Journal of Mathematical Physics, Analysis, Geometry
2012, vol. 8, No. 3, pp. 221–239

Spectral Mapping Theorem for the Davies–Helffer–Sjöstrand Functional Calculus

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Received December 21, 2010, revised February 20, 2012

We give a direct non-abstract proof of the spectral mapping theorem for the Davies–Helffer–Sjöstrand functional calculus for linear operators on Banach spaces with real spectra and consequently give a new non-abstract direct proof for the spectral mapping theorem for self-adjoint operators on Hilbert spaces. Our exposition is closer in spirit to the proof by explicit construction of the existence of the Functional Calculus given by Davies. We apply an extension theorem of Seeley to derive a functional calculus for semi-bounded operators.

Key words: Functional calculus, spectral mapping theorem, spectrum.

Mathematics Subject Classification 2010: 47A60.

1. Introduction

The Helffer–Sjöstrand formula was established in [1] in the following proposition:

Proposition 1.1. ([1] Proposition 7.2) *Let H be a self-adjoint operator (not necessarily bounded) on a Hilbert space $\mathcal{H}$. Suppose f is in $C_0^\infty(\mathbb{R})$ and $\tilde{f}$ in $C_0^\infty(\mathbb{C})$ is an extension of f such that $\frac{\partial \tilde{f}}{\partial \bar{z}} = 0$ on $\mathbb{R}$. Then we have*

$$f(H) = -\frac{1}{\pi} \iint\limits_{\mathbb{C}} \frac{\partial \tilde{f}(z)}{\partial \overline{z}} (z-H)^{-1} \, dxdy, \tag{1.1}$$

where $z = x + iy$.

The existence of the functional calculus was assumed by the authors. Davies [2] showed that the formula (Equation 1.1) yielded a new approach to constructing the functional calculus for linear operators on Banach spaces under the following hypothesis:

Hypothesis 1.2. *H is a closed densely defined operator on a Banach space $\mathcal{B}$ with spectrum $\sigma(H) \subseteq \mathbb{R}$. The resolvent operators $(z-H)^{-1}$ are defined and bounded for all $z \notin \mathbb{R}$ and*

$$\|(z-H)^{-1}\| \leq c|\mathrm{Im}\, z|^{-1} \left(\frac{\langle z \rangle}{|\mathrm{Im}\, z|}\right)^{\alpha} \tag{1.2}$$

for some $\alpha \geq 0$ and all $z \notin \mathbb{R}$, where $\langle z \rangle := (1+|z|^2)^{\frac{1}{2}}$.

His functional calculus for operators on Banach spaces was defined for an algebra of slowly decreasing smooth functions. Davies [2] pointed out that a functional calculus based upon almost analytic extensions was also constructed by Dyn'kin [3]. However, the two approaches were quite different and that Davies' approach was more appropriate for differential operators.

A spectral mapping theorem for the Davies–Helffer–Sjöstrand functional calculus was proved by Bátkai and Fašanga [4]. They applied methods from abstract functional analysis and their primary tool was an existing abstract spectral mapping theorem from the theory of Banach algebras:

Theorem 1.3. ([4] Theorem 4.1) *Let $\mathcal{B}_1$ be a commutative, semisimple, regular Banach algebra, $\mathcal{B}_2$ be a Banach algebra with a unit, $\Theta : \mathcal{B}_1 \to \mathcal{B}_2$ be a continuous algebra homomorphism and $a \in \mathcal{B}_1$. Then*

$$\sigma_{\mathcal{B}_2}(\Theta(a)) = \overline{\hat{a}(\mathrm{Sp}(\theta))} \quad , \textit{where } \mathrm{Sp}(\Theta) := \cap_{b \in Ker\Theta} Ker\, \hat{b}$$

and $\hat{}$ denotes the Gelfand transform.

Our exposition of the spectral mapping theorem, part of the Ph.D thesis referred to in the introduction of [4], takes a very non-abstract and direct approach to the proof. In particular, an existing spectral mapping is not assumed. Our sole ingredients, supplementing the tools provided by Davies in [2], are the very elementary observations:

- ([5] Problem 8.1.11) If H is a closed operator and λ lies in the topological boundary of the spectrum of H, then for every $\epsilon > 0$ there is a vector v with norm 1 such that $\|Hv - \lambda v\| < \epsilon$.
- Stokes' Formula has similarities to the Cauchy Integral Formula.

In the last part of our exposition we derive a functional calculus for operators with spectra bounded on one side. Our main tool here is an extension operator of Seeley,

$$\mathcal{E} : C^{\infty}[0,\infty) \longrightarrow C^{\infty}(\mathbb{R}).$$

1.1. Functional Calculus

We summarize some of the main aspects of the Davies–Helffer–Sjöstrand functional calculus presented in [6] and some properties of the algebra of functions. Let $\psi_{a,\epsilon}$ be a smooth function such that

$$\psi_{a,\epsilon}(x) := \begin{cases} 1 & \text{if } x \geq a \\ 0 & \text{if } x \leq a-\epsilon. \end{cases}$$

Then given an interval $[a,b]$, we define the approximate characteristic function $\Psi_{[a,b],\epsilon}$

$$\Psi_{[a,b],\epsilon}(x) = \psi_{a,\epsilon}(x) - \psi_{b+\epsilon,\epsilon}(x)$$

which has a support $[a-\epsilon, b+\epsilon]$ and is equal to 1 in $[a,b]$ and is smooth.

Definition 1.4. *For $\beta \in \mathbb{R}$ let S^{β} be the set of all complex-valued smooth functions defined on $\mathbb{R}$, where for every $n \in \mathbb{N} \cup \{0\}$ there is a positive constant c_n such that*

$$\left|\frac{d^n f(x)}{dx^n}\right| \leq c_n \langle x \rangle^{\beta-n}.$$

We then define the algebra $\mathcal{A} := \bigcup_{\beta<0} S^{\beta}$.

