Journal of Mathematical Physics, Analysis, Geometry
2012, vol. 8, No. 2, pp. 119–134

General Boundary Value Problem for the Third Order Linear Differential Equation of Composite Type

A. Delshad Gharehgheshlaghi

Institute of Mathematics and Mechanics of NAS of Azerbaijan
E-mail: ali-delshad@rambler.ru

N. Aliyev

Baku State University, Baku, Azerbaijan

Received September 14, 2010, revised October 11, 2011

The boundary value problem is considered for the linear two-dimensional integro-differential loaded third order equation of composite type with nonlocal terms in the boundary conditions. The principal part of the equation is a derivative of the two-dimensional Laplace equation with respect to the variable x_2. Taking into account the ill-posedness of boundary value problems for hyperbolic differential equations, the principal part of the boundary conditions is chosen in a special form dictated by the obtained necessary conditions.

Key words: composite type equations, nonlocal boundary conditions, global boundary conditions, necessary conditions, regularization, Fredholm property.

Mathematics Subject Classification 2000: 35J25, 35J40.

1. Introduction

It is well known that in the case of ordinary differential operators, the Lagrange formula is the main tool [1] to solve the boundary value problems. But when the operator is generated by the boundary value problem for partial equations, Green's second formula [2, 3] becomes basic. For each concrete case [3–6], some potentials (with unknown densities) that are the solutions of the considered problems are derived proceeding from boundary conditions.

The form of the kernel of the potential is determined by Green's formula mentioned above. The study of the properties of the constructed potentials enables to define the unknown densities of some integral equations. By studying the properties of simple and double layers, it was possible to investigate the solutions of

Dirichlet and Neumann problems. Despite of being known, limit theorems both for normal derivatives of double layer potentials and tangential derivatives of simple and double layers [2] for some reasons were not often applied to boundary value problems.

To solve the boundary value problems with oblique derivatives, a jump formula obtained in [6–8] is usually used.

Notice that the problem under consideration is new. In the classic papers, boundary value problems are mainly considered for even-order elliptic equations. On the other hand, the mathematical model of the nuclear reactor in some cases is described by the integro-differential equation of first order with the boundary condition given on a part of the boundary [3]. As A.V. Bitsadze noted at one of the Steklov seminars, in connection with the Tricomi problems, the whole boundary should be present in the boundary condition. Therefore, the boundary condition given in [3] is not appropriate. If we replace the boundary condition given in [3] by the Dirichlet condition (given on the whole boundary), the obtained problem will have no solution.

The boundary conditions given here (nonlocal conditions) correspond to the first-order derivative.

The applied method is a continuation and generalization of the potential theory. The solution is sought in the form of Green's discrete second formula. Therefore, in our case we always have jumps.

If the potential of the simple layer is continuous, then we have jumps because of the double layer potential. If the double layer is continuous, then we have a jump because of the simple layer potential. In the case of the problem with non-local conditions, both potentials have jumps.

In the classic papers, in the case of inclined derivatives, if the inclination is tangential, then we do not have a jump, i.e., we get the Fredholm integral equation of first kind.

In our case such difficulties do not arise. Therefore, we can consider a problem with oblique derivatives of arbitrary form.

Notice that in our case the whole boundary is a support for each boundary condition.

Finally, we consider an example corresponding to the stated problem. Here the problem is discretized in a certain sense, the system of 39 algebraic equations with 39 variables is solved, and the errors of the obtained solution are shown.

2. Problem Statement

Let D be a bounded convex in the direction of x_2 plane domain with a Lyapunov-type boundary Γ [3]. When the domain D is orthogonally projected on the axis x_1 (parallel to x_2), the boundary Γ is divided into the parts Γ_1 and Γ_2. The equations of these lines are denoted by $x_2 = \gamma_k(x_1)\,,\ \ k = 1, 2;\ \ x_1 \in [a_1, b_1]$.

Consider the boundary value problem

$$lu \equiv \frac{\partial^3 u(x)}{\partial x_2^3} + \frac{\partial^3 u(x)}{\partial x_1^2 \partial x_2} + \sum_{k=0}^{2} a_{2k}(x) \frac{\partial^2 u(x)}{\partial x_1^k \partial x_2^{2-k}}$$

$$+ \sum_{k=1}^{2} a_{1k}(x) \frac{\partial u(x)}{\partial x_k} + a_0(x) u(x)$$

$$+ \sum_{m=0}^{2} \sum_{n=1}^{2} \int_{a_1}^{b_1} K_{2mn}(x, \eta_1) \left. \frac{\partial^2 u(\eta)}{\partial \eta_1^m \partial \eta_2^{2-m}} \right|_{\eta_2 = \gamma_n(\eta_1)} d\eta_1$$

$$+ \sum_{m=1}^{2} \sum_{n=1}^{2} \int_{a_1}^{b_1} K_{1mn}(x, \eta_1) \left. \frac{\partial u(\eta)}{\partial \eta_m} \right|_{\eta_2 = \gamma_n(\eta_1)} d\eta_1$$

$$+ \sum_{n=1}^{2} \int_{a_1}^{b_1} K_{0n}(x, \eta_1)\; u(\eta_1, \gamma_n(\eta_1))\, d\eta_1 = f(x), \quad x \in D \subset \mathbb{R}^2, \tag{1}$$

$$l_k u \equiv \left. \frac{\partial^2 u(x)}{\partial x_2^2} \right|_{x_2 = \gamma_k(x_1)} - \sum_{p=1}^{2} \sum_{j=1}^{2} \alpha_{kjp}(x_1) \left. \frac{\partial u(x)}{\partial x_j} \right|_{x_2 = \gamma_p(x_1)}$$

$$- \sum_{p=1}^{2} \alpha_{kp}(x_1) u(x_1, \gamma_p(x_1)) - \sum_{p=1}^{2} \sum_{j=1}^{2} \int_{a_1}^{b_1} \alpha_{kjp}(x_1, \eta_1) \left. \frac{\partial u(\eta)}{\partial \eta_j} \right|_{\eta_2 = \gamma_p(\eta_1)} d\eta_1$$

$$- \sum_{p=1}^{2} \int_{a_1}^{b_1} \alpha_{kp}(x_1, \eta_1)\; u(\eta_1, \gamma_p(\eta_1))\, d\eta_1 = f_k(x_1), \quad k = 1, 2; \quad x_1 \in [a_1, b_1], \tag{2}$$

$$l_3 u \equiv \left. \frac{\partial^2 u(x)}{\partial x_1^2} \right|_{x_2 = \gamma_2(x_1)} - \sum_{p=1}^{2} \sum_{j=1}^{2} \alpha_{3jp}(x_1) \left. \frac{\partial u(x)}{\partial x_j} \right|_{x_2 = \gamma_p(x_1)}$$

$$- \sum_{p=1}^{2} \alpha_{3p}(x_1) u(x_1, \gamma_p(x_1)) + \sum_{p=1}^{2} \sum_{j=1}^{2} \int_{a_1}^{b_1} \alpha_{3jp}(x_1, \eta_1) \left. \frac{\partial u(\eta)}{\partial \eta_j} \right|_{\eta_2 = \gamma_p(\eta_1)} d\eta_1$$

$$- \sum_{p=1}^{2} \int_{a_1}^{b_1} \alpha_{3p}(x_1, \eta_1)\; u(\eta_1, \gamma_p(\eta_1))\, d\eta_1 = f_3(x_1), \quad x_1 \in [a_1, b_1], \tag{3}$$

where all data of equation (1) and boundary conditions (2), (3) are assumed to be continuous functions.

