *For given A, B and arbitrary Q, $L_n(A, B, Q)$ is increasing for $Q \in (0, Q^*)$ and decreasing for $Q \in (Q^*, \infty)$.*

P r o o f. Recall that for different values of Q the corresponding polynomials H_n only differ in the factor L_n with which T_n is multiplied. Moreover, T_n has all its zeros in $[B, A]$, which means that the only points of intersection of the graphs of two extremal polynomials for different values of Q can be in $[B, A]$.

Suppose there exists $Q \in (0, Q^*)$ such that $L_n(A, B, Q) = L_n(A, B, Q^*)$, which means we have only one polynomial to consider. For this polynomial it holds that $H_n(-Q^*) = Q^* - L_n$ as well as $H_n(-Q) = Q - L_n$. However, this is a contradiction to the monotonicity of H_n' since we know that $H_n'(-Q^*) = -1$.

Now suppose there exists $Q \in (0, Q^*)$ such that $L_n(A,B,Q) > L_n(A,B,Q^*)$. This implies that $H_n(B,A,B,Q) > H_n(B,A,B,Q^*)$ and also that $H(-Q,A,B,Q) < H(-Q^*,A,B,Q^*)$. This means that there exists a point of intersection of the two graphs in $(-Q^*, B)$. However, this is a contradiction to the fact that T_n has all its zeros in $[B,A]$. Hence, for all $Q \in (0, Q^*)$ it holds that $L_n(A,B,Q) < L_n(A,B,Q^*)$.

Note that for each $Q \neq Q^*$ there exists a unique $\tilde{Q}$ such that $L_n(A,B,Q) = L_n(A,B,\tilde{Q})$. Therefore if L_n is increasing for $Q \in (0, Q^*)$, it is decreasing for $Q \in (Q^*, \infty)$. ■

Theorem 3. *For given A, B let C^*, Q^*, D^* be as defined above. For arbitrary $C > 0$ there exists $\mathfrak{D}_1 \geq C$ such that for any D with $\mathfrak{D}_1 \geq D \geq C$ there exists an extremal polynomial $H_n(x,\alpha,\beta,\gamma)$ with $y(x,\alpha,\beta) = -1 + \frac{x-\beta}{\alpha-\beta} \cdot 2$ on $I = [-D,-C] \cup [B,A]$.*

More precisely, set $\alpha = A$, $\beta = B$. Then:

(I) *if $C \in (0, C^*)$, $\mathfrak{D}_1$ is such that $H_n(-C,A,B,\mathfrak{D}_1) = C + L_n(A,B,\mathfrak{D}_1)$. For any D with $\mathfrak{D}_1 \geq D \geq C$, we set $\gamma = D$, and the extremal polynomial on I is given by $H_n(x,A,B,D)$.*

(II) *if $C \in [C^*, Q^*]$, $\mathfrak{D}_1 = D^*$.*

- (II.a) *For $Q^* > D \geq C$, we set $\gamma = D$, and the extremal polynomial on I is given by $H_n(x,A,B,D)$.*
- (II.b) *For $D^* \geq D \geq Q^*$, we set $\gamma = Q^*$, and the extremal polynomial on I is given by $H_n(x,A,B,Q^*)$.*

(III) *if $C \in (Q^*, \infty)$, $\mathfrak{D}_1$ is such that $H_n(-\mathfrak{D}_1,A,B,C) = \mathfrak{D}_1 + L_n(A,B,C)$. For any D with $\mathfrak{D}_1 \geq D \geq C$, we set $\gamma = C$, and the extremal polynomial on I is given by $H_n(x,A,B,C)$.*

Remark 1. *For an illustration of Case 1, see Figure 3 and Figure 4. For an example of a diagram of the maximal first extensions for given A, B, n and different values of C see Figure 5(a).*

P r o o f. **Case 1.** Let $C \in (0, C^*)$. Note that $C < C^*$ implies $\mathfrak{D}_1 < Q^*$. We have to show that for any D with $\mathfrak{D}_1 \geq D \geq C$

$$H_n(-C,A,B,D) \leq C + L_n(A,B,D).$$

Due to Proposition 4, we have

$$H_n(-x,A,B,\mathfrak{D}_1) \geq H_n(-x,A,B,C)$$

for $x \in (0, \mathfrak{D}_1)$.

