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Andreev–Korkin Identity, Saigo Fractional Integration Operator and $\mathrm{Lip}_L(\alpha)$ Functions

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The Andreev–Korkin identity for the Chebyshev functional is treated by Hölder inequality, when the functional consists of $\mathrm{Lip}_L(\alpha)$ functions. The derived upper bound is applied to the so-called Chebyshev–Saigo functional, built by Saigo fractional integral operator — recently introduced by Saxena *et al.* (*R.K. Saxena, J. Ram, J. Daiya, and T.K. Pogány. — Integral Tranforms Spec. Funct.* **22** (2011), 671–680).

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1. Introduction

Let $w\colon [0,1] \mapsto \mathbb{R}_+$, $w \in L_1[0,1]$ be a normalized weight function, that is,

$$\int_0^1 w(x)dx = 1\,.$$

The *weighted Chebyshev functional* is defined by

$$\mathfrak{T}(w; f, g) := \mathfrak{M}(w; fg) - \mathfrak{M}(w; f)\mathfrak{M}(w; g)\,, \tag{1.1}$$

where $\mathfrak{M}(w; f)$ denotes the integral mean

$$\mathfrak{M}(w; f) = \int_0^1 w(x) f(x) dx \,. \tag{1.2}$$

We point out that using another support interval $\mathrm{supp}(f) = [a, b] \subset \mathbb{R}$, say, different from the unit one, we only achieve an artificial extension of (1.3), since obvious substitution

$$\frac{x-a}{b-a} \colon [a, b] \mapsto [0, 1]$$

leads us to (1.1).

Now let us recall in short the Andreev–Korkin identity for the weighted Chebyshev functional. In our setting this celebrated relation reads as follows:

$$\mathfrak{T}(w; f, g) = \frac{1}{2} \int_0^1 \int_0^1 w(x) w(y) (f(x) - f(y))(g(x) - g(y)) \, dxdy \,. \tag{1.3}$$

R e m a r k 1.1. According to [1, pp. 6–7] starting from the finite sums identity

$$\frac{1}{n} \sum_{j=1}^n x_j y_j = \Big(\frac{1}{n} \sum_{j=1}^n x_j \Big) \Big(\frac{1}{n} \sum_{j=1}^n y_j \Big) + \frac{1}{n^2} \sum_{1 \le i < j \le n} (x_i - x_j)(y_i - y_j) \,, \tag{1.4}$$

Korkin proved in 1882 Chebyshev's integral inequality [1, p. 2, Eq. (0.3)] in his letter to Bugaev [2] (in Russian) and presented the same procedure in the letter to Hermite [3] (in French), see also the most familiar source [4, pp. 242–243].

The identity analogous to Korkin's (1.4), where integrals replaced finite sums, was obtained in the next year by Andreev [5], another mathematician from the celebrated Kharkiv Mathematical Society.

It seems that A. Winckler (1884) and F. Franklin (1885) rediscovered independently Korkin's and Andreev's identities, respectively [1, p. 8], but we prefer to call (1.3) the Andreev–Korkin identity.

2. Andreev–Korkin Identity Built in Lipschitz Function Class

Given two metric spaces (Ξ, d) and (Υ, d), where $d(x, y) = |x - y|$, $x, y \in \Xi, \Upsilon \subseteq \mathbb{R}$. A function $f \colon \Xi \mapsto \Upsilon$ is said to be *uniform Lipschitz of order* α on Ξ if there exists an absolute constant $L > 0$ such that

$$|f(x) - f(y)| \le L|x - y|^\alpha \qquad 0 < \alpha \le 1, \; x, y \in \Xi \,. \tag{2.1}$$

Here L is the Lipschitz constant, and the class consisting of such functions we write $\mathrm{Lip}_L(\alpha)$.

Theorem 2.1. *Let r, s, $r^{-1}+s^{-1}=1$, $r>1$, be conjugated Hölder exponents. Assume that $f \in \mathrm{Lip}_{L_f}(\alpha_f)$, $f \in \mathrm{Lip}_{L_g}(\alpha_g)$. Then*

$$\big|\mathfrak{T}(w; f, g)\big| \le \frac{L_f L_g}{2} \min\big\{M_1, M_2\big\}, \tag{2.2}$$

where

$$M_1 = \left(\int_0^1\int_0^1 w(x)w(y)|x-y|^{\alpha_f r}dxdy\right)^{1/r}\left(\int_0^1\int_0^1 w(x)w(y)|x-y|^{\alpha_g s}dxdy\right)^{1/s}, \tag{2.3}$$

$$M_2 = \frac{1}{\big(\alpha_f r+1\big)^{1/r}(\alpha_g s+1)^{1/s}}\left(\int_0^1 x^{\alpha_f r+1}\big\{w^r(x)+w^r(1-x)\big\}dx\right)^{1/r}$$
$$\times\left(\int_0^1 x^{\alpha_g s+1}\big\{w^s(x)+w^s(1-x)\big\}dx\right)^{1/s}. \tag{2.4}$$