Lemma 1.5. (Davies [2, 6]) *$\mathcal{A}$ is an algebra under point-wise multiplication. For any f in $\mathcal{A}$ the expression*

$$\|f\|_n := \sum_{r=0}^{n} \int_{-\infty}^{\infty} \left|\frac{d^r f(x)}{dx^r}\right| \langle x \rangle^{r-1} dx \tag{1.3}$$

defines a norm on $\mathcal{A}$ for each n. Moreover, $C_0^{\infty}(\mathbb{R})$ is dense in $\mathcal{A}$ with this norm.

Lemma 1.6. *The function $\langle x \rangle^{\beta}$ is in $\mathcal{A}$ for each $\beta < 0$.*

P r o o f. The statement follows from the observations that if $\beta < 0$ and $m \geq n$, then

$$x^n \langle x \rangle^{\beta-m} \leq \langle x \rangle^{\beta}$$

and

$$\frac{d\left(x^n \langle x \rangle^{\beta-m}\right)}{dx} = n x^{n-1} \langle x \rangle^{\beta-m} + (\beta-m)\, x^{n+1} \langle x \rangle^{\beta-m-2}.$$

■

Lemma 1.7. *Let $s \in \mathbb{R}$. If f is in $\mathcal{A}$, then the function*

$$g_s(x) := \begin{cases} \frac{f(x)-f(s)}{x-s} & x \neq s \\ f'(s) & x = s \end{cases}$$

is also in $\mathcal{A}$.

Proof. When $|x-s|$ is large, then

$$\frac{1}{|x-s|} \leq c_s \langle x \rangle^{-1}$$

for some $c_s > 0$. Moreover,

$$g_s^{(r)}(x) = \sum_{m=0}^{r} c_r f^{(m)}(x)\ (x-s)^{m-r-1} + cf(s)(x-s)^{-r-1}$$

and

$$\lim_{x \to s} g_s^{(m)}(x) = \frac{1}{m+1} f^{(m+1)}(s).$$

■

Lemma 1.8. *If $f \in S^{\beta}$ for $\beta < 0$ and $g \in S^0$, then $fg \in \mathcal{A}$.*

Proof.

$$|(fg)^{(r)}(x)| \leq c_r \sum_{m=0}^{r} |g^{(r-m)}(x)|\,|f^{(m)}(x)| \leq c_{r,\phi} \langle x \rangle^{\beta - r}.$$

■

The following concept of almost analytic extensions is due to Hörmander [7, p. 63].

Definition 1.9. *Let $\tau(x,y)$ be a smooth function such that*

$$\tau(x,y) := \begin{cases} 1 & \text{if } |y| \leq \langle x \rangle \\ 0 & \text{if } |y| \geq 2\langle x \rangle. \end{cases}$$

Then given $f \in \mathcal{A}$ we define an almost analytic extension $\tilde{f}$ as

$$\tilde{f}(x,y) := \left(\sum_{r=0}^{n} \frac{d^r f(x)}{dx^r} \frac{(iy)^r}{r!} \right) \tau(x,y) \tag{1.4}$$

for some $n \in \mathbb{N}$. Moreover, we define

$$\frac{\partial \tilde{f}}{\partial \overline{z}} := \frac{1}{2} \left(\frac{\partial \tilde{f}}{\partial x} + i \frac{\partial \tilde{f}}{\partial y} \right). \tag{1.5}$$

The specific choices of τ and n are not critical.

The following lemma establishes the construction of the new functional calculus:

Lemma 1.10. (Davies [2]) *Let $f \in \mathcal{A}$, then define*

$$f(H) := -\frac{1}{\pi} \iint_{\mathbb{C}} \frac{\partial \tilde{f}(z)}{\partial \overline{z}} (z - H)^{-1} \, dxdy, \tag{1.6}$$

where $\tilde{f}$ is an almost-analytic version of f as defined in definition 1.9 *and $z = x + iy$. Then*

i. If $n > \alpha$, then subject to hypothesis 1.2 *the integral* (1.6) *is norm convergent for all f in $\mathcal{A}$ and*

$$\|f(H)\| \le c\|f\|_{n+1}.$$

ii. The operator $f(H)$ is independent of n and the cut-off function τ, subject to $n > \alpha$.

iii. If f is a smooth function of compact support disjoint from the spectrum of H, then $f(H) = 0$.

iv. If f and g are in $\mathcal{A}$, then $(fg)(H) = f(H)\, g(H)$.

v. If $z \notin \mathbb{R}$ and $g_z(x) := (z - x)^{-1}$ for all $x \in \mathbb{R}$, then $g_z \in \mathcal{A}$ and $g_z(H) = (z - H)^{-1}$.

1.2. Preliminaries

Definition 1.11. *Given z, ω in $\mathbb{C}$, we define the curve Γ in the complex plane*

$$\Gamma(z, \omega, \alpha) := ((1-\alpha)|z| + \alpha|\omega|)\, e^{i(1-\alpha)Arg(z) + i\alpha Arg(\omega)},$$

where $\alpha \in [0, 1]$.

The important property of Γ is that it is able to connect two non-zero points in the complex plane without intersection with the origin.

Theorem 1.12. *Let $\lambda \in \mathbb{C}$. If f is a smooth complex valued function in the interval $[a, b]$, where $f(a) \neq \lambda$ and $f(b) \neq \lambda$, then there is a smooth function h in $C^{\infty}([a, b])$ such that*

$$\{x \in [a, b] : h(x) = \lambda\} \text{ is empty },$$

and $f - h$ and all derivatives of $f - h$ vanish at a and b.