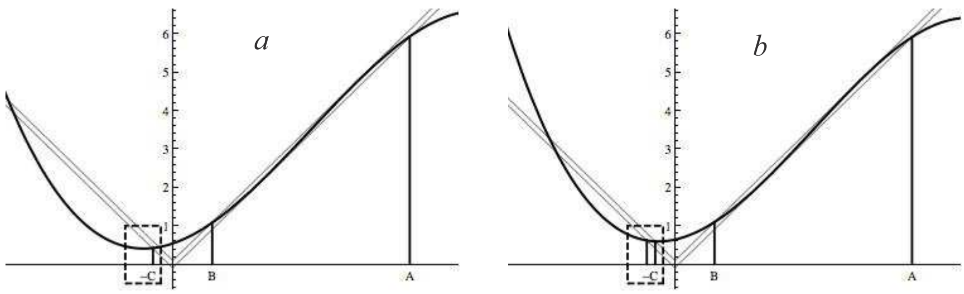

Fig. 3. Illustration of Case 1 for $A = 6$, $B = 1$, $C = 0.5$. Solution on $\{-C\} \cup [B, A]$ (a). Extension to $[-\mathfrak{D}_1, -C] \cup [B, A]$ (b).

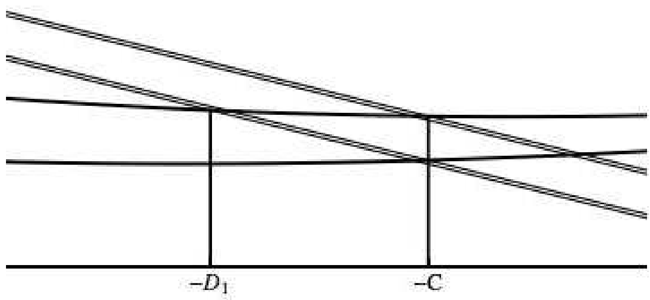

Fig. 4. Illustration of Case 1 for $A = 6$, $B = 1$, $C = 0.5$. Zoom (Dashed Box) of Figure 3.

Suppose that there exists γ with $\mathfrak{D}_1 \geq \gamma \geq C$ such that

$$H_n(-C, A, B, \gamma) > C + L_n(A, B, \gamma),$$

then

$$\begin{aligned}(-1)^n\, T_n(y(-C, A, B))(L_n(A, B, \mathfrak{D}_1) - L_n(A, B, \gamma)) &= \\ &= H_n(-C, A, B, \mathfrak{D}_1) - H_n(-C, A, B, \gamma) \\ &< C + (L_n(A, B, \mathfrak{D}_1) - C - (L_n(A, B, \gamma) \\ &= L_n(A, B, \mathfrak{D}_1) - L_n(A, B, \gamma).\end{aligned}$$

This is a contradiction to the fact that $(-1)^n T_n(y(-C, A, B)) > 1$ since $-C < B$. Therefore we have shown

$$H_n(-C, A, B, D) \leq C + L_n(A, B, D).$$

Case 2. Let $C \in [C^*, Q^*]$. We have to show that for $\gamma = D$ and $Q^* > D \geq C$

$$H_n(-C, A, B, D) \leq C + L_n(A, B, D).$$

We can use a similar argument as in the first case. Due to Proposition 4, we know that

$$H_n(-x, A, B, Q^*) \geq H_n(-x, A, B, C)$$

for $x \in (0, Q^*)$. Suppose there exists γ with $Q^* > \gamma \geq C$ such that