P r o o f. By the triangle inequality and since r, s, $r>1$ are conjugated, we conclude by virtue of the weighted Hölder inequality from the Andreev–Korkin identity the following estimates:

$$\begin{aligned}\big|\mathfrak{T}(w; f, g)\big| &\le \frac{1}{2}\int_0^1\int_0^1 \big\{w(x)w(y)\big\}^{1/r+1/s}\big|f(x)-f(y)\big|\,\big|g(x)-g(y)\big|\,dxdy\\ &\le \frac{1}{2}\left(\int_0^1\int_0^1 w(x)w(y)\big|f(x)-f(y)\big|^r\,dxdy\right)^{1/r}\\ &\qquad\times\left(\int_0^1\int_0^1 w(x)w(y)\big|g(x)-g(y)\big|^s\,dxdy\right)^{1/s}.\end{aligned} \tag{2.5}$$

Because $f \in \mathrm{Lip}_{L_f}(\alpha_f), g \in \mathrm{Lip}_{L_g}(\alpha_g)$, we estimate the right–hand side of (2.5) by

$$\left|\mathfrak{T}(w;f,g)\right| \le \frac{L_f L_g}{2}\left(\int_0^1\int_0^1 w(x)w(y)|x-y|^{\alpha_f r}\,dxdy\right)^{1/r} \times \left(\int_0^1\int_0^1 w(x)w(y)|x-y|^{\alpha_g s}\,dxdy\right)^{1/s}$$

which is evidently (2.3) up to the constant.

To prove (2.4), we begin with regrouping the integrand in (2.5) separating two weight functions and employ the classical Hölder inequality with the same couple of conjugated parameters r, s getting

$$\begin{aligned}\left|\mathfrak{T}(w;f,g)\right| &\le \frac{1}{2}\int_0^1\int_0^1 \left\{w(x)\left|f(x)-f(y)\right|\right\}\cdot\left\{w(y)\left|g(x)-g(y)\right|\right\}dxdy \\ &\le \frac{1}{2}\left(\int_0^1\int_0^1 w^r(x)\left|f(x)-f(y)\right|^r dxdy\right)^{1/r} \\ &\quad\times\left(\int_0^1\int_0^1 w^s(y)\left|g(x)-g(y)\right|^s dxdy\right)^{1/s}.\end{aligned}$$

Estimating the increments of f and g by their $\mathrm{Lip}_L(\alpha)$ definition, we conclude

$$\left|\mathfrak{T}(w;f,g)\right| \le \frac{L_f L_g}{2}\left(\int_0^1\int_0^1 w^r(x)|x-y|^{\alpha_f r}\,dxdy\right)^{1/r} \times \left(\int_0^1\int_0^1 w^s(y)|x-y|^{\alpha_g s}\,dxdy\right)^{1/s}. \tag{2.6}$$

Since

$$\begin{aligned}\int_0^1 |x-y|^{\alpha_f r}\mathrm{d}y &= \int_0^x (x-y)^{\alpha_f r}dy + \int_x^1 (y-x)^{\alpha_f r}dy \\ &= \frac{1}{\alpha_f r+1}\left(x^{\alpha_f r+1} + (1-x)^{\alpha_f r+1}\right),\end{aligned}$$

the substitution of arguments leads us *via* (2.6) to the stated upper bound (2.4). ■

Now we will apply the result obtained to the weight function case closely connected to the Chebyshev–Saigo functional associated with the Saigo fractional integral operator.

3. Andreev–Korkin Identity for Chebyshev–Saigo Functional

The Saigo hypergeometric fractional integral of the function $f : \mathbb{R}_+ \mapsto \mathbb{R}$ is defined for all $\eta > 0, \sigma \in \mathbb{R}$ as

$$I^{\rho,\sigma,\eta}_{0,t}\big[f\big]=\begin{cases} \dfrac{t^\sigma}{\Gamma(\rho)}\displaystyle\int_0^1 (1-x)^{\rho-1}\, {}_2F_1\Big[\begin{matrix}\rho-\sigma,-\eta\\ \rho\end{matrix}\Big|1-x\Big]f(tx)dx & \Re(\rho)>0\\ \dfrac{\mathrm{d}^n}{\mathrm{d}t^n} I^{\rho+n,-\sigma-n,\eta-n}_{0,t}\big[f\big] & \Re(\rho)\le 0,\ n=\big[\Re(-\rho)\big]+1\,, \end{cases} \tag{3.1}$$

where $\Gamma(\cdot)$ stands for the Euler gamma function, compare, for instance, [6, p. 104, Definition 3.20].