If we consider the sufficiently smooth data of boundary value problem (1)–(3), then this problem reduces to the Fredholm integral equation of the second kind with respect to the function $u(x)$. Otherwise, we get the system of the Fredholm integral equations of the second kind with respect to the unknown function $u(x)$ and its derivatives. The kernels of these equations or of the obtained system have no singularities.

3. Fundamental Solutions and Their Basic Properties

Applying the Fourier transformations [2, 3] for the principal part of equation (1), namely, for the first two terms, we get the fundamental solution in the form

$$U(x-\xi) = \frac{1}{4\pi^2} \int_{\mathbb{R}^2} \frac{e^{i(\alpha, x-\xi)}}{\alpha_2 \left(\alpha_1^2 + \alpha_2^2\right)} d\alpha, \tag{4}$$

where $x - \xi = (x_1 - \xi_1, x_2 - \xi_2)$, and $(\alpha, x-\xi) = \alpha_1 (x_1 - \xi_1) + \alpha_2 (x_2 - \xi_2)$, $\mathbb{R}^2$ is a real plane.

Then, by means of Hormander's ladder method [9], for the integral (4) we obtain

$$U(x-\xi) = \frac{x_2 - \xi_2}{2\pi} \left[\ln \sqrt{|x_1 - \xi_1|^2 + (x_2 - \xi_2)^2} - 1 \right] + \frac{|x_1 - \xi_1|}{2\pi} \mathrm{arctg} \frac{x_2 - \xi_2}{|x_1 - \xi_1|}. \tag{5}$$

Differentiating (4) or (5), one can easily get

$$\frac{\partial^3 U(x-\xi)}{\partial x_2^3} + \frac{\partial^3 U(x-\xi)}{\partial x_1^2 \partial x_2} = \delta(x-\xi), \tag{6}$$

where

$$\frac{\partial U(x-\xi)}{\partial x_1} = \frac{e(x_1 - \xi_1)}{\pi} \mathrm{arctg} \frac{x_2 - \xi_2}{|x_1 - \xi_1|}, \tag{7}$$

$$\frac{\partial U(x-\xi)}{\partial x_2} = \frac{1}{2\pi} \ln \sqrt{|x_1 - \xi_1|^2 + (x_2 - \xi_2)^2}, \tag{8}$$

$$\frac{\partial^2 U(x-\xi)}{\partial x_1^2} = e(x_2 - \xi_2)\,\delta(x_1 - \xi_1) - \frac{1}{2\pi} \frac{x_2 - \xi_2}{|x_1 - \xi_1|^2 + (x_2 - \xi_2)^2}, \tag{9}$$

$$\Delta_x U(x-\xi) = e(x_2 - \xi_2)\,\delta(x_1 - \xi_1). \tag{10}$$

Here $e(t)$ is the symmetric Heaviside function, $\delta(x-\xi) = \delta(x_1 - \xi_1)\,\delta(x_2 - \xi_2)$ is a two-dimensional Dirac delta function [2, 3].

4. Basic Relations

Using fundamental solution (5), its property (6) and considering equations (1), we get Green's second formula [2–6]. From these formulas we get representations for any solution of equation (1) and expressions for the boundary values of this solution

$$\int_{\Gamma} \frac{\partial^2 u(x)}{\partial x_2^2} U(x-\xi) \cos(\nu, x_2)\, dx + \int_{\Gamma} \frac{\partial^2 u(x)}{\partial x_1^2} U(x-\xi) \cos(\nu, x_2)\, dx$$

$$- \int_{\Gamma} \frac{\partial u(x)}{\partial x_2} \frac{\partial U(x-\xi)}{\partial x_2} \cos(\nu, x_2)\, dx - \int_{\Gamma} \frac{\partial u(x)}{\partial x_1} \frac{\partial U(x-\xi)}{\partial x_2} \cos(\nu, x_1)\, dx$$

$$+ \int_{\Gamma} u(x) \frac{\partial^2 U(x-\xi)}{\partial x_1 \partial x_2} \cos(\nu, x_1)\, dx + \int_{\Gamma} u(x) \frac{\partial^2 U(x-\xi)}{\partial x_2^2} \cos(\nu, x_2)\, dx$$

$$+ \int_{D} l_0 u \cdot U(x-\xi)\, dx - \int_{D} f(x) U(x-\xi)\, dx = \begin{cases} u(\xi), & \xi \in D, \\ \frac{1}{2} u(\xi), & \xi \in \Gamma, \end{cases} \tag{11}$$

where

$$l_0 u \equiv \sum_{k=0}^{2} a_{2k}(x) \frac{\partial^2 u(x)}{\partial x_1^k \partial x_2^{2-k}} + \sum_{k=1}^{2} a_{1k}(x) \frac{\partial u(x)}{\partial x_k} + a_0(x) u(x)$$

$$+ \sum_{m=0}^{2} \sum_{n=1}^{2} \int_{a_1}^{b_1} K_{2mn}(x, \eta_1) \left. \frac{\partial^2 u(\eta)}{\partial \eta_1^m \partial \eta_2^{2-m}} \right|_{\eta_2 = \gamma_n(\eta_1)} d\eta_1$$

$$+ \sum_{m=1}^{2} \sum_{n=1}^{2} \int_{a_1}^{b_1} K_{1mn}(x, \eta_1) \left. \frac{\partial u(\eta)}{\partial \eta_m} \right|_{\eta_2 = \gamma_n(\eta_1)} d\eta_1 + \sum_{n=1}^{2} \int_{a_1}^{b_1} K_{0n}(x, \eta_1)\, u(\eta_1, \gamma_n(\eta_1))\, d\eta_1 . \tag{12}$$

Then applying the schemes used in [10–14], we obtain the remaining basic relations that give representations for the derivative of the unknown function and the boundary values of these derivatives

$$- \int_{\Gamma} \frac{\partial^2 u(x)}{\partial x_2^2} \frac{\partial U(x-\xi)}{\partial x_2} \cos(\nu, x_2)\, dx - \int_{\Gamma} \frac{\partial^2 u(x)}{\partial x_1 \partial x_2} \frac{\partial U(x-\xi)}{\partial x_2} \cos(\nu, x_1)\, dx$$

$$+ \int_{\Gamma} \frac{\partial u(x)}{\partial x_2} \frac{\partial^2 U(x-\xi)}{\partial x_2^2} \cos(\nu, x_2)\, dx + \int_{\Gamma} \frac{\partial u(x)}{\partial x_2} \frac{\partial^2 U(x-\xi)}{\partial x_1 \partial x_2} \cos(\nu, x_1)\, dx$$

$$- \int_{D} l_0 u \cdot \frac{\partial U(x-\xi)}{\partial x_2} dx + \int_{D} f(x) \frac{\partial U(x-\xi)}{\partial x_2} dx = \begin{cases} \frac{\partial u(\xi)}{\partial \xi_2}, & \xi \in D, \\ \frac{1}{2} \frac{\partial u(\xi)}{\partial \xi_2}, & \xi \in \Gamma, \end{cases} \tag{13}$$