$$H_n(-C, A, B, \gamma) > C + L_n(A, B, \gamma).$$

Then we also have that

$$H_n(-C^*, A, B, \gamma) > C^* + L_n(A, B, \gamma)$$

due to the monotonicity of H'_n. Hence,

$$\begin{aligned}(-1)^n T_n(y(-C^*, A, B))(L_n(A, B, Q^*) - L_n(A, B, \gamma)) =& \\ & H_n(-C^*, A, B, Q^*) - H_n(-C^*, A, B, \gamma) \\ & < C^* + (L_n(A, B, Q^*) - C^* - (L_n(A, B, \gamma) \\ & = L_n(A, B, Q^*) - L_n(A, B, \gamma).\end{aligned}$$

This yields the same contradiction as before. Note that for $\gamma = Q^*$ we always have

$$x - L_n(A, B, Q^*) \leq H_n(x, A, B, Q^*) \leq x + L_n(A, B, Q^*)$$

for $x \in (C, D^*)$. Hence, $H_n(x, A, B, Q^*)$ is maximal for any D with $D^* \geq D \geq Q^*$ on I.

Case 3. Let $C \in (Q^*, \infty)$. In this case we can only trivially extend the solution of $\{-C\} \cup [B, A]$ to the point $\mathfrak{D}_1$ that is defined via

$$H_n(-\mathfrak{D}_1, A, B, C) = \mathfrak{D}_1 + L_n(A, B, C).$$

Due to Proposition 3, this point always exists and, since $H'_n(-C, A, B, C) < -1$, we always have that

$$x - L_n(A, B, C) \leq H_n(x, A, B, C) \leq x + L_n(A, B, C)$$

for $x \in (C, \mathfrak{D}_1)$. ∎

Remark 2. *We can see that it is possible to extend $I = \{-C\} \cup [B, A]$ if we change the parameter γ and, therefore, the factor L_n in the representation of the extremal polynomial. Since L_n also denotes the approximation error and it is increasing for increasing $C \in (0, Q^*)$ and decreasing for increasing $C \in (Q^*, \infty)$, it is not surprising that we can only trivially extend our interval (to the left) for any $C \in (Q^*, \infty)$.*

4. Further Extension of I

In the previous section we extended I by changing the parameter γ of the extremal polynomial. At the end of the first extension step we obtained a new alternation point in each of the three cases, so that we had 2 in the subset of the negative half-axes and still $(n+1)$ in the subset of the positive half-axes. In this section we show that we can extend I further by changing the parameters α and β accordingly. That is, we change the interval on which the Chebyshev polynomial T_n is extremal. The idea is that since we have one alternation point more after the first extension than we need for applying the alternation theorem, we will try to extend our interval by shifting the rightmost resp. the leftmost alternation point on the positive half-axes. Of course, there is a natural limitation in this shift, namely, we can only shift as long as we do not lose a second alternation point in I. Formally, we define the right limit point $\bar{A} > A$ for given A, B, C in the following way:

$$H_n(A, \bar{A}, B, C) = A - (-1)^n L_n(\bar{A}, B, C), \tag{11}$$
$$H_n'(A, \bar{A}, B, C) = 1, \tag{12}$$

and $H_n'(x, \bar{A}, B, C) > 1$ or < 1 for all $x \in (A, \infty)$. This means we shift the rightmost critical point of the polynomial $H_n(x, A, B, C)$ to the position A and obtain a new polynomial $H_n(x, \bar{A}, B, C)$. Note that for any α with $A \leq \alpha \leq \bar{A}$ the graph of the corresponding polynomial $H_n(x, \alpha, B, C)$ has $n-1$ points of intersection with the graph of $H_n(x, A, B, C)$ in the interval $[B, A]$. Analogously, we can define, if possible, a point $\bar{B} > 0$ by shifting the leftmost critical point to the position B. In the case when this point does not exist, we consider β with $0 < \beta \leq B$ instead of $\bar{B} \leq \beta \leq B$.