The Riemann–Liouville and Erdélyi–Kober fractional integration operators follow respectively as special cases of (3.1), *viz.*

$$I^{\rho,\rho,\eta}_{0,t}\big[f\big] = I^{\rho}_{0,t}\big[f\big] = \frac{t^\rho}{\Gamma(\rho)}\int_0^1 (1-x)^{\rho-1}\, f(tx)\, dx \qquad \Re(\rho)>0, \tag{3.2}$$

$$I^{\rho,0,\eta}_{0,t}\big[f\big] = I^{\rho,\eta}_{0,t}\big[f\big] = \frac{1}{\Gamma(\rho)}\int_0^1 (1-x)^{\rho-1}x^\eta\, f(tx)\, dx \qquad \Re(\rho),\eta>0\,. \tag{3.3}$$

The hypergeometric term in the Saigo operator's integrand is strictly positive [7, p. 35, Theorem 2, Eqs. (3.3), (3.4)]

$${}_2F_1\Big[\begin{matrix}\rho-\sigma,\ -\eta\\ \rho\end{matrix}\Big|x\Big]>0, \qquad x\in(0,1)\,.$$

Hence, for all $\sigma > -1$, the related weight function

$$w_S(x) = \frac{\Gamma(1+\sigma)\Gamma(1+\rho+\eta)}{\Gamma(\rho)\Gamma(1+\sigma+\eta)}(1-x)^{\rho-1}{}_2F_1\Big[\begin{matrix}\rho-\sigma,-\eta\\ \rho\end{matrix}\Big|1-x\Big] \tag{3.4}$$

is well-defined. Moreover, the associated weight functions, relative to the Riemann–Liouville and the Erdélyi–Kober operators, are

$$w_{RL}(x) = \rho\,(1-x)^{\rho-1} \qquad \rho>0, \tag{3.5}$$

$$w_{EK}(x) = \frac{(1-x)^{\rho-1}x^\eta}{\mathrm{B}(\rho,1+\eta)} \qquad \min\{\rho,1+\eta\}>0, \tag{3.6}$$

respectively. Display (3.5) is obvious, while by virtue of the fact

$$ {}_2F_1\left[\begin{matrix}\rho,\, -\eta\\ \rho\end{matrix}\,\Big|\, \cdot\,\right] = {}_1F_0\left[\begin{matrix}-\eta\\ -\end{matrix}\,\Big|\, \cdot\,\right] = (1-\cdot)^\eta\,, $$

we deduce (3.6). (We point out that all three considered weight functions are independent of any scaling parameter t). Now we are ready to introduce the *scaled integral mean associated with the Saigo fractional integral operator* in the form

$$ \mathfrak{M}_t(w_S; f) := \int_0^1 w_S(x) f(tx)\, dx \qquad t > 0. \tag{3.7} $$

Of course $t = 1$, that is, $\mathfrak{M}_1 \equiv \mathfrak{M}$ gives a link to the integral mean (1.2).

Definition 3.1. *The Chebyshev weighted scaled functionals*

$$ T_S(f,g) := \mathfrak{M}_t(w_S; fg) - \mathfrak{M}_t(w_S; f)\mathfrak{M}_t(w_S; g), \tag{3.8} $$
$$ T_{RL}(f,g) := \mathfrak{M}_t(w_{RL}; fg) - \mathfrak{M}_t(w_{RL}; f)\mathfrak{M}_t(w_{RL}; g), \tag{3.9} $$
$$ T_{EK}(f,g) := \mathfrak{M}_t(w_{EK}; fg) - \mathfrak{M}_t(w_{EK}; f)\mathfrak{M}_t(w_{EK}; g) \tag{3.10} $$

we call, by convention, the Chebyshev–Saigo, the Riemann–Liouville and the Erdélyi–Kober functionals, respectively, where $\mathfrak{M}_t(w;\cdot)$ is given by (3.7).

R e m a r k 2.1. The Chebyshev–Saigo functional was introduced in a somewhat different manner by Saxena *et al.* in [8, Eq. (2.8)].

Theorem 3.1. *Let $r, s,\ r^{-1}+s^{-1} = 1,\ r > 1$, be conjugated Hölder parameters. Then for all $f \in \mathrm{Lip}_{L_f}(\alpha_f), g \in \mathrm{Lip}_{L_g}(\alpha_g)$ and $\min\{t, \Re(\rho), \eta\} > 0, \sigma \in \mathbb{R}$, we have*

$$ \big|T_S(f,g)\big| \le \frac{L_f L_g\, \Gamma^2(1+\sigma)\, \Gamma^2(1+\rho+\eta)\, \mathcal{E}^{1/r}(r,\alpha_f)\, \mathcal{E}^{1/s}(s,\alpha_g)}{2\big(\alpha_f r+1\big)^{1/r}\big(\alpha_g s+1\big)^{1/s}\, \Gamma^2(\rho)\, \Gamma^2(1+\sigma+\eta)}\, t^{\alpha_f+\alpha_g}\,, \tag{3.11} $$

where

$$ \mathcal{E}(u,v) = \int_0^1 x^{uv+1}\Big\{(1-x)^{u(\rho-1)}\, {}_2F_1^r\left[\begin{matrix}\rho-\sigma,\, -\eta\\ \rho\end{matrix}\,\Big|\, 1-x\right] + x^{u(\rho-1)}\, {}_2F_1^r\left[\begin{matrix}\rho-\sigma,\, -\eta\\ \rho\end{matrix}\,\Big|\, x\right]\Big\}\, dx\,. $$