$$-\int_{\Gamma}\frac{\partial^2 u(x)}{\partial x_2^2}\frac{\partial U(x-\xi)}{\partial x_1}\cos(\nu,x_2)\,dx-\int_{\Gamma}\frac{\partial^2 u(x)}{\partial x_1^2}\frac{\partial U(x-\xi)}{\partial x_1}\cos(\nu,x_2)\,dx$$
$$+\int_{\Gamma}\frac{\partial u(x)}{\partial x_2}\frac{\partial^2 U(x-\xi)}{\partial x_1\partial x_2}\cos(\nu,x_2)\,dx+\int_{\Gamma}\frac{\partial u(x)}{\partial x_1}\frac{\partial^2 U(x-\xi)}{\partial x_1\partial x_2}\cos(\nu,x_1)\,dx$$
$$-\int_{\Gamma}\frac{\partial u(x)}{\partial x_2}\frac{\partial^2 U(x-\xi)}{\partial x_2^2}\cos(\nu,x_1)\,dx-\int_{\Gamma}\frac{\partial u(x)}{\partial x_1}\frac{\partial^2 U(x-\xi)}{\partial x_2^2}\cos(\nu,x_1)\,dx$$
$$-\int_{D}l_0u\cdot\frac{\partial U(x-\xi)}{\partial x_1}dx+\int_{D}f(x)\frac{\partial U(x-\xi)}{\partial x_1}dx=\begin{cases}\dfrac{\partial u(\xi)}{\partial \xi_1}, & \xi\in D,\\[2mm] \dfrac{1}{2}\dfrac{\partial u(\xi)}{\partial \xi_1}, & \xi\in\Gamma,\end{cases}\tag{14}$$

$$\int_{\Gamma}\frac{\partial^2 u(x)}{\partial x_2^2}\frac{\partial^2 U(x-\xi)}{\partial x_1^2}\cos(\nu,x_2)\,dx+\int_{\Gamma}\frac{\partial^2 u(x)}{\partial x_1^2}\frac{\partial^2 U(x-\xi)}{\partial x_1^2}\cos(\nu,x_2)\,dx$$
$$-\int_{\Gamma}\frac{\partial^2 u(x)}{\partial x_2^2}\frac{\partial^2 U(x-\xi)}{\partial x_1\partial x_2}\cos(\nu,x_1)\,dx+\int_{\Gamma}\frac{\partial^2 u(x)}{\partial x_1\partial x_2}\frac{\partial^2 U(x-\xi)}{\partial x_1\partial x_2}\cos(\nu,x_2)\,dx$$
$$-\int_{\Gamma}\frac{\partial^2 u(x)}{\partial x_1\partial x_2}\frac{\partial^2 U(x-\xi)}{\partial x_2^2}\cos(\nu,x_1)\,dx+\int_{\Gamma}\frac{\partial^2 u(x)}{\partial x_1^2}\frac{\partial^2 U(x-\xi)}{\partial x_2^2}\cos(\nu,x_2)\,dx$$
$$+\int_{D}l_0u\cdot\frac{\partial^2 U(x-\xi)}{\partial x_1^2}dx-\int_{D}f(x)\frac{\partial^2 U(x-\xi)}{\partial x_1^2}dx=\begin{cases}\dfrac{\partial^2 u(\xi)}{\partial \xi_1^2}, & \xi\in D,\\[2mm] \dfrac{1}{2}\dfrac{\partial^2 u(\xi)}{\partial \xi_1^2}, & \xi\in\Gamma.\end{cases}\tag{15}$$

Notice that in the remaining two expressions (see also [9–11]) the derivatives higher than third order in the domain D (both for $u(x)$ and for $U(x-\xi)$) and the derivatives higher than second order on the boundary Γ do not appear in the integrand, i.e.,

$$\int_{\Gamma}\frac{\partial^2 u(x)}{\partial x_2^2}\frac{\partial U^2(x-\xi)}{\partial x_2^2}\cos(\nu,x_2)\,dx+\int_{\Gamma}\frac{\partial^2 u(x)}{\partial x_1\partial x_2}\frac{\partial^2 U(x-\xi)}{\partial x_2^2}\cos(\nu,x_1)\,dx$$
$$-\int_{\Gamma}\frac{\partial^2 u(x)}{\partial x_1\partial x_2}\frac{\partial^2 U(x-\xi)}{\partial x_1\partial x_2}\cos(\nu,x_2)\,dx+\int_{\Gamma}\frac{\partial^2 u(x)}{\partial x_2^2}\frac{\partial^2 U(x-\xi)}{\partial x_1\partial x_2}\cos(\nu,x_1)\,dx$$
$$+\int_{D}l_0u\cdot\frac{\partial^2 U(x-\xi)}{\partial x_2^2}dx-\int_{D}f(x)\frac{\partial^2 U(x-\xi)}{\partial x_2^2}dx=\begin{cases}\dfrac{\partial^2 u(\xi)}{\partial \xi_2^2}, & \xi\in D,\\[2mm] \dfrac{1}{2}\dfrac{\partial^2 u(\xi)}{\partial \xi_2^2}, & \xi\in\Gamma,\end{cases}\tag{16}$$

$$\int_{\Gamma} \frac{\partial^2 u(x)}{\partial x_2^2} \frac{\partial^2 U(x-\xi)}{\partial x_1 \partial x_2} \cos(\nu, x_2)\, dx + \int_{\Gamma} \frac{\partial^2 u(x)}{\partial x_1 \partial x_2} \frac{\partial^2 U(x-\xi)}{\partial x_1 \partial x_2} \cos(\nu, x_1)\, dx$$
$$-\int_{\Gamma} \frac{\partial^2 u(x)}{\partial x_2^2} \frac{\partial^2 U(x-\xi)}{\partial x_2^2} \cos(\nu, x_1)\, dx + \int_{\Gamma} \frac{\partial^2 u(x)}{\partial x_1 \partial x_2} \frac{\partial^2 U(x-\xi)}{\partial x_2^2} \cos(\nu, x_2)\, dx$$
$$+\int_{D} l_0 u \cdot \frac{\partial^2 U(x-\xi)}{\partial x_1 \partial x_2} dx - \int_{D} f(x) \frac{\partial^2 U(x-\xi)}{\partial x_1 \partial x_2} dx = \begin{cases} \dfrac{\partial^2 u(\xi)}{\partial \xi_1 \partial \xi_2}, & \xi \in D, \\ \dfrac{1}{2} \dfrac{\partial^2 u(\xi)}{\partial \xi_1 \partial \xi_2}, & \xi \in \Gamma. \end{cases} \tag{17}$$

Thus we establish the following

Theorem 1. *If $D \subset \mathbb{R}^2$ is a bounded convex domain with Lyapunov boundary Γ, the data of equation* (1)*, $a_{2k}(x)$, $k = \overline{0,2}$, $x \in D$; $a_{1k}(x)$, $k = 1,2$, $x \in D$; $a_0(x)$, $x \in D$; $K_{2mn}(x, \eta_1)$, $m = \overline{0,2}$, $n = \overline{1,2}$, $x \in D$, $\eta_1 \in (a_1, b_1)$; $K_{1mn}(x, \eta_1)$, $m = \overline{1,2}$, $n = \overline{1,2}$, $x \in D$, $\eta_1 \in (a_1, b_1)$; $K_{on}(x, \eta_1)$, $x \in D$, $\eta_1 \in (a_1, b_1)$, $f(x)$, $x \in D$ are continuous functions, then every solution of equation* (1) *determined in the domain D satisfies basic relations* (11), (13)–(17).