Proposition 5. *For given A, B, α and β with $A \leq \alpha \leq \bar{A}$ and $\bar{B} \leq \beta \leq B$, we have*

$$Q^*(A, B) \leq Q^*(\alpha, B) \quad \text{and} \quad D^*(A, B) \leq D^*(\alpha, B), \tag{13}$$
$$C^*(A, \beta) \leq C^*(A, B). \tag{14}$$

P r o o f. To prove the first part, note that $L_n(A, B, Q^*(A, B)) < L_n(\alpha, B, Q^*(\alpha, B))$ and hence

$$H_n(B, \alpha, B, Q^*(\alpha, B)) > H_n(B, A, B, Q^*(A, B)),$$
$$H_n(-Q^*(\alpha, B), \alpha, B, Q^*(\alpha, B)) < H_n(-Q^*(\alpha, B), A, B, Q^*(A, B)),$$

which means that the two graphs intersect in the interval $(-Q^*(\alpha, B), B)$. Moreover, there are $n-1$ points of intersection in $[B, A]$, and since $H_n(x, \alpha, B, Q^*(\alpha, B))$

$-H_n(x, A, B, Q^*(A, B))$ is a polynomial of degree n, these are all intersections. Furthermore, we have for all $x \in (-\infty, B)$ that

$$(-1)^n T_n'(y(x, A, B)) \leq (-1)^n T_n'(y(x, \alpha, B))$$

and therefore

$$L_n(A, B, Q^*(A, B))(-1)^n T_n'(y(x, A, B)) \leq L_n(\alpha, B, Q^*(\alpha, B))(-1)^n T_n'(y(x, \alpha, B)).$$

This implies

$$-1 = H_n'(-Q^*(A, B), A, B, Q^*(A, B)) \leq H_n'(-Q^*(A, B), \alpha, B, Q^*(\alpha, B)),$$

and due to the monotonicity of the derivative,

$$Q^*(A, B) \leq Q^*(\alpha, B).$$

Moreover, since $H_n(x, \alpha, B, C) < H_n(x, A, B, C)$ for all $x \in (-\infty, -Q^*(\alpha, B))$, it follows that

$$D^*(A, B) < D^*(\alpha, B).$$

For the second part, note that $L_n(A, B, Q^*(A, B)) < L_n(A, \beta, Q^*(A, \beta))$ and that

$$\begin{aligned} H_n(B, A, \beta, Q^*(A, \beta)) &< H_n(B, A, B, Q^*(A, B)), \\ H_n(-Q^*(A, \beta), A, \beta, Q^*(A, \beta)) &< H_n(-Q^*(A, \beta), A, B, -Q^*(A, B)). \end{aligned}$$

Hence, there is no point of intersection of the two graphs in $(-Q^*(A, \beta), B)$, (if there were one, this would imply that there has to be another one to satisfy these relations) and

$$\begin{aligned} H_n(-C^*(A, B), A, \beta, Q^*(A, \beta)) &< H_n(-C^*(A, B), A, B, Q^*(A, B)) \\ &= C^*(A, B) + L_n(A, B, Q^*(A, B)), \end{aligned}$$

such that

$$C^*(A, \beta) < C^*(A, B). \qquad \blacksquare$$

This technical proposition is the base of the following considerations. We improve each case of Theorem 3 in a separate lemma from which our main theorem, as stated in the introduction, immediately follows.

Lemma 1. *For given A, B and $C \in (Q^*(\bar{A}, B), \infty)$, there exists $\mathfrak{D}_2 > \mathfrak{D}_1$ such that $H_n(x, \bar{A}, B, C)$ is maximal on $[-\mathfrak{D}_2, -C] \cup [B, A]$.*