P r o o f. A straightforward application of Theorem 2.1 results in (3.11). Indeed, following the lines of the proving procedure of Theorem 2.1 we have

$$ \big|T_S(f,g)\big| \le \frac{1}{2}\int_0^1\!\!\int_0^1 \big\{w_S(x)\big|f(tx)-f(ty)\big|\big\}\cdot\big\{w_S(y)\big|g(tx)-g(ty)\big|\big\}\, dxdy $$

$$\leq \frac{1}{2}\left(\int_0^1\int_0^1 w_S^r(x)\big|f(tx)-f(ty)\big|^r\,dxdy\right)^{1/r}$$

$$\times\left(\int_0^1\int_0^1 w_S^s(y)\big|g(tx)-g(ty)\big|^s\,dxdy\right)^{1/s} =: U\,.$$

Both f and g being Lipschitz, we may conclude

$$U \leq \frac{L_fL_g}{2}\left(\int_0^1\int_0^1 w_S^r(x)|x-y|^{\alpha_f r}\,dxdy\right)^{1/r}$$

$$\times\left(\int_0^1\int_0^1 w_S^s(y)|x-y|^{\alpha_g s}\,dxdy\right)^{1/s}\cdot t^{\alpha_f+\alpha_g}\,.$$

Now obvious further calculation leads to (3.11). ■

The next results show how to reduce upper bounds for the modulus of the Chebyshev–Saigo functional to the bounds when the Saigo hypergeometric fractional integration operator is replaced by the Riemann–Liouville and the Erdélyi–Kober operators.

Corollary 3.1. *Let r, s, $r^{-1}+s^{-1}=1$, $r>1$, be conjugated Hölder parameters. Then for all $f\in \mathrm{Lip}_{L_f}(\alpha_f), g\in \mathrm{Lip}_{L_g}(\alpha_g)$ and $\min\{t,\Re(\rho)\}>0$, we have*

$$\big|T_{RL}(f,g)\big| \leq \frac{L_fL_g\,\rho^2\,\mathcal{G}^{1/r}(r,\alpha_f)\,\mathcal{G}^{1/s}(s,\alpha_f)}{2\big(\alpha_f r+1\big)^{1/r}\big(\alpha_g s+1\big)^{1/s}}\,t^{\alpha_f+\alpha_g}\,, \tag{3.12}$$

where

$$\mathcal{G}(u,v) = \mathrm{B}(uv+2, u(\rho-1)+1) + \big((v+\rho-1)u+2\big)^{-1}\,.$$

Corollary 3.2. *Let r, s, $r^{-1}+s^{-1}=1$, $r>1$, be conjugated Hölder exponents. Then for all $f\in \mathrm{Lip}_{L_f}(\alpha_f), g\in \mathrm{Lip}_{L_g}(\alpha_g)$ and $\min\{t,\Re(\rho),\eta\}>0$, we have*

$$\big|T_{EK}(f,g)\big| \leq \frac{L_fL_g\,\mathcal{H}^{1/r}(r,\alpha_f)\,\mathcal{H}^{1/s}(s,\alpha_g)}{2\big(\alpha_f r+1\big)^{1/r}\big(\alpha_g s+1\big)^{1/s}\,\mathrm{B}^2(\rho,1+\eta)}\,t^{\alpha_f+\alpha_g}\,, \tag{3.13}$$

where

$$\mathcal{H}(u,v) = \mathrm{B}\big((v+\eta)u+2, u(\rho-1)+1\big) + \mathrm{B}\big(u(v+\rho-1)+2, \eta u+1\big)\,.$$

4. Further Bounds for the Chebyshev–Saigo Functional

To get a more sophisticated bound for $T_S(f,g,)$, we need an auxiliary upper bound inequality for the hypergeometric function appearing in the Saigo fractional integral operator. A similar upper bound was given by Carlson [7, p. 35, Theorem 2, Eqs. (2.13), (2.14)]:

Lemma. *Let $c > b > 0$ and $x < 1, x \neq 0$. Then*

$$ {}_2F_1\left[\begin{matrix} a, b \\ c \end{matrix} \,\Big|\, x \right] < \begin{cases} J & a < -1, \\ \min\{H,\, J_1\} & a \in (-1, 0), \end{cases} \tag{4.1} $$

where

$$ \begin{aligned} J &:= (1-b/c) + (b/c)(1-x)^{-a}, \\ H &:= \big(1 - bx/c\big)^{-a}, \\ J_1 &:= (b/c)(1-x)^{c-a-b} + (1-b/c)(1-x)^{-b}. \end{aligned} $$

If $a \leq c-1$, then $\min\{H, J_1\} = H$.