5. Necessary conditions

Considering the second expressions of basic relations (11), (13)–(17) and passing from the integrals over the boundary Γ to those over its parts $\Gamma_k\,(k = 1,2)$, one obtains

$$\begin{aligned}
&u(\xi_1, \gamma_k(\xi_1)) = 2\int_{\Gamma} \tfrac{\partial^2 u(x)}{\partial x_2^2} U(x_1-\xi_1, x_2-\gamma_k(\xi_1)) \cos(\nu, x_2) dx \\
&+2\int_{\Gamma} \tfrac{\partial^2 u(x)}{\partial x_1^2} U(x_1-\xi_1, x_2-\gamma_k(\xi_1)) \cos(\nu, x_2) dx \\
&-2\int_{\Gamma} \tfrac{\partial u(x)}{\partial x_2} \tfrac{\partial U(x-\xi)}{\partial x_2}\Big|_{\xi_2=\gamma_k(\xi_1)} \cos(\nu, x_2) dx - 2\int_{\Gamma} \tfrac{\partial u(x)}{\partial x_1} \tfrac{\partial U(x-\xi)}{\partial x_2}\Big|_{\xi_2=\gamma_k(\xi_1)} \cos(\nu, x_1) dx \\
&+2\int_{\Gamma} u(x) \tfrac{\partial^2 u(x-\xi)}{\partial x_1 \partial x_2}\Big|_{\xi_2=\gamma_k(\xi_1)} \cos(\nu, x_1) dx + 2\int_{\Gamma} u(x) \tfrac{\partial^2 U(x-\xi)}{\partial x_2^2}\Big|_{\xi_2=\gamma_k(\xi_1)} \cos(\nu, x_2) dx \\
&+2\int_{D} l_0 u U(x_1-\xi_1, x_2-\gamma_k(\xi_1)) dx \\
&-2\int_{D} f(x) U(x_1-\xi_1, x_2-\gamma_k(\xi_1)) dx, k = 1,2, \xi_1 \in [a_1, b_1],
\end{aligned} \tag{18}$$

$$\frac{\partial u(\xi)}{\partial \xi_1}\bigg|_{\xi_2=\gamma_k(\xi_1)} = -2\int\limits_{\Gamma} \frac{\partial^2 u(x)}{\partial x_2^2}\frac{\partial U(x-\xi)}{\partial x_1}\bigg|_{\xi_2=\gamma_k(\xi_1)} \cos(\nu, x_2)dx$$
$$-2\int\limits_{\Gamma} \frac{\partial^2 u(x)}{\partial x_1^2}\frac{\partial U(x-\xi)}{\partial x_1}\bigg|_{\xi_2=\gamma_k(\xi_1)} \cos(\nu, x_2)dx + 2\int\limits_{\Gamma} \frac{\partial u(x)}{\partial x_2}\frac{\partial^2 U(x-\xi)}{\partial x_1 \partial x_2}\bigg|_{\xi_2=\gamma_k(\xi_1)} \cos(\nu, x_2)dx$$
$$+2\int\limits_{\Gamma} \frac{\partial u(x)}{\partial x_1}\frac{\partial^2 U(x-\xi)}{\partial x_1 \partial x_2}\bigg|_{\xi_2=\gamma_k(\xi_1)} \cos(\nu, x_1)dx - 2\int\limits_{\Gamma} \frac{\partial u(x)}{\partial x_2}\frac{\partial^2 U(x-\xi)}{\partial x_2^2}\bigg|_{\xi_2=\gamma_k(\xi_1)} \cos(\nu, x_1)dx$$
$$-2\int\limits_{\Gamma} \frac{\partial u(x)}{\partial x_1}\frac{\partial^2 U(x-\xi)}{\partial x_2^2}\bigg|_{\xi_2=\gamma_k(\xi_1)} \cos(\nu, x_1)dx - 2\int\limits_{D} l_0 u \frac{\partial U(x-\xi)}{\partial x_1}dx$$
$$+2\int\limits_{D} f(x)\frac{\partial U(x-\xi)}{\partial x_1}dx, \quad k=1,2, \xi_1 \in [a_1, b_1] \tag{19}$$

and

$$\frac{\partial u(\xi)}{\partial \xi_2}\bigg|_{\xi_2=\gamma_k(\xi_1)} = -2\int\limits_{\Gamma} \frac{\partial^2 u(x)}{\partial x_2^2}\frac{\partial U(x-\xi)}{\partial x_2}\bigg|_{\xi_2=\gamma_k(\xi_1)} \cos(\nu, x_2)dx$$
$$-2\int\limits_{\Gamma} \frac{\partial^2 u(x)}{\partial x_1 \partial x_2}\frac{\partial U(x-\xi)}{\partial x_2}\bigg|_{\xi_2=\gamma_k(\xi_1)} \cos(\nu, x_1)dx + \int\limits_{\Gamma} \frac{\partial u(x)}{\partial x_2}\frac{\partial U^2(x-\xi)}{\partial x_2^2}\bigg|_{\xi_2=\gamma_k(\xi_1)} \cos(\nu, x_2)dx$$
$$+2\int\limits_{\Gamma} \frac{\partial u(x)}{\partial x_2}\frac{\partial U^2(x-\xi)}{\partial x_1 \partial x_2}\bigg|_{\xi_2=\gamma_k(\xi_1)} \cos(\nu, x_1)dx - 2\int\limits_{D} l_0 u \frac{\partial U(x-\xi)}{\partial x_2}\bigg|_{\xi_2=\gamma_k(\xi_1)} dx$$
$$+2\int\limits_{D} f(x)\frac{\partial U(x-\xi)}{\partial x_2}\bigg|_{\xi_2=\gamma_k(\xi_1)} dx, \quad k=1,2, \xi_1 \in [a_1, b_1], \tag{20}$$

where $\xi_2 = \gamma_k(\xi_1)$, $k = 1, 2$, are the equations of Γ_k, and the "dots" denote the sums of nonsingular terms.

As it is seen from the fundamental solution (5), for the boundary values of the second derivative we have

$$\frac{\partial^2 U(x-\xi)}{\partial x_2^2}\bigg|_{\substack{x_2=\gamma_p(x_1)\\ \xi_2=\gamma_p(\xi_1)}} = \frac{1}{2\pi}\frac{\gamma'_p(\sigma_p(x_1,\xi_1))}{(x_1-\xi_1)\left[1+\gamma'^2_p(\sigma_p)\right]}, \quad p=1,2; \tag{21}$$

$$\frac{\partial^2 U(x-\xi)}{\partial x_1 \partial x_2}\bigg|_{\substack{x_2=\gamma_p(x_1)\\ \xi_2=\gamma_p(\xi_1)}} = \frac{1}{2\pi}\frac{1}{(x_1-\xi_1)\left[1+\gamma'^2_p(\sigma_p)\right]}, \quad p=1,2; \tag{22}$$

$$\frac{\partial^2 U(x-\xi)}{\partial x_1^2}\bigg|_{\substack{x_2=\gamma_p(x_1)\\ \xi_2=\gamma_p(\xi_1)}} = -\frac{1}{2\pi}\frac{\gamma'_p(\sigma_p)}{(x_1-\xi_1)\left[1+\gamma'^2_p(\sigma_p)\right]}, \quad p=1,2; \tag{23}$$

$$\frac{\partial^2 U(x-\xi)}{\partial x_1^2}\bigg|_{\substack{x_2=\gamma_p(x_1)\\ \xi_2=\gamma_q(\xi_1)}} = \delta(x_1-\xi_1)\, e\left(\gamma_p(x_1)-\gamma_q(\xi_1)\right), \quad p,q=1,2; \quad p \neq q, \tag{24}$$

where $\sigma_p(x_1, \xi_1)$ is located between x_1 and ξ_1.