P r o o f. As in the previous proposition, we can see that $H_n(x,\alpha,B,C) - H_n(x,A,B,C)$ for $A \le \alpha \le \bar{A}$ is a polynomial of degree n with $n-1$ zeros in $[B,A]$ and an additional zero in $[-C,B]$. This implies

$$H_n(x,\alpha,B,C) \le H_n(x,A,B,C)$$

for all $x \in (-\infty,-C)$, and

$$H_n(-\mathfrak{D}_1,\alpha,B,C) \le H_n(-\mathfrak{D}_1,A,B,C) = L_n(A,B,C) + \mathfrak{D}_1 < L_n(\alpha,B,C) + \mathfrak{D}_1.$$

Consequently, due to Proposition 3, there exists $\mathfrak{D}_2 > \mathfrak{D}_1$ such that

$$H_n(-\mathfrak{D}_2,\alpha,B,C) = L_n(\alpha,B,C) + \mathfrak{D}_2.$$

To get a polynomial which is maximal for any D with $\mathfrak{D}_1 \le D \le \mathfrak{D}_2$, one has to chose an appropriate α with $A \le \alpha \le \bar{A}$. ■

Lemma 2. *For given A, B and $C \in (C^*(A,B), Q^*(\bar{A},B))$, there exists α, with $A \le \alpha \le \bar{A}$ and $\mathfrak{D}_2 > \mathfrak{D}_1$ such that $H_n(x,\alpha,B,Q^*(\alpha,B))$ is maximal on $[-\mathfrak{D}_2,-C] \cup [B,A]$.*

P r o o f. **Case 1.** Let $C^*(A,B) \le C^*(\bar{A},B)$. For A,B,C we know from Theorem 3 that $\mathfrak{D}_1 = D^*(A,B)$.

However, for all $C \in (C^*(A,B), C^*(\bar{A},B))$, there exists α, with $A \le \alpha \le \bar{A}$, such that

$$H_n(-C,\alpha,B,Q^*(\alpha,B)) = C + L_n(\alpha,B,Q^*(\alpha,B)).$$

Due to Proposition 5, $\mathfrak{D}_2 = D^*(\alpha,B) \ge D^*(A,B) = \mathfrak{D}_1$.

Now consider $C \in (C^*(\bar{A},B), Q^*(\bar{A},B))$. In this case we can again apply Case 2 of Theorem 3 and see due to Proposition 5 that the maximal extension $\mathfrak{D}_2 = D^*(\bar{A},B)$ for $H_n(x,\bar{A},B,Q^*(\bar{A},B))$.

Case 2. Let $C^*(\bar{A},B) \le C^*(A,B)$. Like in the second subcase of the first case, we can apply Case 2 of Theorem 3 and see due to Proposition 5 that the maximal extension $\mathfrak{D}_2 = D^*(\bar{A},B)$ for $H_n(x,\bar{A},B,Q^*(\bar{A},B))$. ■

Lemma 3. *For given A, B and $C \in (0, C^*(A,B))$, there exists β, with $\bar{B} \le \beta \le B$, and $\mathfrak{D}_2 > \mathfrak{D}_1$ such that $H_n(x,A,\beta,Q^*(A,\beta))$ is maximal on $[-\mathfrak{D}_2,-C] \cup [B,A]$.*

P r o o f. **Case 1.** Let $C \in (C^*(A,\bar{B}), C^*(A,B))$. For all C there exists β, with $\bar{B} \le \beta \le B$, such that

$$H_n(-C,A,\beta,Q^*(A,\beta)) = C + L_n(A,\beta,Q^*(A,\beta)).$$

Note that $C = C^*(A, \beta)$ and, therefore, $\mathfrak{D}_2 = D^*(A, \beta)$ due to Case 2 of Theorem 3. From Case 1 of Theorem 3 we know that $\mathfrak{D}_1 < Q^*(A, B)$. So we have to show that $\mathfrak{D}_2 = D^*(A, \beta) \geq Q^*(A, B)$. Suppose $D^*(A, \beta) < Q^*(A, B)$. This would mean that the point of intersection of the graphs of the two polynomials is in $(0, Q^*(A, B))$. However, as we can see from Proposition 5, this point lies in $(Q^*(A, B), \infty)$. Hence, $D^*(A, \beta) \geq Q^*(A, B)$ and, therefore, $\mathfrak{D}_2 > \mathfrak{D}_1$, and the extremal polynomial is given by $H_n(x, A, \beta, Q^*(A, \beta))$.