P r o o f. Let

$$g(x) := \Gamma\left(f(a) - \lambda, f(b) - \lambda, \frac{x-a}{b-a}\right) + \lambda. \tag{1.7}$$

Since f is continuous, we know there is an $0 < \epsilon < \frac{b-a}{2}$ such that

$$\{x \in [a,b]/(a+\epsilon, b-\epsilon) : f(x) = \lambda\} = \emptyset.$$

Then we can define

$$h := \left(1 - \Psi_{[a+\epsilon,b-\epsilon],\epsilon}\right) f + \Psi_{[a+\epsilon,b-\epsilon],\epsilon}\, g.$$

∎

Lemma 1.13. *Given $f \in \mathcal{A}$, let λ be a non-zero point in $\mathbb{C}$ and let $A_\lambda := \{x : f(x) = \lambda\}$.*
If $A_\lambda \cap \sigma(H)$ is empty, then there is a function $h \in \mathcal{A}$ such that $h(x) \neq \lambda$ for all $x \in \mathbb{R}$ and

$$h(H) = f(H).$$

P r o o f. If A_λ is empty, then we put $h = f$.
If A_λ is not empty, then A_λ is a compact subset of $\rho(H)$. Moreover, A_λ can be covered by a finite set of closed disjoint intervals $[a_i, b_i]$ which are also subsets of $\rho(H)$. By applying Theorem 1.12 to each interval, we can find a function h in $\mathcal{A}$ such that

$$h(x) = f(x) \text{ for all } x \in \sigma(H)$$

and $h(x) \neq \lambda$ for all $x \in \mathbb{R}$. Moreover, since $(f - h)$ has a compact support in $\rho(H)$, then it follows from Lemma 1.10 (iii) that $h(H) = f(H)$. ∎

2. Bounded Operators

We let B be a bounded operator satisfying hypothesis (1.2). Moreover, let

$$u := \sup \sigma(B) \quad \text{and} \quad l := \inf \sigma(B).$$

Lemma 2.1. *For any $f \in \mathcal{A}$ and $\epsilon > 0$*

$$f\Psi_{[l',u'],\epsilon}(B) = f(B),$$

where $l' < l$ and $u' > u$.

P r o o f. Suppose f has a compact support, then $f - f\Psi_{[l',u'],\epsilon}$ has a compact support disjoint from the spectrum of B, hence, by Lemma 1.10 (iii), the statement of the lemma is true for functions in $C_0^\infty(\mathbb{R})$. The statement for all $f \in \mathcal{A}$ follows from the density of $C_0^\infty(\mathbb{R})$ in $\mathcal{A}$. ∎

Lemma 2.2. *Let $f \in \mathcal{A}$. If $\epsilon > 0$ and*

$$D_\epsilon := \{z : |z - \tfrac{u+l}{2}| < \tfrac{u-l}{2} + \epsilon\} \quad \textit{and} \quad \partial D_\epsilon := \{z : |z - \tfrac{u+l}{2}| = \tfrac{u-l}{2} + \epsilon\},$$

then

$$f(B) = \frac{1}{2\pi i} \int\limits_{\partial D_\epsilon} \tilde{f}(z)(z-B)^{-1}\, dz - \frac{1}{\pi} \int\limits_{D_\epsilon} \frac{\partial \tilde{f}}{\partial \overline{z}} (z-B)^{-1}\, dxdy.$$

P r o o f. By Lemma 2.1, we can assume that f has a compact support in $[l-\epsilon, u+\epsilon]$.
If $R > \frac{u-l}{2} + \epsilon$ and A_R is the annulus $\{z : \frac{u-l}{2} + \epsilon < |z - \frac{u+l}{2}| < R\}$, then

$$\int\limits_{|z-\frac{u+l}{2}|<R} \frac{\partial \tilde{f}}{\partial \overline{z}} (z-B)^{-1}\, dxdy = \int\limits_{A_R} \frac{\partial \tilde{f}}{\partial \overline{z}} (z-B)^{-1}\, dxdy + \int\limits_{D_\epsilon} \frac{\partial \tilde{f}}{\partial \overline{z}} (z-B)^{-1}\, dxdy.$$

Applying Stokes' theorem

$$\int\limits_{A_R} \frac{\partial \tilde{f}}{\partial \overline{z}} (z-B)^{-1}\, dxdy = \frac{1}{2i} \int\limits_{|z-\frac{u+l}{2}|=R} \tilde{f}(z-B)^{-1}\, dz - \frac{1}{2i} \int\limits_{\partial D_\epsilon} \tilde{f}(z-B)^{-1}\, dz$$

and letting R be large enough for $\tilde{f}$ to vanish on $\{z : |z - \frac{u+l}{2}| = R\}$ completes the proof. ∎

Lemma 2.3. *Let $\epsilon > 0$. If $l' < l$ and $u' > u$, then*

$$\Psi_{[l',u'],\epsilon}(B) = 1.$$

P r o o f. Let $0 < \delta < 1$, and define Ω as the open rectangle

$$\{z \in \mathbb{C} : |\mathrm{Re}\, z - \tfrac{u'+l'}{2}| < \tfrac{u'-l'}{2}, \quad |\mathrm{Im}\, z| < \delta\},$$

as illustrated in Fig. 1. Using a similar argument to that given in the proof of Lemma 2.2, we see that

$$\Psi_{[l',u'],\epsilon}(B) = \frac{1}{2\pi i} \int\limits_{\partial\Omega} \widetilde{\Psi}_{[l',u'],\epsilon}(z,\overline{z})(z-B)^{-1}\, dz - \frac{1}{\pi} \int\limits_{\Omega} \frac{\partial \widetilde{\Psi}_{[l',u'],\epsilon}}{\partial \overline{z}} (z-B)^{-1}\, dxdy.$$