$$
\begin{aligned}
&\left.\frac{\partial u^2(\xi)}{\partial \xi_2^2}\right|_{\xi_2=\gamma_k(\xi_1)} = -2\int_{a_1}^{b_1} \left.\frac{\partial^2 u(x)}{\partial x_2^2}\right|_{x_2=\gamma_1(x_1)} \left.\frac{\partial^2 U(x-\xi)}{\partial x_2^2}\right|_{\substack{x_2=\gamma_1(x_1)\\ \xi_2=\gamma_k(\xi_1)}} dx_1 \\
&+2\int_{a_1}^{b_1} \left.\frac{\partial^2 u(x)}{\partial x_2^2}\right|_{x_2=\gamma_2(x_1)} \left.\frac{\partial^2 U(x-\xi)}{\partial x_2^2}\right|_{\substack{x_2=\gamma_2(x_1)\\ \xi_2=\gamma_k(\xi_1)}} dx_1 \\
&+2\int_{a_1}^{b_1} \left.\frac{\partial^2 u(x)}{\partial x_1 \partial x_2}\right|_{x_2=\gamma_1(x_1)} \left.\frac{\partial^2 U(x-\xi)}{\partial x_2^2}\right|_{\substack{x_2=\gamma_1(x_1)\\ \xi_2=\gamma_k(\xi_1)}} \gamma_1'(x_1)dx_1 \\
&-2\int_{a_1}^{b_1} \left.\frac{\partial^2 u(x)}{\partial x_1 \partial x_2}\right|_{x_2=\gamma_2(x_1)} \left.\frac{\partial^2 U(x-\xi)}{\partial x_2^2}\right|_{\substack{x_2=\gamma_2(x_1)\\ \xi_2=\gamma_k(\xi_1)}} \gamma_2'(x_1)dx_1 \\
&+2\int_{a_1}^{b_1} \left.\frac{\partial^2 u(x)}{\partial x_1 \partial x_2}\right|_{x_2=\gamma_1(x_1)} \left.\frac{\partial^2 U(x-\xi)}{\partial x_1 \partial x_2}\right|_{\substack{x_2=\gamma_1(x_1)\\ \xi_2=\gamma_k(\xi_1)}} dx_1 \\
&-2\int_{a_1}^{b_1} \left.\frac{\partial^2 u(x)}{\partial x_1 \partial x_2}\right|_{x_2=\gamma_2(x_1)} \left.\frac{\partial^2 U(x-\xi)}{\partial x_1 \partial x_2}\right|_{\substack{x_2=\gamma_2(x_1)\\ \xi_2=\gamma_k(\xi_1)}} dx_1 \\
&+2\int_{a_1}^{b_1} \left.\frac{\partial^2 u(x)}{\partial x_2^2}\right|_{x_2=\gamma_1(x_1)} \left.\frac{\partial^2 U(x-\xi)}{\partial x_1 \partial x_2}\right|_{\substack{x_2=\gamma_1(x_1)\\ \xi_2=\gamma_k(\xi_1)}} \gamma_1'(x_1)dx_1 \\
&-2\int_{a_1}^{b_1} \left.\frac{\partial^2 u(x)}{\partial x_2^2}\right|_{x_2=\gamma_2(x_1)} \left.\frac{\partial^2 U(x-\xi)}{\partial x_1 \partial x_2}\right|_{\substack{x_2=\gamma_2(x_1)\\ \xi_2=\gamma_k(\xi_1)}} \gamma_2'(x_1)dx_1 \\
&+2\int_D l_0 u \frac{\partial^2 U(x-\xi)}{\partial x_2^2} dx + 2\int_D f(x)\frac{\partial^2 U(x-\xi)}{\partial x_2^2} dx \\
&= \frac{(-1)^{k-1}}{\pi}\int_{a_1}^{b_1} \left.\frac{\partial^2 u(x)}{\partial x_1 \partial x_2}\right|_{x_2=\gamma_k(x_1)} \frac{dx_1}{x_1-\xi_1} + \cdots, \quad k=1,2.
\end{aligned} \tag{25}
$$

$$
\begin{aligned}
&\left.\frac{\partial^2 u(\xi)}{\partial \xi_1^2}\right|_{\xi_2=\gamma_k(\xi_1)} = -2\int_{a_1}^{b_1} \left.\frac{\partial^2 u(x)}{\partial x_2^2}\right|_{x_2=\gamma_1(x_1)} \left.\frac{\partial^2 U(x-\xi)}{\partial x_1^2}\right|_{\substack{x_2=\gamma_1(x_1)\\ \xi_2=\gamma_k(\xi_1)}} dx_1 \\
&+2\int_{a_1}^{b_1} \left.\frac{\partial^2 u(x)}{\partial x_2^2}\right|_{x_2=\gamma_2(x_1)} \left.\frac{\partial^2 U(x-\xi)}{\partial x_1^2}\right|_{\substack{x_2=\gamma_2(x_1)\\ \xi_2=\gamma_k(\xi_1)}} dx_1 \\
&-2\int_{a_1}^{b_1} \left.\frac{\partial^2 u(x)}{\partial x_1^2}\right|_{x_2=\gamma_1(x_1)} \left.\frac{\partial^2 U(x-\xi)}{\partial x_1^2}\right|_{\substack{x_2=\gamma_1(x_1)\\ \xi_2=\gamma_k(\xi_1)}} dx_1 \\
&+2\int_{a_1}^{b_1} \left.\frac{\partial^2 u(x)}{\partial x_1^2}\right|_{x_2=\gamma_2(x_1)} \left.\frac{\partial^2 U(x-\xi)}{\partial x_1^2}\right|_{\substack{x_2=\gamma_2(x_1)\\ \xi_2=\gamma_k(\xi_1)}} dx_1 \\
&-2\int_{a_1}^{b_1} \left.\frac{\partial^2 u(x)}{\partial x_2^2}\right|_{x_2=\gamma_1(x_1)} \left.\frac{\partial^2 U(x-\xi)}{\partial x_1 \partial x_2}\right|_{\substack{x_2=\gamma_1(x_1)\\ \xi_2=\gamma_k(\xi_1)}} \gamma_1'(x_1)dx_1 \\
&+2\int_{a_1}^{b_1} \left.\frac{\partial^2 u(x)}{\partial x_2^2}\right|_{x_2=\gamma_2(x_1)} \left.\frac{\partial^2 U(x-\xi)}{\partial x_1 \partial x_2}\right|_{\substack{x_2=\gamma_2(x_1)\\ \xi_2=\gamma_k(\xi_1)}} \gamma_2'(x_1)dx_1 \\
&-2\int_{a_1}^{b_1} \left.\frac{\partial^2 u(x)}{\partial x_1 \partial x_2}\right|_{x_2=\gamma_1(x_1)} \left.\frac{\partial^2 U(x-\xi)}{\partial x_1 \partial x_2}\right|_{\substack{x_2=\gamma_1(x_1)\\ \xi_2=\gamma_k(\xi_1)}} dx_1
\end{aligned}
$$