Case 2. Let $C \in (0, C^*(A, \bar{B}))$. In this case we have to apply Case 1 of Theorem 3 twice. First, we get the relation $H_n(-C, A, B, \mathfrak{D}_1) = C + L_n(A, B, \mathfrak{D}_1)$ for $\mathfrak{D}_1$. Secondly, we get the relation $H_n(-C, A, \bar{B}, \mathfrak{D}_2) = C + L_n(A, \bar{B}, \mathfrak{D}_2)$ for $\mathfrak{D}_2$. Suppose that $\mathfrak{D}_2 < \mathfrak{D}_1$, due to the relations for these two values this would again yield a contradiction to the fact that the point of intersection of the two graphs is in the interval $(Q^*(A, B), \infty)$.

If $\bar{B}$ does not exist, we can apply Case 1 for all C. ■

These three lemmata can be combined to obtain Theorem 2. According to our observations, we can classify C for given A, B as shown in Table 1. For an example of a diagram of the maximal second extensions for given A, B, n and different values of C see Figure 5(b).

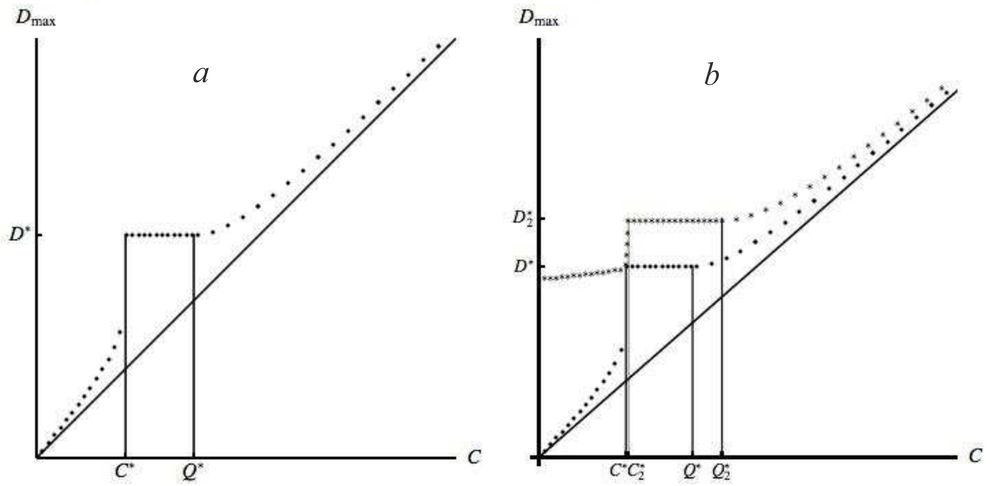

Fig. 5. Maximal interval extensions. Maximal first extension for $A = 6$, $B = 1$ (a). Maximal first and second extension for $A = 6$, $B = 1$ (b).

To sum up, we have proven that we can further extend our first maximal set for each case of Theorem 3 using this classification of C and applying Lemmas 1,2,3.

	Theorem 3	# Alt. Pts		Lemma 1/2/3	# Alt. Pts	
		$[-D,-C]$	$[B,A]$		$[-D,-C]$	$[B,A]$
$C\in$	$(0,C^*(A,B))$	2	$(n+1)$	$(0,C^*(A,\bar{B}))$	2	n
				$(C^*(A,\bar{B}),C^*(A,B))$	3	n
$C\in$	$(C^*(A,B),Q^*(A,B))$	2	$(n+1)$	$(C^*(A,B),C^*(\bar{A},B))$	3	n
				$(C^*(\bar{A},B),Q^*(A,B))$	2	n
$C\in$	$(Q^*(A,B),\infty)$	2	$(n+1)$	$(Q^*(A,B),Q^*(\bar{A},B))$	2	n
				$(Q^*(\bar{A},B),\infty)$	2	n