In order to present the results, we need a definition of the Fox–Wright function ${}_p\Psi_q^*$ which is a generalization of the familiar generalized hypergeometric function ${}_pF_q$ [6, 9],

$$ \begin{aligned} {}_p\Psi_q^*\left[\begin{matrix} (a_1, A_1), \cdots, (a_p, A_p) \\ (b_1, B_1), \cdots, (b_q, B_q) \end{matrix} \,\Big|\, z \right] &= {}_p\Psi_q^*\left[\begin{matrix} (a_p, A_p) \\ (b_q, B_q) \end{matrix} \,\Big|\, z \right] \\ &:= \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p} (a_j)_{A_j n}}{\prod_{j=1}^{q} (b_j)_{B_j n}} \frac{z^n}{n!}, \end{aligned} \tag{4.2} $$

where $(\tau)_T$ is the Pochhammer symbol (or shifted factorial), with $(1)_n = n!$, $n \in \mathbb{N}_0$, defined in terms of gamma function by

$$ (\tau)_T = \frac{\Gamma(\tau+T)}{\Gamma(\tau)} = \begin{cases} 1 & T = 0,\, \tau \in \mathbb{C} \setminus \{0\}, \\ \tau(\tau+1)\cdots(\tau+T-1) & T \in \mathbb{N},\, \tau \in \mathbb{C}, \end{cases} $$

where, as understood conventionally, $(0)_0 := 1$.

In (4.2) $a_j,\, b_k \in \mathbb{C}$, $A_j,\, B_k > 0$, $j = \overline{1,p}$, $k = \overline{1,q}$ and

$$ \Delta = 1 + \sum_{j=1}^{q} B_j - \sum_{j=1}^{p} A_j \geq 0; \tag{4.3} $$

for $\Delta = 0$ the convergence holds for suitably bounded values of $|z|$, given by $|z| < \nabla$, where

$$ \nabla = \prod_{j=1}^{q} B_j^{B_j} \cdot \prod_{j=1}^{p} A_j^{-A_j}. \tag{4.4} $$

Theorem 4.1. *Let r, s, $r^{-1}+s^{-1}=1$, $r>1$, be conjugated Hölder parameters. Then for all $f \in \mathrm{Lip}_{L_f}(\alpha_f), g \in \mathrm{Lip}_{L_g}(\alpha_g)$, $0<\sigma<\rho<2\sigma$ we have:*

(i) *for $\eta > 1$,*

$$\big|T_S(f,g)\big| \le \frac{L_f L_g\,\Gamma^2(1+\sigma)\,\Gamma^2(1+\rho+\eta)\,\mathcal{I}^{1/r}(r,\alpha_f)\,\mathcal{I}^{1/s}(s,\alpha_g)}{2\big(\alpha_f r+1\big)^{1/r}\big(\alpha_g s+1\big)^{1/s}\,\Gamma^2(\rho)\,\Gamma^2(1+\sigma+\eta)}\,t^{\alpha_f+\alpha_g}\,, \quad (4.5)$$

where

$$\mathcal{I}(u,v) := \Big(\frac{\sigma}{\rho}\Big)^u \Big\{\mathrm{B}(uv+2,u(\rho-1)+1)\,{}_2\Psi_1^*\Big[\begin{array}{c}(-u,1),(uv+2,\eta)\\ \big((v+\rho-1)u+3,\eta\big)\end{array}\Big|\,1-\frac{\rho}{\sigma}\Big] + \frac{1}{(v+\rho-1)u+2}\,{}_2\Psi_1^*\Big[\begin{array}{c}(-u,1),(1,\eta)\\ \big((v+\rho-1)u+3,\eta\big)\end{array}\Big|\,1-\frac{\rho}{\sigma}\Big]\Big\}\,;$$

(ii) *for $\eta \in (0,1)$,*

$$\big|T_S(f,g)\big| \le \frac{L_f L_g\,\Gamma^2(1+\sigma)\,\Gamma^2(1+\rho+\eta)\,\mathcal{J}^{1/r}(r,\alpha_f)\,\mathcal{J}^{1/s}(s,\alpha_g)}{2\big(\alpha_f r+1\big)^{1/r}\big(\alpha_g s+1\big)^{1/s}\,\Gamma^2(\rho)\,\Gamma^2(1+\sigma+\eta)}\,t^{\alpha_f+\alpha_g}\,, \quad (4.6)$$

where

$$\mathcal{J}(u,v) := \Big(\frac{\sigma}{\rho}\Big)^{\eta u} \mathrm{B}(uv+2,u(\rho-1)+1)\,{}_2F_1\Big[\begin{array}{c}-\eta u,\ \ uv+2\\ (v+\rho-1)u+3\end{array}\Big|\,1-\frac{\rho}{\sigma}\Big] + \frac{1}{(v+\rho-1)u+2}\,{}_2F_1\Big[\begin{array}{c}-\eta u,\ \ (v+\rho-1)u+2\\ (v+\rho-1)u+3\end{array}\Big|\,1-\frac{\sigma}{\rho}\Big]\,.$$

P r o o f. Taking $a=-\eta, b=\rho-\sigma, c=\rho$, the conditions of Lemma are fulfilled with $0<\sigma<\rho$, so by (4.1) we have

$${}_2F_1\Big[\begin{array}{c}a,\ b\\ c\end{array};x\Big] < \begin{cases}\sigma/\rho+(1-\sigma/\rho)(1-x)^\eta & \eta>1,\\ \min\big\{H,\ J_1\big\} & \eta\in(0,1),\end{cases} \quad (4.7)$$

where

$$H = \big(1-(1-\sigma/\rho)x\big)^\eta,$$
$$J_1 = (1-\sigma/\rho)(1-x)^{\eta+\sigma} + (\sigma/\rho)(1-x)^{\sigma-\rho},$$

and for $\eta+\sigma\ge 1$ it is $\min\{H,J_1\}=H$.