$$
\begin{aligned}
&+2\int\limits_{a_1}^{b_1} \frac{\partial^2 u(x)}{\partial x_1 \partial x_2}\Big|_{x_2=\gamma_2(x_1)} \frac{\partial^2 U(x-\xi)}{\partial x_1 \partial x_2}\Big|_{\substack{x_2=\gamma_2(x_1)\\ \xi_2=\gamma_k(\xi_1)}} dx_1\\
&-2\int\limits_{a_1}^{b_1} \frac{\partial^2 u(x)}{\partial x_1 \partial x_2}\Big|_{x_2=\gamma_1(x_1)} \frac{\partial^2 U(x-\xi)}{\partial x_1 \partial x_2}\Big|_{\substack{x_2=\gamma_1(x_1)\\ \xi_2=\gamma_k(\xi_1)}} \gamma_1'(x_1)dx_1\\
&+2\int\limits_{a_1}^{b_1} \frac{\partial^2 u(x)}{\partial x_1 \partial x_2}\Big|_{x_2=\gamma_2(x_1)} \frac{\partial^2 U(x-\xi)}{\partial x_2^2}\Big|_{\substack{x_2=\gamma_2(x_1)\\ \xi_2=\gamma_k(\xi_1)}} \gamma_2'(x_1)dx_1\\
&-2\int\limits_{a_1}^{b_1} \frac{\partial^2 u(x)}{\partial x_1^2}\Big|_{x_2=\gamma_1(x_1)} \frac{\partial^2 U(x-\xi)}{\partial x_2^2}\Big|_{\substack{x_2=\gamma_1(x_1)\\ \xi_2=\gamma_k(\xi_1)}} dx_1\\
&+2\int\limits_{a_1}^{b_1} \frac{\partial^2 u(x)}{\partial x_1^2}\Big|_{x_2=\gamma_2(x_1)} \frac{\partial^2 U(x-\xi)}{\partial x_2^2}\Big|_{\substack{x_2=\gamma_2(x_1)\\ \xi_2=\gamma_k(\xi_1)}} dx_1\\
&+2\int\limits_D l_0 u \frac{\partial^2 U(x-\xi)}{\partial x_1^2}dx - 2\int\limits_D f(x)\frac{\partial^2 U(x-\xi)}{\partial x_1^2}dx, \quad k=1,2, \xi_1\in[a_1,b_1].
\end{aligned} \tag{26}
$$

Then the remaining necessary conditions take the form

$$
\begin{aligned}
&\frac{\partial^2 u(\xi)}{\partial \xi_1^2}\Big|_{\xi_2=\gamma_1(\xi_1)} - \frac{\partial^2 u(\xi)}{\partial \xi_1^2}\Big|_{\xi_2=\gamma_2(\xi_1)} - \frac{\partial^2 u(\xi)}{\partial \xi_2^2}\Big|_{\xi_2=\gamma_2(\xi_1)}\\
&= -\frac{1}{\pi}\int_{a_1}^{b_1} \frac{\partial^2 u(x)}{\partial x_1 \partial x_2}\Big|_{x_2=\gamma_1(x_1)} \frac{dx_1}{x_1-\xi_1} + \dots
\end{aligned} \tag{27}
$$

$$
\begin{aligned}
&\frac{\partial^2 u(\xi)}{\partial \xi_1^2}\Big|_{\xi_2=\gamma_2(\xi_1)} - \frac{\partial^2 u(\xi)}{\partial \xi_1^2}\Big|_{\xi_2=\gamma_1(\xi_1)} - \frac{\partial^2 u(\xi)}{\partial \xi_2^2}\Big|_{\xi_2=\gamma_1(\xi_1)}\\
&= \frac{1}{\pi}\int_{a_1}^{b_1} \frac{\partial^2 u(x)}{\partial x_1 \partial x_2}\Big|_{x_2=\gamma_2(x_1)} \frac{dx_1}{x_1-\xi_1} + \dots
\end{aligned} \tag{28}
$$

$$
\begin{aligned}
\frac{\partial^2 u(\xi)}{\partial \xi_1 \partial \xi_2}\Big|_{\xi_2=\gamma_k(\xi_1)} &= -2\int\limits_{a_1}^{b_1} \frac{\partial^2 u(x)}{\partial x_2^2}\Big|_{x_2=\gamma_1(x_1)} \frac{\partial^2 U(x-\xi)}{\partial x_1 \partial x_2}\Big|_{\substack{x_2=\gamma_1(x_1)\\ \xi_2=\gamma_k(\xi_1)}} dx_1\\
&+2\int\limits_{a_1}^{b_1} \frac{\partial^2 u(x)}{\partial x_2^2}\Big|_{x_2=\gamma_2(x_1)} \frac{\partial^2 U(x-\xi)}{\partial x_1 \partial x_2}\Big|_{\substack{x_2=\gamma_2(x_1)\\ \xi_2=\gamma_k(\xi_1)}} dx_1\\
&+2\int\limits_{a_1}^{b_1} \frac{\partial^2 u(x)}{\partial x_1 \partial x_2}\Big|_{x_2=\gamma_1(x_1)} \frac{\partial^2 U(x-\xi)}{\partial x_1 \partial x_2}\Big|_{\substack{x_2=\gamma_1(x_1)\\ \xi_2=\gamma_k(\xi_1)}} \gamma_1'(x_1)dx_1\\
&-2\int\limits_{a_1}^{b_1} \frac{\partial^2 u(x)}{\partial x_1 \partial x_2}\Big|_{x_2=\gamma_2(x_1)} \frac{\partial^2 U(x-\xi)}{\partial x_1 \partial x_2}\Big|_{\substack{x_2=\gamma_2(x_1)\\ \xi_2=\gamma_k(\xi_1)}} \gamma_2'(x_1)dx_1\\
&-2\int\limits_{a_1}^{b_1} \frac{\partial^2 u(x)}{\partial x_2^2}\Big|_{x_2=\gamma_1(x_1)} \frac{\partial^2 U(x-\xi)}{\partial x_2^2}\Big|_{\substack{x_2=\gamma_1(x_1)\\ \xi_2=\gamma_k(\xi_1)}} \gamma_1'(x_1)dx_1
\end{aligned}
$$

$$
\begin{aligned}
&+2\int\limits_{a_1}^{b_1} \frac{\partial^2 u(x)}{\partial x_2^2}\Big|_{x_2=\gamma_2(x_1)} \frac{\partial^2 U(x-\xi)}{\partial x_2^2}\Big|_{\substack{x_2=\gamma_2(x_1)\\ \xi_2=\gamma_k(\xi_1)}} \gamma_2'(x_1)dx_1\\
&-2\int\limits_{a_1}^{b_1} \frac{\partial^2 u(x)}{\partial x_1\partial x_2}\Big|_{x_2=\gamma_1(x_1)} \frac{\partial^2 U(x-\xi)}{\partial x_2^2}\Big|_{\substack{x_2=\gamma_1(x_1)\\ \xi_2=\gamma_k(\xi_1)}} dx_1\\
&+2\int\limits_{a_1}^{b_1} \frac{\partial^2 u(x)}{\partial x_1\partial x_2}\Big|_{x_2=\gamma_2(x_1)} \frac{\partial^2 U(x-\xi)}{\partial x_2^2}\Big|_{\substack{x_2=\gamma_2(x_1)\\ \xi_2=\gamma_k(\xi_1)}} dx_1\\
&+2\int\limits_D l_0 u\frac{\partial^2 U(x-\xi)}{\partial x_1\partial x_2}dx - 2\int\limits_D f(x)\frac{\partial^2 U(x-\xi)}{\partial x_1\partial x_2}dx\\
&= \frac{(-1)^k}{\pi}\int\limits_{a_1}^{b_1} \frac{\partial^2 u(x)}{\partial x_2^2}\Big|_{x_2=\gamma_k(x_1)} \frac{dx_1}{x_1-\xi_1} + \cdots, \quad k=1,2, \xi_1\in[a_1,b_1].
\end{aligned} \tag{29}
$$

Thus we obtained the following statements.

Theorem 2. *Under the conditions of Theorem* 1 *every solution of equation* (1) *satisfies the necessary regular conditions* (18)–(20).

Theorem 3. *Under the conditions of Theorem 1 every solution of equation* (1) *satisfies the necessary singular conditions* (25)–(29).