When $l' \leq x \leq u'$, then $\Psi_{[l',u'],\epsilon}(x) = 1$. Moreover, when $l' \leq x \leq u'$, then $\Psi^{(n)}_{[l',u'],\epsilon}(x) = 0$ for all $n > 0$. Recalling definition (1.4), we can see that

$$\widetilde{\Psi}_{[l',u'],\epsilon}(z, \overline{z}) = 1 \quad \text{for all } z \in \Omega,$$

hence

$$\Psi_{[l',u'],\epsilon}(B) = \frac{1}{2\pi i} \int\limits_{\partial\Omega} (z - B)^{-1} \, dz$$

and we conclude with an application of Cauchy's integral formula.

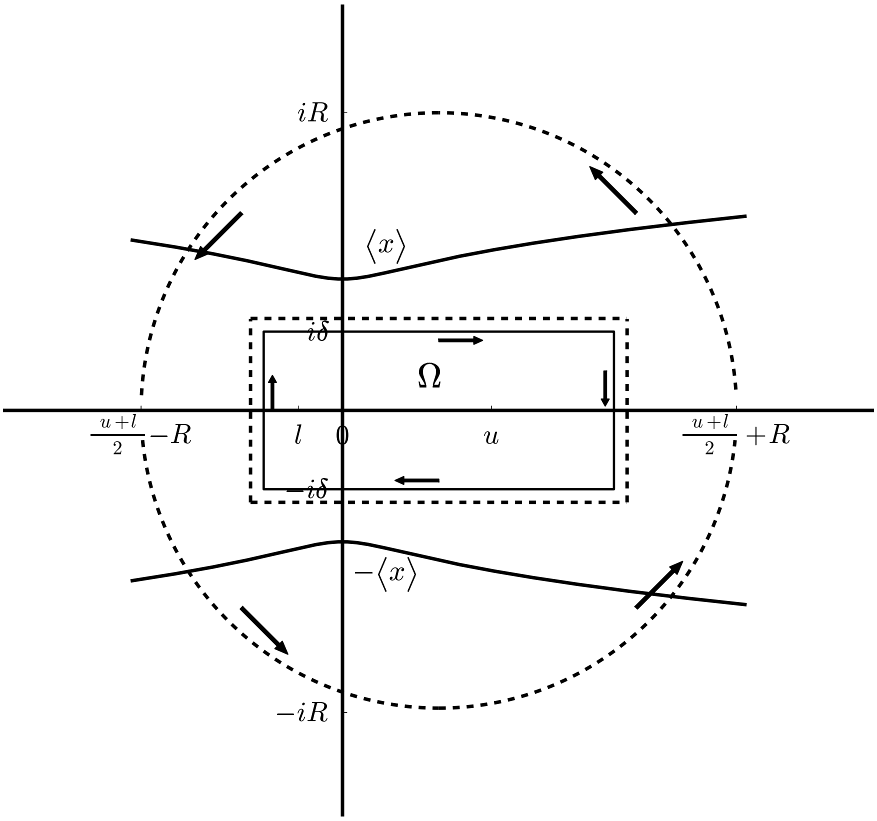

Fig. 1. Integral domain for Lemma 2.3.

■

3. Enlargement of $\mathcal{A}$

We extend the algebra $\mathcal{A}$ of slow decaying functions in a trivial but necessary way. The purpose of the extension is to provide a multiplicative identity, the constant 1 function.

Definition 3.1. *Let*

$$\hat{\mathcal{A}} := \{(z, f) : z \in \mathbb{C}, f \in \mathcal{A}\},$$

where for each $x \in \mathbb{R}$ we define

$$(z, f)(x) := z + f(x).$$

Moreover, we define the point-wise addition and multiplication:

$$\begin{aligned} (\omega, f) \circ (z, g) &:= (\omega z, \omega g + zf + fg), \\ (\omega, f) + (z, g) &:= (\omega + z, f + g). \end{aligned}$$

It is clear that $(1, 0)$, the multiplicative identity, and $(0, 0)$, the additive identity, are in $\hat{\mathcal{A}}$, and the algebra is closed under these operations.
For any $z \in \mathbb{C}$ we will denote $(z, 0) \in \hat{\mathcal{A}}$ simply by z.
Given $\phi = (z, f) \in \hat{\mathcal{A}}$, let

$$\pi_{\mathcal{A},\phi} := f \quad \text{and} \quad \pi_{C,\phi} := z$$

and let

$$\|\phi\|_n := |\pi_{C,\phi}| + \|\pi_{\mathcal{A},\phi}\|_n.$$

Definition 3.2. *We have the extended functional calculus. For $\phi \in \hat{\mathcal{A}}$, let*

$$\phi(H) := \pi_{\mathcal{A},\phi}(H) + \pi_{C,\phi} I$$

along with the implied norm

$$\begin{aligned} \|\phi(H)\| &:= |\pi_{C,\phi}| + \|\pi_{\mathcal{A},\phi}(H)\| \\ &\leq |\pi_{C,\phi}| + k\|\pi_{\mathcal{A},\phi}\|_{n+1} \\ &\leq k\|\phi\|_{n+1} \end{aligned}$$

for some $k > 1$.