Table 1. Classification of C

Remark 3. *Note that $C^*(A,B) \le C^*(\alpha,B)$ and $D^*(A,\beta) \le D^*(A,B)$ for all $A \le \alpha \le \bar{A}$ and $\bar{B} \le \beta \le B$ immediately imply that our second extension step is the maximal possible extension such that P_n can be written in terms of Chebyshev polynomials. We have computational evidence, but no proof that these relations hold in general. If these relations do not hold (this can be computationally checked in an easy way for arbitrary values), we have to be (a little) more careful with the values C in the interval $(C^*(\alpha,B),C^*(A,B))$. In this case we can, for given A,B,C, change either the parameter α or β and apply the ideas from Lemmas $1,2,3$. The maximal extension is then given by the maximum of the two solutions we get. Note that if $\bar{B}$ exists, we can change both parameters for the unique value $C = C^*(\bar{A},\bar{B})$ and therefore get a third case to consider when (computationally) looking for the maximal extension.*

5. The Special Case $B = 0$

With respect to the original Bernstein problem, it is interesting to consider the special case $B = 0$. Note that our considerations of Section 2 still hold true if we set $B = 0$. In this case we have $C^*(A,0) = 0$ for any $A > 0$ since

$$H_n(0,A,0,Q^*(A,0)) = L_n(A,0,Q^*(A,0))(-1)^nT_n(-1)+0 = L_n(A,0,Q^*(A,0))+0.$$

Hence, we can reduce our set I to one interval if we choose $C = C^* = 0$ and we can apply Case 2 of Theorem 3 to see that $\mathfrak{D}_1 = D^*$. In this case we have $n+3$ alternation points and can further extend our interval. It follows from above that $C^*(A,0) = C^*(\bar{A},0) = 0$ and, therefore, we can apply the ideas of Lemma 2 to obtain $\mathfrak{D}_2 = D^*(\bar{A},0) > \mathfrak{D}_1$. It is easy to see that this is the maximal possible extension for P_n to be written in terms of Chebyshev polynomials on I.

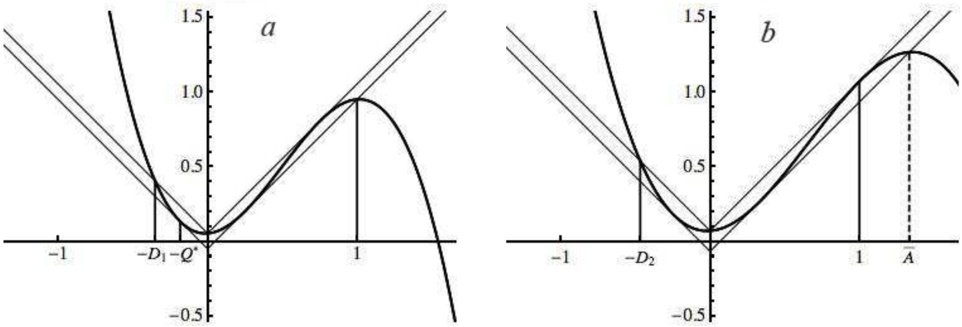

Fig. 6. Solution of the point-interval problem for $n = 3$, $A = 1$, $B = 0$ and $Q = Q^*(1, 0)$ (a). Maximal extension for $C = 0$ (b).

Let us discuss the special example with $A = 1, B = 0, n = 3$. We get $Q^* = 1/4(-1+\sqrt{3}) = 0.183\ldots$ and $D^* = 0.3509\ldots$ (see Figure 6). More computations show that the values Q^* and D^* decrease for increasing n. For our example we get $\bar{A} = 4/3, Q^*(\bar{A}, 0) = 1/3(-1+\sqrt{3}) = 0.244\ldots$ and $\mathfrak{D}_2 = D^*(\bar{A}, 0) = 0.4678\ldots$.

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