(i) $\eta>1$. By (3.11) and (4.7), we conclude

$$\big|T_S(f,g)\big| \le \frac{L_f L_g t^{\alpha_f+\alpha_g}\,\Gamma^2(1+\sigma)\,\Gamma^2(1+\rho+\eta)}{2\big(\alpha_f r+1\big)^{1/r}\big(\alpha_g s+1\big)^{1/s}\,\Gamma^2(\rho)\,\Gamma^2(1+\sigma+\eta)}$$

$$\times\left(\int_0^1 x^{\alpha_f r+1}\Big\{(1-x)^{r(\rho-1)}\big(\sigma/\rho+(1-\sigma/\rho)x^\eta\big)^r + x^{r(\rho-1)}\big(\sigma/\rho+(1-\sigma/\rho)(1-x)^\eta\big)^r\Big\}dx\right)^{1/r}$$
$$\times\left(\int_0^1 x^{\alpha_g s+1}\Big\{(1-x)^{s(\rho-1)}\big(\sigma/\rho+(1-\sigma/\rho)x^\eta\big)^s + x^{s(\rho-1)}\big(\sigma/\rho+(1-\sigma/\rho)(1-x)^\eta\big)^s\Big\}dx\right)^{1/s}. \tag{4.8}$$

Since

$$\begin{aligned}\mathcal{I}_1(r) &:= \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}(1+\gamma x^\delta)^r dx = \sum_{n=0}^\infty \binom{r}{n}\gamma^n \int_0^\infty x^{\alpha+\delta n-1}(1-x)^{\beta-1}dx \\ &= \Gamma(\beta)\sum_{n=0}^\infty \frac{(-r)_n\Gamma(\alpha+\delta n)}{\Gamma(\alpha+\beta+\delta n)}\,\frac{(-\gamma)^n}{n!} = \mathrm{B}(\alpha,\beta)\sum_{n=0}^\infty \frac{(-r)_n(\alpha)_{\delta n}}{(\alpha+\beta)_{\delta n}}\,\frac{(-\gamma)^n}{n!} \\ &= \mathrm{B}(\alpha,\beta)\cdot {}_2\Psi_1^*\Big[\begin{matrix}(-r,1),\,(\alpha,\delta)\\ (\alpha+\beta,\delta)\end{matrix}\;\Big|\,-\gamma\Big],\end{aligned} \tag{4.9}$$

where $\mathrm{B}(\cdot,\cdot)$ denotes the Eulerian beta function, and because

$$\begin{aligned}\mathcal{I}_2(r) &:= \int_0^1 x^{\nu-1}(1+\gamma(1-x)^\delta)^r dx = \sum_{n=0}^\infty \binom{r}{n}\gamma^n \int_0^\infty x^{\nu-1}(1-x)^{\delta n}dx \\ &= \Gamma(\nu)\sum_{n=0}^\infty \frac{(-r)_n\Gamma(1+\delta n)}{\Gamma(\nu+1+\delta n)}\,\frac{(-\gamma)^n}{n!} = \frac{1}{\nu}\sum_{n=0}^\infty \frac{(-r)_n(1)_{\delta n}}{(\nu+1)_{\delta n}}\,\frac{(-\gamma)^n}{n!} \\ &= \frac{1}{\nu}\cdot {}_2\Psi_1^*\Big[\begin{matrix}(-r,1),\,(1,\delta)\\ (\nu+1,\delta)\end{matrix}\;\Big|\,-\gamma\Big],\end{aligned}$$

for $\delta>0$, in both cases $\Delta = 1+\delta-1-\delta = 0$, therefore the series $I_{1,2}(r)$ converge in the whole range of $|\gamma| < \nabla = \delta^\delta\cdot\delta^{-\delta} = 1$. Hence, the first integral in (4.8) becomes

$$\begin{aligned}\mathcal{I}(r,\alpha_f) = \Big(\frac{\sigma}{\rho}\Big)^r\Big\{&\mathrm{B}(\alpha_f r+2, r(\rho-1)+1)\,{}_2\Psi_1^*\Big[\begin{matrix}(-r,1),(\alpha_f r+2,\eta)\\ \big((\alpha_f+\rho-1)r+3,\eta\big)\end{matrix}\;\Big|\,1-\frac{\rho}{\sigma}\Big] \\ &+\frac{1}{(\alpha_f+\rho-1)r+2}\,{}_2\Psi_1^*\Big[\begin{matrix}(-r,1),(1,\eta)\\ \big((\alpha_f+\rho-1)r+3,\eta\big)\end{matrix}\;\Big|\,1-\frac{\rho}{\sigma}\Big]\Big\};\end{aligned}$$

both series converge for $|1-\rho/\sigma|<1$, that is, in the assumed range $\sigma<\rho<2\sigma$. By this conclusion the case **(i)** is proved.