Definition 3.3. *For $\phi \in \hat{\mathcal{A}}$, let*

$$\mu(\phi) := \frac{1}{\pi_{C,\phi} + \pi_{\mathcal{A},\phi}} - \frac{1}{\pi_{C,\phi}}.$$

Lemma 3.4. *If $\phi \in \hat{\mathcal{A}}$ and $-\pi_{C,\phi}$ is not in $\overline{\mathrm{Ran}(\pi_{\mathcal{A},\phi})}$, then $\mu(\phi)$ is in $\mathcal{A}$, and*

$$\phi^{-1} = \left(\frac{1}{\pi_{C,\phi}}, \mu(\phi)\right).$$

P r o o f. By re-writing

$$\mu(\phi) = \frac{1}{\pi_{C,\phi} + \pi_{\mathcal{A},\phi}} - \frac{1}{\pi_{C,\phi}} = \frac{-\pi_{\mathcal{A},\phi}}{\pi_{C,\phi}\left(\pi_{C,\phi} + \pi_{\mathcal{A},\phi}\right)},$$

then it is routine exercise in differentiation to show that

$$\frac{-1}{\pi_{C,\phi}\left(\pi_{C,\phi} + \pi_{\mathcal{A},\phi}\right)}$$

is in S^0. Then, since $\pi_{\mathcal{A},\phi}$ is in $\mathcal{A}$, Lemma 1.8 implies the statement. ■

Corollary 3.5. *Given $\phi = (z, f) \in \hat{\mathcal{A}}$ and $\lambda \in \mathbb{C}$ such that $-(z-\lambda)$ is not in the closure of the range of f, then we have*

$$(\phi - \lambda)^{-1} \in \hat{\mathcal{A}}.$$

4. Spectral Mapping Theorem

Lemma 4.1. *If ϕ is in $\hat{\mathcal{A}}$, then*

$$\sigma(\phi(H)) \subseteq \overline{\mathrm{Ran}(\phi)}.$$

P r o o f. Given $\lambda \in \mathbb{C}$ which is not in $\overline{\mathrm{Ran}(\phi)}$, we have by Corollary 3.5

$$(\phi - \lambda)^{-1} \in \hat{\mathcal{A}},$$

hence $(\phi(H) - \lambda)^{-1}$ exists and is bounded and therefore $\lambda \notin \sigma(\phi(H))$. ■

Lemma 4.2. *If ϕ is in $\hat{\mathcal{A}}$, then*

$$\sigma(\phi(H)) \subseteq \phi(\sigma(H)) \cup \{\pi_{\mathbb{C},\phi}\}.$$

P r o o f. Let $\lambda \in \mathbb{C}$ be such that $\lambda \neq \pi_{\mathbb{C},\phi}$ and let

$$A_\lambda = \{x : \phi(x) = \lambda\}.$$

If $A_\lambda \cap \sigma(H) = \emptyset$, then by Lemma 1.13, we have that there is function h in $\mathcal{A}$ such that

$$h(x) = \pi_{\mathcal{A},\phi}(x) \text{ for all } x \in \sigma(H),$$

and

$$h(x) \neq \lambda - \pi_{\mathbb{C},\phi} \text{ for all } x \in \mathbb{R},$$

moreover,

$$h(H) = \pi_{\mathcal{A},\phi}(H).$$

Since $\theta := \left(\pi_{\mathbb{C},\phi}, h\right) \in \hat{\mathcal{A}}$, it follows from the definition of the enlargement of the algebra that

$$\phi(H) = \theta(H).$$

Since $\lambda \notin \overline{\mathrm{Ran}(\theta)}$, the statement of the lemma follows from Lemma 4.1. ■

Lemma 4.3. *Let $\phi \in \hat{\mathcal{A}}$. If H is bounded and*

$$\{x : \phi(x) = \pi_{\mathbb{C},\phi}\} \cap \sigma(H) \text{ is empty },$$

then $\pi_{\mathbb{C},\phi} \notin \sigma(\phi(H))$.

P r o o f. Let $u := \sup \sigma(H)$ and $l := \inf \sigma(H)$.
Let $0 < \epsilon \ll 1$ such that $\pi_{\mathcal{A},\phi}$ is not zero on $[l-\epsilon, l]$ and on $[u, u+\epsilon]$.
Then let $u' := u + \epsilon$ and $l' := l - \epsilon$.
The set

$$\{x \in [l', u'] : \pi_{\mathcal{A},\phi}(x) = 0\}$$

can be covered by a finite number of disjoint intervals $[a_i, b_i]$ which are all disjoint from $\sigma(H)$ and are all in $[l', u']$. Applying Lemma 1.12 to each $[a_i, b_i]$, we can find a function $f \in \mathcal{A}$ such that

$$\{x \in [l', u'] : f(x) = 0\} = \emptyset$$

and $f = \pi_{\mathcal{A},\phi}$ for all x in $\mathbb{R}/[l', u']$.
Let g be any function in $\mathcal{A}$ such that $g(x) = \frac{1}{f(x)}$ for all $x \in [l', u']$.
By Lemma 1.10 (iii), we have

$$\pi_{\mathcal{A},\phi}(H)\, g(H) = f(H)\, g(H)$$

and by Lemma 2.1, we have

$$f(H)\, g(H) = (fg\Psi_{[l',u'],\epsilon})(H) = \Psi_{[l',u'],\epsilon}(H),$$

hence by Lemma 2.3, we have

$$\pi_{\mathcal{A},\phi}(H)\, g(H) = 1$$

and, consequently,

$$\left(-\pi_{\mathbb{C},\phi} + \phi(H)\right) g(H) = 1.$$

■

Theorem 4.4. *If ϕ in $\hat{\mathcal{A}}$, then $\sigma(\phi(H)) \subseteq \overline{\phi(\sigma(H))}$.*