6. Fredholm Property

Considering the necessary singular conditions (29) and taking into account the boundary conditions (2), we get

$$
\begin{aligned}
\frac{\partial^2 u(\xi)}{\partial \xi_1 \partial \xi_2}\Big|_{\xi_2=\gamma_k(\xi_1)} &= \frac{(-1)^k}{\pi}\int_{a_1}^{b_1} \frac{dx_1}{x_1-\xi_1}\Bigg\{ f_k(x_1) + \sum_{p=1}^{2}\sum_{j=1}^{2} \alpha_{kjp}(x_1)\frac{\partial u(x)}{\partial x_j}\Big|_{x_2=\gamma_p(x_1)}\\
&+\sum_{p=1}^{2}\alpha_{kp}(x_1)\,u(x_1,\gamma_p(x_1)) + \sum_{p=1}^{2}\sum_{j=1}^{2}\int_{a_1}^{b_1}\alpha_{kjp}(x_1,\eta_1)\frac{\partial u(\eta)}{\partial \eta_j}\Big|_{\eta_2=\gamma_p(\eta_1)}\\
&+\sum_{p=1}^{2}\int_{a_1}^{b_1}\alpha_{kp}(x_1,\eta_1)\,u(\eta_1,\gamma_p(\eta_1))\,d\eta_1\Bigg\}.
\end{aligned} \tag{30}
$$

The first term in the right-hand side is easily regularized if

$$
f_k(x) \in C^{(1)}[a_1,b_1], \quad f_k(a_1)=f_k(b_1)=0, \quad k=1,2, \tag{31}
$$

as shown in [15].

Concerning the second and the third terms in the right-hand side of (30), they are regularized by using (18)–(20).

After substituting (18)–(20) into (30), it suffices to replace the regular integrals in (18)–(20) by the singular integrals from (30). For the last two terms in the right-hand side of (30) it suffices to replace the integrals contained in it.

Finally, consider the necessary condition (27). Its right-hand side contains singular integrals, and we substitute $\left.\frac{\partial^2 u(x)}{\partial x_1 \partial x_2}\right|_{x_2=\gamma_1(x_1)}$ by its regular expression obtained by means of (30). Then in the left-hand side of expression (27), instead of $\left.\frac{\partial^2 u(\xi)}{\partial \xi_1^2}\right|_{\xi_2=\gamma_2(\xi_1)}$ and $\left.\frac{\partial^2 u(\xi)}{\partial \xi_2^2}\right|_{\xi_2=\gamma_2(\xi_1)}$ we use their expressions from the boundary conditions (2) and (3). Thus we get the regular relation for $\left.\frac{\partial^2 u(\xi)}{\partial \xi_1^2}\right|_{\xi_2=\gamma_1(\xi_1)}$ as well.

Thus we proved.

Theorem 4. *Under the conditions of Theorem* 1, (31) *and if* $\alpha_{kjp}(x_1)$, $k=\overline{1,3}$, $j=1,2$, $p=1,2$; $\alpha_{kp}(x_1)$, $k=\overline{1,3}$, $p=1,2$, $x_1\in[a_1,b_1]$; $\alpha_{kjp}(x_1,\eta_1)$, $k=\overline{1,3}$, $j=1,2$, $p=1,2$; $x_1\in[a_1,b_1]$, $\eta_1\in[a_1,b_1]$; $\alpha_{kp}(x_1,\eta_1)$, $k=\overline{1,3}$, $p=1,2$; $x_1\in[a_1,b_1]$, $\eta_1\in[a_1,b_1]$ *and* $f_3(x_1)$, $x_1\in[a_1,b_1]$ *are continuous functions, then for the boundary values of* $u(x)$ *and its derivatives up to the second order we get a normal system of the second order integral equations whose Fredholm kernel formulas do not contain singularities (i.e., the singularity in the trace formula is weak).*

If all boundary values up to the second order are determined by the above mentioned system of integral equations, then after substitution these boundary values into the left-hand side of (11), (13)–(17), we obtain for the unknown function $u(x)$ and its derivatives up to the second order, inclusively for $\xi\in D$, the system of the Fredholm integral equations of the second kind with regular kernels.

Theorem is proved.

Theorem 5. *Under the conditions of Theorem* 4*, the stated boundary value problem is reduced to the system of the Fredholm integral equations of the second kind whose Fredholm kernel does not contain singular integrals.*

A boundary value problem for the second order equation of composite type was studied in [16–21].

Various special cases of boundary value problems for the composite type equations of third order were considered in [22–24].

7. Example

Consider the domain D in the upper half-plane enclosed by the abscissa axis $(x_2 = 0)$ and the parabola $(x_2 = 1 - x_1^2)$,

$$D = \left\{ x = (x_1, x_2),\ x_2 \in (0, 1 - x_1^2),\ x_1 \in (-1, 1) \right\}.$$

Consider the boundary value problem with the solution $u(x) = x_1(x_2^2 - x_1^2)$

$$\frac{\partial^3 u(x)}{\partial x_2^3} + \frac{\partial^3 u(x)}{\partial x_1^2 \partial x_2} = 0, \quad x \in D, \tag{32}$$

under the boundary conditions

$$\left. \frac{\partial^2 u(x)}{\partial x_2^2} \right|_{x_2=0} = 2x_1, \tag{33}$$

$$\left. \frac{\partial^2 u(x)}{\partial x_2^2} \right|_{x_2=1-x_1^2} = 2x_1, \tag{34}$$

and

$$\left. \frac{\partial^2 u(x)}{\partial x_1^2} \right|_{x_2=1-x_1^2} = -6x_1. \tag{35}$$

Notice that for discretization we use the following pattern. If discretization is performed at the point m, n, then it is necessary to know the values of this function on the four layers

$$\begin{aligned} &(m, n-1); (m-1, n), (m, n), (m+1, n); \\ &(m-1, n+1), (m, n+1), (m+1, n+1); (m, n+2). \end{aligned} \tag{36}$$

Then equation (32) yields the following discrete equations at the point (m, n):

$$\begin{aligned} y_{m+1,n+1} - y_{m+1,n} + y_{m,n+2} - 5y_{m,n+1} + 5y_{m,n} \\ - y_{m,n-1} + y_{m-1,n+1} - y_{m-1,n} = 0, \end{aligned} \tag{37}$$

where, $m = \overline{-3, 3}$, $n = 1$; $m = \overline{-2, 2}$, $n = 2$.

Similarly, the equations obtained from the boundary condition (33) are in the form (notice that here $h = \frac{1}{4}$, $x_1 = \frac{1}{4}m$)

$$y_{m,n+1} - 2y_{m,n} + y_{m,n-1} = \frac{1}{32}m, \tag{38}$$

where, $m = \overline{-4, 4}$, $n = 1$.

Equations (38) are obtained from the boundary condition (34) with $m = -3, n = 2; m = \overline{-2, 2}, n = 3; m = 3, n = 2$.

Finally, the algebraic equations are obtained from the boundary condition (35)

$$y_{m+1,n} - 2y_{m,n} + y_{m-1,n} = -\frac{3}{32}m. \qquad (39)$$

Here

$$m = -3, n = \overline{0,2}; m = 3, n = \overline{0,2}; m = \overline{-1,1}, n = 4; m = -2, 2, n = 3$$

are obtained. So, we get a system of 39 variables and 39 linear algebraic equations. Since, $u(x) = x_1(x_2^2 - x_1^2)$, then,

$$y_{m,n} = y(mh, nh) = mh(n^2h^2 - m^2h^2) = mh^3(n^2 - m^2) = \frac{m}{64}(n^2 - m^2). \qquad (40)$$

In the considered example we give the exact values at the knot points of the domain, the approximate values of the solution of the system of linear algebraic equations, and finally their absolute error in the table below.

It should be noted that here $e = |y(\text{Exact}) - y(\text{App})|$.