(ii) $\eta\in(0,1)$, $\eta+\sigma\geq 1$. In this case H and J_1 possess the common tangent $x\mapsto 1-\eta(1-\sigma/\rho)x$ at the origin. Being

$$H''(x)=\eta(\eta-1)\Big(1-\frac{\sigma}{\rho}\Big)^2\big(1-(1-\sigma/\rho)x\big)^{\eta-2}<0$$

$$J_1''(x)=\Big(1-\frac{\sigma}{\rho}\Big)(1-x)^{\sigma-\rho-2}\big[(\eta+\sigma)(\eta+\sigma-1)(1-x)^{\eta+\rho}+\sigma(\rho+1-\sigma)\big]>0\,,$$

we clearly conclude that H is concave and J_1 is convex in the unit interval. Thus, it is $\min\{H,J_1\}=H$ according to Carlson's Lemma too.

Without condition $\eta+\sigma\geq 1$, as *mutatis mutandis* $\min\{H,J_1\}\leq H$, we conclude the case **(ii)** by (3.11).

Now in both cases $\eta\in(0,1),\sigma>0$ and by (4.7) we have

$$\begin{aligned}\big|T_S(f,g)\big|\leq{}&\frac{L_fL_g\,\Gamma^2(1+\sigma)\,\Gamma^2(1+\rho+\eta)}{2\big(\alpha_fr+1\big)^{1/r}\big(\alpha_gs+1\big)^{1/s}\,\Gamma^2(\rho)\,\Gamma^2(1+\sigma+\eta)}\,t^{\alpha_f+\alpha_g}\\
&\times\Bigg(\int_0^1 x^{\alpha_fr+1}\Big\{(1-x)^{r(\rho-1)}\big(\sigma/\rho+(1-\sigma/\rho)x\big)^{\eta r}\\
&\qquad+x^{r(\rho-1)}\big(1-(1-\sigma/\rho)x\big)^{\eta r}\Big\}dx\Bigg)^{1/r}\\
&\times\Bigg(\int_0^1 x^{\alpha_gs+1}\Big\{(1-x)^{s(\rho-1)}\big(\sigma/\rho+(1-\sigma/\rho)x\big)^{\eta s}\\
&\qquad+x^{s(\rho-1)}\big(1-(1-\sigma/\rho)x\big)^{\eta s}\Big\}dx\Bigg)^{1/s},\end{aligned}\tag{4.10}$$

and the first integral in (4.10) is equal to

$$\begin{aligned}\mathcal{J}(r,\alpha_f)={}&\Big(\frac{\sigma}{\rho}\Big)^{\eta r}\,\mathrm{B}(\alpha_fr+2,r(\rho-1)+1)\,{}_2F_1\Big[\begin{matrix}-\eta r,\ \alpha_fr+2\\ (\alpha_f+\rho-1)r+3\end{matrix}\Big|\,1-\frac{\rho}{\sigma}\Big]\\
&+\frac{1}{(\alpha_f+\rho-1)r+2}\,{}_2F_1\Big[\begin{matrix}-\eta r,\ (\alpha_f+\rho-1)r+2\\ (\alpha_f+\rho-1)r+3\end{matrix}\Big|\,1-\frac{\sigma}{\rho}\Big],\end{aligned}\tag{4.11}$$

where both hypergeometric series converge in the range of $0<\sigma<\rho<2\sigma$.

Indeed, we have

$$\mathcal{J}(r,\alpha_f) = \Big(\frac{\sigma}{\rho}\Big)^{\eta r}\mathcal{I}_1(\eta r) + \int_0^1 x^{\alpha-1}\big(1-(1-\sigma/\rho)x\big)^{\eta r}\,dx = \Big(\frac{\sigma}{\rho}\Big)^{\eta r}\mathcal{I}_1(\eta r) + \mathcal{I}_3(r)\,,$$

when in (4.9) one specifies $\alpha = \alpha_f r + 2, \beta = r(\rho-1)+1, \gamma = \rho/\sigma - 1$ and $\delta = 1$, while further short calculation gives us

$$\mathcal{I}_3(r) = \sum_{n=0}^{\infty}\binom{\eta r}{n}\frac{(-1)^n(1-\sigma/\rho)^n}{(\alpha_f+\rho-1)r+2+n}\,.$$

Using the transformation

$$\frac{1}{A+n} = \frac{\Gamma(A+n)}{\Gamma(A+1+n)} = \frac{1}{A}\cdot\frac{(A)_n}{(A+1)_n}\,,$$

where $A = (\alpha_f+\rho-1)r+2$, we conclude

$$\begin{aligned}\mathcal{I}_3(r) &= \frac{1}{(\alpha_f+\rho-1)r+2}\sum_{n=0}^{\infty}\frac{(-\eta r)_n\big((\alpha_f+\rho-1)r+2\big)_n}{\big((\alpha_f+\rho-1)r+3\big)_n}\,\frac{(1-\sigma/\rho)^n}{n!} \\ &= \frac{1}{(\alpha_f+\rho-1)r+2}\,{}_2F_1\Big[\begin{matrix}-\eta r,\ (\alpha_f+\rho-1)r+2\\ (\alpha_f+\rho-1)r+3\end{matrix}\Big|\,1-\frac{\sigma}{\rho}\Big].\end{aligned}$$

So is the proof of (4.12). ■

Finally, let us present two more results in which we discuss the hypergeometric kernel function appearing in the Saigo fractional integration operator: the first with the integer value parameters $\eta r, \eta s$, where r, s form the conjugated Hölder exponents, the second estimating functions H, J_1 in Carlson's Lemma by uniform upper bound equal to 1 on the whole unit interval.