P r o o f. If H is unbounded, then $\overline{\phi(\sigma(H))} = \phi(\sigma(H)) \cup \{\pi_{\mathbb{C},\phi}\}$ and the theorem follows from Lemma 4.2. If H is bounded and there is an $x \in \sigma(H)$ such that $\phi(x) = \pi_{\mathbb{C},\phi}$, then $\overline{\phi(\sigma(H))} = \phi(\sigma(H)) \cup \{\pi_{\mathbb{C},\phi}\}$ and again the theorem follows from 4.2. If H is bounded and

$$\phi(x) \neq \pi_{\mathbb{C},\phi}$$

for all $x \in \sigma(H)$, then $\overline{\phi(\sigma(H))} = \phi(\sigma(H))$ by Lemmas 4.2 and 4.3. ■

Lemma 4.5. *Given $s \in \mathbb{R}$ and a function $f \in \mathcal{A}$, let $k_s(x) := \left(1, -\frac{s+i}{x+i}\right) \in \hat{\mathcal{A}}$ and let the function g_s be defined as in Lemma 1.7, then*

$$(f(H) - f(s))(H+i)^{-1} = g_s(H)\, k_s(H).$$

P r o o f. This statement follows directly from the functional calculus and the observation

$$(-f(s), f(x))\left(0, (x+i)^{-1}\right) = \left(0, \frac{f(x) - f(s)}{x-s}\right)\left(1, -\frac{s+i}{x+i}\right).$$

■

Theorem 4.6. *Let f be a function in $\mathcal{A}$, then*

$$\overline{f(\sigma(H))} \subseteq \sigma(f(H)).$$

P r o o f. We observe the identity

$$H - x = (H+i) - (x+i) = \left(1 - (x+i)(H+i)^{-1}\right)(H+i) = k_x(H)(H+i) \tag{4.1}$$

for some $x \in \mathbb{R}$. Let $s \in \mathbb{R}$. Suppose there is a sequence of unit norm vectors $\{v_m\} \subset Dom(H)$ such that $\lim_{m\to\infty}(H-s)v_m = 0$. Using identity (4.1), we have $\lim_{m\to\infty} g_s(H)\, k_s(H)(H+i)\, v_m = 0$. By applying Lemma 4.5, we can conclude that $\lim_{m\to\infty}(f(H) - f(s))\, v_m = 0$. The accumulation points of $f(\sigma(H))$ are in $\sigma(f(H))$ since the latter is closed. ■

R e m a r k 4.7. In the proof of Theorem 4.6, $\sigma(H)$ is equal to the approximate point spectrum of H and it is proved that if s is in the approximate point spectrum of H, then $f(s)$ is in the approximate point spectrum of $f(H)$.

Corollary 4.8. *Let ϕ be a function $\hat{\mathcal{A}}$, then*

$$\overline{\phi(\sigma(H))} \subseteq \sigma(\phi(H)).$$

5. Self-Adjoint Operators

We now assume that H is self-adjoint and $\mathcal{B}$ is a Hilbert space. The following theorem of Davies extends the Davies–Helffer–Sjöstrand functional calculus to $C_0(\mathbb{R})$ for self-adjoint operators.

Theorem 5.1. (Davies [2] Theorem 9) *The functional calculus may be extended to a map from $f \in C_0(\mathbb{R})$ to $f(H) \in \mathcal{L}(\mathcal{B})$ with the following properties:*

i. $f \to f(H)$ *is an algebra homomorphism.*

ii. $\overline{f}(H) = f(H)^*$.

iii. $\|f(H)\| \leq \|f\|_\infty$.

iv. If $z \notin \mathbb{R}$ *and* $g_z(x) := (z-x)^{-1}$ *for all* $x \in \mathbb{R}$, *then* $g_z(H) = (z-H)^{-1}$.

Moreover, the functional calculus is unique subject to these conditions.

Lemma 5.2. *If* $f \in C_0(\mathbb{R})$, *then*

$$\overline{f(\sigma(H))} \subseteq \sigma(f(H)).$$

P r o o f. This is a consequence of the density of $\mathcal{A}$ in $C_0(\mathbb{R})$. By the Stone–Weierstrass theorem, the linear subspace

$$\{\sum_{i=1}^{n} \tfrac{\lambda_i}{x-\omega_i} : \lambda_i \in \mathbb{C} \quad \omega_i \notin \mathbb{R}\}$$

is dense in $C_0(\mathbb{R})$. If $f_\epsilon \in \mathcal{A}$ is close to f and if $v \in \mathcal{B}$ is of norm 1, then

$$\|f(H)v - f(s)v\| \leq \|f(H) - f_\epsilon(H)\| + \|f_\epsilon(H)v - f_\epsilon(s)v\| + \|f_\epsilon - f\|_\infty.$$

The statement then follows from Lemma 5.1 (iii). ■

Lemma 5.3. *If* $f \in C_0(\mathbb{R})$, *then*

$$\sigma(f(H)) \subseteq \overline{f(\sigma(H))}.$$

P r o o f. Let f_n be a sequence converging to f in $C_0(\mathbb{R})$ such that

$$f_n(x) := \sum_{i=1}^{n} \tfrac{\lambda_{n,i}}{x-\omega_{n,i}}, \quad \omega_{n,i} \notin \mathbb{R}.$$

The existence of such a sequence follows from the Stone–Weierstrass theorem as explained in the proof of Lemma 5.2.

Suppose $\lambda \in \mathbb{C}$ is not in the closure of $f(\sigma(H))$. Then there is $\delta > 0$ such that

$$\inf_{s \in \sigma(H)} |f(s) - \lambda| = \delta.$$

Also for all large enough n, we have $\|f_n - f\|_\infty < \frac{\delta}{2}$. Then from

$$|f(s) - f_n(s) + f_n(s) - \lambda| \geq \delta$$

we can deduce that

$$|f_n(s) - \lambda| > \delta - \|f_n - f\|_\infty,$$

hence

$$\inf_{s \in \sigma(H)} |f_n(s) - \lambda| > \frac{\delta}{2}$$

and $\lambda \notin \sigma(f_n(H))$.