Table

$y_{m,n}$	Exact	App	Err
$y_{-4,0}$	1	0.1107	0.8893
$y_{-3,0}$	0.4218	0.1141	0.3077
$y_{-2,0}$	0.125	-0.0076	0.1326
$y_{-1,0}$	0.0126	-0.1461	0.1617
$y_{0,0}$	0	-0.0511	0.0511
$y_{1,0}$	-0.0156	-0.0497	0.0341
$y_{2,0}$	-0.125	-0.1422	0.0172
$y_{3,0}$	-0.4218	-0.1166	0.3052
$y_{4,0}$	-1	0.0072	1.0072
$y_{-4,1}$	0.9375	0.0684	0.8691
$y_{-3,1}$	0.375	0.0677	0.3073
$y_{-2,1}$	0.0937	0.0045	0.0892
$y_{-1,1}$	0	-0.0852	0.0852
$y_{0,1}$	0	-0.0271	0.0271
$y_{1,1}$	0	-0.0002	0.0002
$y_{2,1}$	-0.0937	-0.0023	0.9347
$y_{3,1}$	-0.375	-0.0357	0.3393
$y_{4,1}$	-0.9375	0.0399	0.9774
$y_{-4,2}$	0.75	0.0256	0.7244
$y_{-3,2}$	0.2343	0.0113	0.223
$y_{-2,2}$	0	0.0099	0.0099
$y_{-1,2}$	-0.0468	0.0084	0.0552
$y_{0,2}$	0	0.1650	0.1550
$y_{1,2}$	0.0468	0.0852	0.3828

$y_{2,2}$	0	0.0367	0.0367
$y_{3,2}$	-0.2343	0.0488	0.2831
$y_{4,2}$	-0.75	0.0922	0.8422
$y_{-3,3}$	0	0.1964	0.1964
$y_{-2,3}$	-0.1562	0.1192	0.2754
$y_{-1,3}$	-0.125	0.1045	0.2295
$y_{0,3}$	0	0.1597	0.1597
$y_{1,3}$	0.125	0.2773	0.1523
$y_{2,3}$	0.1562	-0.0343	0.1905
$y_{3,3}$	0	-0.0225	0.0225
$y_{-2,4}$	-0.375	0.0831	0.4581
$y_{1,4}$	-0.2343	0.2823	0.5176
$y_{0,4}$	0	-0.4307	0.4307
$y_{1,4}$	0.2343	-0.2748	0.5088
$y_{2,4}$	0.375	0.0061	0.3689

As it is seen from the table, for the given test the errors are as given in the table. It should also be noted that the maximal error is 1.0072 and the minimal error is 0.0002. Subsequently this problem will be reduced to the integral equation and solved approximately, and the error will be found. Finally, the errors obtained in this way will be compared.

References

[1] *M.A. Naimark*, Linear Differential Operators. Moscow, 1969. (Russian)

[2] *R. Courant and D. Hilbert*, Methods of Mathematical Physics. Partial Differential Equations, No. 2, Interscience, 1969.

[3] *V.S. Vladimirov*, Equations of Mathematical Physics. Mir Publishers, Moscow, 1984.

[4] *A.V. Bitsadze*, Equations of Mathematical Physics. Mir Publishers, Moscow, 1980.

[5] *S.G. Mikhlin*, Mathematical Physics. North-Holland, Amsterdam, 1980.

[6] *I.G. Petrovskii*, Lectures on Partial Differential Equations. Willy-Intersciense, New York, 1955.

[7] *A.V. Bitsadze*, To the problem on oblique derivative for harmonic function in three-dimensional domains. Materials of the Joint Soviet-American Symposium on Partial Equations. Siberian branch of AS of the USSR, Novosibirsk, 1963.

[8] *A.V. Bitsadze*, Some Classes of Partial Equations. Nauka, Moscow, 1981.

[9] *G.E. Shilov*, Mathematical Analysis. Second Special Course. Nauka, Moscow, 1969.

[10] *N.A. Aliev and M. Jahanshahi*, Solution of Poisson's Equation with Global, Local and Nonlocal Boundary Conditions. — *Int. J. Math. Educ. Sci. Technol.* **33** (2002), No. 2, 241–247.

[11] *N.A. Aliyev and S.M. Hosseini*, An Analysis of a Parabolic Problem with a General (Nonlocal and Global) Supplementary Linear Conditions-II. — *Italian J. Pure Appl. Math.* (2003), No. 13, 116–127.

[12] *A.M. Aliyev and N.A. Aliyev*, On Boundary Value Problem with Nonlocal Boundary Conditions. 37-th Annual Iranian Mathematics Conference, 2006, pp. 265–268.

[13] *A.V. Bitsadze and M.S. Salahaddinov*, About the Theory of Mixed and Composite Equation. — *SMJ* **2** (1961), No. 1, 7–19.

[14] *S.M. Hosseini and N.A. Aliyev*, Sufficient Conditions for the Reduction of a BVP for PDE with Nonlocal and Global Boundary Conditions to Fredholm Integral Equations (on a rectangular domain). — *Appl. Math. Comput.* **147** (2004), 669–686.

[15] *F.O. Gakhov*, Boundary Value Problems. Pergamon, 1966.

[16] *N.A. Aliyev and Mahammad Jahanshahi*, Sufficient Conditions for Reduction of the BVP Including a Mixed PDE with Nonlocal Boundary Conditions to Fredholm Integral Equations. — *Int. J. Math. Educ. Sci. Technol.* **28** (1997), No. 3, 419–425.

[17] *A.M. Aliyev and N.A. Aliyev*, On Fredholm Property of a Boundary Value Problem for Composite Type Equation. Contemporary Problems of Computational Mathematics and Mathematical Physics. To the 90-th anniversary of acad. A.A. Samarskii, Moscow, 2009, pp. 117–118. (Russian)

[18] *N. Aliyev, Sh. Rezapour, and M. Jahanshahi*, On a Mixed Problem for Navier–Stokes System in the Unit Cube. — *Math. Moravica* **13-1** (2009), 13–24.

[19] *A.Y. Delshad Gharehgheshlaghi*, Investigation of Boundary Value Problems for a Composite Type Integro-differential Equation with Non-local and Golobal Terms in the Boundary Conditions. — *Proc. NAS Azerbaijan* **31** (2009), 23–30.

[20] *M.R. Fatemi and N.A. Aliyev*, General Linear Boundary Value Problem for the Second-Order Integro-Differential Loaded Equation with Boundary Conditions Containing Both Nonlocal and Global Terms. — *Abstr. Appl. Anal.* Vol. 2010 (2010) Article ID 547526, 12 pp., doi:10.1155/2010/547526.

[21] *M. Jahanshahi, N.A. Aliyev, and S.M. Hosseini*, An Analytic Method for Investigation and Solving Two-Dimensional Steady State Navier–Stokes Equations (I). — *Southeast Asian Bulletin of Mathematics* **33** (2009), No. 6, 1075–1089.

[22] *A.M. Aliyev and A.A. Niftiyev*, The Inverse Boundary Problem Relative Domain for the Composition Type Equation and its Solving Algoritm. — *J. Math. Phys.* (2006), No. 6, 358.

[23] *F. Bahrami, N. Aliev, and S.M. Hosseini*, A Method for the Reduction of Four Dimensional Mixed Problems with General Boundary Conditions to a System of Second kind Fredholm Integral Equations. — *Italian J. Pure Appl. Math.* (2005), No. 17, 91–104.

[24] *A.Y. Delshad Gharehgheshlaghi and N.A. Aliyev*, A Problem for a Composite Type Equation of Third Order with General Linear Boundary Conditions. — *Transact. NAS Azerbaijan* **29** (2009), No. 4, 36–46.