Corollary 4.1. *Let $r,\ s,\ r^{-1}+s^{-1}=1,\ r>1$, be conjugated Hölder parameters, $\eta\in(0,1)$, $\eta r, \eta s\in\mathbb{N}$. Then for all $f\in \mathrm{Lip}_{L_f}(\alpha_f)$, $g\in\mathrm{Lip}_{L_g}(\alpha_g)$, $0<\sigma<\rho<2\sigma$ we have*

$$\big|T_S(f,g)\big| \le \frac{L_fL_g\,\Gamma^2(1+\sigma)\,\Gamma^2(1+\rho+\eta)\,\mathcal{K}^{1/r}(r,\alpha_f)\,\mathcal{K}^{1/s}(s,\alpha_g)}{2\big(\alpha_f r+1\big)^{1/r}\big(\alpha_g s+1\big)^{1/s}\,\Gamma^2(\rho)\,\Gamma^2(1+\sigma+\eta)}\,t^{\alpha_f+\alpha_g}\,, \tag{4.12}$$

where

$$\mathcal{K}(u,v) := \Big(\frac{\sigma}{\rho}\Big)^{\eta u}\mathrm{B}(uv+2,u(\rho-1)+1)\,P_{\eta u}\Big(1-\frac{\rho}{\sigma}\Big) + \frac{Q_{\eta u}(1-\sigma/\rho)}{(v+\rho-1)u+2}\,,$$

and

$$P_{\eta u}(z) = {}_2F_1\left[\begin{matrix} -\eta u, & uv+2 \\ (v+\rho-1)u+3 \end{matrix} \,\middle|\, z\right] = \sum_{n=0}^{\eta u} \frac{(-\eta u)_n (uv+2)_n}{\big((v+\rho-1)u+3\big)_n n!} z^n,$$

$$Q_{\eta u}(z) = {}_2F_1\left[\begin{matrix} -\eta u, & (v+\rho-1)u+2 \\ (v+\rho-1)u+3 \end{matrix} \,\middle|\, z\right] = \sum_{n=0}^{\eta u} \frac{(-\eta u)_n \big((v+\rho-1)u+2\big)_n}{\big((v+\rho-1)u+3\big)_n n!} z^n$$

are polynomials of degree ηu.

P r o o f. Because $\eta r, \eta s \in \mathbb{N}$, the hypergeometric functions

$${}_2F_1\left[\begin{matrix} -\eta u, & uv+2 \\ (v+\rho-1)u+3 \end{matrix} \,\middle|\, \cdot\right],$$

$${}_2F_1\left[\begin{matrix} -\eta u, & (v+\rho-1)u+2 \\ (v+\rho-1)u+3 \end{matrix} \,\middle|\, \cdot\right]$$

reduce to the polynomials $P_{\eta r}(\cdot), Q_{\eta s}(\cdot)$ of degrees $\eta r, \eta s$, respectively. The claim now follows from Theorem 4.1, case **(ii)**. ∎

Corollary 4.2. *Let r, s, $r^{-1}+s^{-1}=1$, $r>1$, be conjugated Hölder parameters. Then for all $f \in \mathrm{Lip}_{L_f}(\alpha_f)$, $g \in \mathrm{Lip}_{L_g}(\alpha_g)$, $0<\sigma<\rho<2\sigma$, $\eta \in (0,1)$, $1+r(\rho-1)>0$, $1+s(\rho-1)>0$ we have*

$$\big|T_S(f,g)\big| \le \frac{L_f L_g \,\Gamma^2(1+\sigma)\,\Gamma^2(1+\rho+\eta)\,\mathcal{N}^{1/r}(r,\alpha_f)\,\mathcal{N}^{1/s}(s,\alpha_g)}{2\big(\alpha_f r+1\big)^{1/r}\big(\alpha_g s+1\big)^{1/s}\,\Gamma^2(\rho)\,\Gamma^2(1+\sigma+\eta)}\, t^{\alpha_f+\alpha_g}, \tag{4.13}$$

where

$$\mathcal{N}(u,v) = \mathrm{B}\big(2+uv, 1+u(\rho-1)\big) + \frac{1}{u(v+\rho-1)+2}.$$

P r o o f. By virtue of the obvious estimate $\min\{J_1, H\} \le 1$, (4.13) is a direct consequence of Theorem 3.1 and Lemma. ∎

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