From the identity

$$\|(f(H) - \lambda)(f_n(H) - \lambda)^{-1} - 1\| = \|(f(H) - f_n(H))(f_n(H) - \lambda)^{-1}\|$$

we can deduce that $\lambda \notin \sigma(f(H))$. ■

6. Functional Calculus for Semi-Bounded Operators

We modify our main hypothesis (1.2) by assuming that the spectrum of H is bounded below and, without loss of generality, $\sigma(H) \subseteq [0, \infty)$. We introduce a new ring of functions $\mathcal{A}^+$.

Definition 6.1. *S_+^β is the set of smooth functions on $\mathbb{R}^+ \cup \{0\}$ with the same decaying property as S^β, that is, for every n there is positive constant c_n such that*

$$\left|\frac{d^n f(x)}{dx^n}\right| \leq c_n \langle x \rangle^{\beta - n}.$$

Then $\mathcal{A}^+$ is defined appropriately and similarly we define the Banach space $\mathcal{A}_n^+$ with norm

$$\|f\|_{\mathcal{A}_n^+} := \sum_{r=0}^{n} \int_0^\infty \left|\frac{d^r f(x)}{dx^r}\right| \langle x \rangle^{r-1} dx. \tag{6.1}$$

We present a theorem due to Seeley [8] which gives a linear extension operator for smooth functions from the half line to the whole line.

Theorem 6.2. (Seeley's Extension Theorem) *There is a linear extension operator*

$$\mathcal{E} : C^{\infty}[0,\infty) \longrightarrow C^{\infty}(\mathbb{R})$$

such that for all $x > 0$

$$(\mathcal{E}f)(x) = f(x).$$

The extension operator is continuous for many topologies including uniform convergence of each derivative. The proof of the theorem relies on the following lemma.

Lemma 6.3. ([8]) *There are sequences $\{a_k\}$, $\{b_k\}$ such that*

i. $b_k < -1$.

ii. $\sum\limits_{k=0}^{\infty} |a_k||b_k|^n < \infty$ *for all non-negative integers n.*

iii. $\sum\limits_{k=0}^{\infty} a_k (b_k)^n = 1$ *for all non-negative integers n.*

iv. $b_k \to -\infty$.

The proof to Seeley's extension theorem is by construction and it is informative to give explicitly the extension. First, we need to define two linear operators.

Definition 6.4. *Given $f \in \mathcal{A}^+$, $\phi \in \mathcal{A}$ and real a, we define*

$$(T_a f)(x) = f(ax),$$

$$(S_\phi f)(x) = \phi(x) f(x).$$

P r o o f. (Proof of Seeley's Extension Theorem.) Let $\phi \in C_c^{\infty}(\mathbb{R})$ such that

$$\phi(x) = \begin{cases} 1, & x \in [0,1], \\ 0, & x \geq 2, \\ 0, & x \leq -1. \end{cases}$$

Then define $\mathcal{E}$ such that

$$(\mathcal{E}f)(x) := \begin{cases} \sum\limits_{k=0}^{\infty} a_k (T_{b_k} S_\phi f)(x), & x < 0, \\ f(x), & x \geq 0. \end{cases}$$

■

Lemma 6.5. *If $a > 1$, then $\|T_a\|_{\mathcal{A}_n^+ \to \mathcal{A}_n^+} \leq a^n$.*

P r o o f. The proof follows from

$$\|T_a f\|_{\mathcal{A}_n^+} = \sum_{r=0}^{n} \int_0^\infty |\frac{d^r f(ax)}{dx^r}| \langle x \rangle^{r-1} dx \leq \sum_{r=0}^{n} a^r \int_0^\infty |\frac{d^r f(x)}{dx^r}| \langle x \rangle^{r-1} dx.$$

■

Lemma 6.6. *If $\phi \in \mathcal{A}$, then S_ϕ is a bounded operator with respect to each norm $\| \, \|_{\mathcal{A}_n^+}$.*

P r o o f. A simple application of Leibniz's rule gives

$$\frac{d^r (\phi(x) f(x))}{dx^r} = \sum_{m=0}^{r} c_r \frac{d^{r-m} (\phi(x))}{dx^{r-m}} \frac{d^m (f(x))}{dx^m},$$

then

$$\begin{aligned} |\frac{d^r (\phi(x) f(x))}{dx^r}| &\leq c_r \sum_{m=0}^{r} d_{r-m,\phi} \langle x \rangle^{\beta-(r-m)} \frac{d^m (f(x))}{dx^m} \\ &\leq c_{r,\phi} \sum_{m=0}^{r} \langle x \rangle^{m-r} \frac{d^m (f(x))}{dx^m} \end{aligned}$$

we integrate to give

$$\begin{aligned} \int_0^\infty |\frac{d^r (\phi(x) f(x))}{dx^r}| \langle x \rangle^{r-1} dx &\leq c_{r,\phi} \sum_{m=0}^{r} \int_0^\infty |\frac{d^m (f(x))}{dx^m}| \langle x \rangle^{m-1} dx \\ &= c_{r,\phi} \|f\|_{\mathcal{A}_r^+}, \end{aligned}$$

and hence we have our estimate

$$\begin{aligned} \|S_\phi f\|_{\mathcal{A}_n^+} &= \sum_{r=0}^{n} \int_0^\infty |\frac{d^r (\phi(x) f(x))}{d^r x}| \langle x \rangle^{r-1} dx \\ &\leq c_{n,\phi} \sum_{r=0}^{n} \|f\|_{\mathcal{A}_r^+} \\ &\leq c_{n,\phi} \|f\|_{\mathcal{A}_n^+}. \end{aligned}$$

■

Theorem 6.7. *For each normed vector space $\mathcal{A}_n^+$ Seeley's Extension Operator is a bounded operator from $\mathcal{A}_n^+$ to $\mathcal{A}_n$.*

P r o o f.

$$\begin{aligned}
\|\mathcal{E}f\|_{\mathcal{A}_n} &= \sum_{r=0}^{n} \int_{-\infty}^{\infty} |\frac{d^r(\mathcal{E}f)}{dx^r}| \langle x\rangle^{r-1} dx \\
&= \sum_{r=0}^{n} \int_{0}^{\infty} |\frac{d^r f(x)}{dx^r}| \langle x\rangle^{r-1} dx + \sum_{r=0}^{n} \int_{-\infty}^{0} |\sum_{0}^{\infty} a_k \frac{d^r(\phi(b_k x) f(b_k x))}{dx^r}| \langle x\rangle^{r-1} dx \\
&= \|f\|_{\mathcal{A}_n^+} + \|\sum_{k=0}^{\infty} a_k T_{-b_k} S_\phi f\|_{\mathcal{A}_n^+} \\
&\leq \|f\|_{\mathcal{A}_n^+} + \sum_{k=0}^{\infty} |a_k| \, \|S_\phi\| \, \||T_{-b_k}\|\|f\|_{\mathcal{A}_n^+} \\
&\leq \|f\|_{\mathcal{A}_n^+} + \left(\sum_{k=0}^{\infty} |a_k| \, |b_k|^n\right) c_{n,\phi} \|f\|_{\mathcal{A}_n^+}
\end{aligned}$$

and hence the extension operator is continuous. ■

If f and g are elements of $\mathcal{A}$ such that $f|_{[0,\infty]} = g|_{[0,\infty]}$ and the spectrum of H is $[0,\infty)$, then it is not necessary that *supp* $(f-g) \cap \sigma(H)$ is empty since $\operatorname{supp}(f-g) \cap \sigma(H) = \{0\}$ is possible.

Lemma 6.8. *If f is a smooth function on $\mathbb{R}$ of a compact support such that*

$$\operatorname{supp}(f) = [-a, 0]$$

and H is an operator satisfying our modified hypothesis with $\sigma(H) \subseteq [0,\infty]$, then

$$f(H) = 0.$$

P r o o f. Let $\epsilon \in (0,1)$. Define

$$f_\epsilon(x) := f(x+\epsilon)$$

so that $\operatorname{supp}(f_\epsilon) = [-(a+\epsilon), -\epsilon]$.

We observe that for all n there are constants $p_n \geq 0$ such that

$$\|\frac{d^n f}{dx^n} - \frac{d^n f_\epsilon}{dx^n}\|_\infty \leq p_n \epsilon.$$

By Lemma 1.10 (iii), we have that $f_\epsilon(H) = 0$, moreover, a further application of Lemma 1.10 implies that for large enough n we have

$$\begin{aligned}
\|f(H)\| &= \|f(H) - f_\epsilon(H)\| \\
&\leq c_n \sum_{r=0}^{n} \int\limits_{-(a+1)}^{0} \left|\frac{d^r f(x)}{dx^r} - \frac{d^r f_\epsilon(x)}{dx^r}\right| \langle x\rangle^{r-1} dx \\
&\leq \epsilon c_n \sum_{r=0}^{n} p_r \int\limits_{-(a+1)}^{0} \langle x\rangle^{r-1} dx \\
&= \epsilon k_{n,f}
\end{aligned}$$

hence our result. ■

Corollary 6.9. *If f and g are in $\mathcal{A}$ such that $f|_{[0,\infty]} = g|_{[0,\infty]}$ and $\sigma(H) \subseteq [0,\infty]$, then $f(H) - g(H) = 0$.*

Theorem 6.10. *If H satisfies our modified hypothesis with spectrum $\sigma(H) \subseteq [0,\infty)$, then there is a functional calculus $\gamma_H : \mathcal{A}^+ \to \mathcal{L}(\mathcal{B})$ such that for all $f \in \mathcal{A}^+ \cap \mathcal{A}$*

$$\gamma_H(f) = -\frac{1}{\pi} \iint\limits_{\mathbb{C}} \frac{\partial \tilde{f}}{\partial \bar{z}} (z - H)^{-1} dxdy.$$

P r o o f. Let $f^+ \in \mathcal{A}^+$, then by Seeley's Extension Theorem there exists an extension $f \in \mathcal{A}$. We define $\gamma_H(f^+) := f(H)$. This definition is independent of the particular extension by Corollary 6.9. The functional analytic properties are inherited from the extension. ■

Theorem 6.11. (Refinement of Theorem 10 of [2]). *Let $n \geq 1$ be an integer and $t > 0$. If we denote the operator $\gamma_H\left(e^{-s^n t}\right)$ by $e^{-H^n t}$, then*

$$e^{-H^n(t_1+t_2)} = e^{-H^n t_1} e^{-H^n t_2}$$

for all $n \geq 1$ and $0 < t \leq 1$.

Acknowledgements. This research was funded by an EPSRC Ph.D grant 95-98 at Kings College, London. I am very grateful to E. Brian Davies for giving me this problem, his encouragement since and for continuing to be a mentor in Mathematics long after having finished supervising my Ph.D. I am indebted to Anita for all her support. I am immensely grateful to the referee for some very helpful comments and suggestions.

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