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An Example of Bianchi Transformation in E^4

V. Gorkavyy

Mathematics Division, B. Verkin Institute for Low Temperature Physics and Engineering
National Academy of Sciences of Ukraine
47 Lenin Ave., Kharkiv 61103, Ukraine

E-mail: gorkaviy@ilt.kharkov.ua

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We describe a particular class of pseudo-spherical surfaces in E^4 which admit Bianchi transformations.

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Introduction

The aim of this note is to describe a particular class of two-dimensional pseudo-spherical surfaces, which admit the Bianchi transformation, in the four-dimensional Euclidean space.

Recall the classical definition of the Bianchi transformation, see [1, 2, 3]. Let F be a pseudo-spherical surface, i.e. a surface of the constant negative Gauss curvature $K \equiv -k^2$, in the three-dimensional Euclidean space E^3. Suppose that F is represented in E^3 by a position-vector $r(\varphi, v)$ in terms of horocyclic coordinates (φ, v), i.e. the metric form of F reads $ds^2 = \frac{1}{k^2} d\varphi^2 + e^{2\varphi} dv^2$. Consider a new surface F^* whose position vector is

$$r^* = r - \partial_\varphi r. \tag{1}$$

It is well known that F^* is pseudo-spherical and has the same Gauss curvature, $K \equiv -k^2$; F^* is called a Bianchi transform of F. Using different horocyclic coordinates and applying the Bianchi transformation, one can construct a one-parameter family of various pseudo-spherical surfaces from a given pseudo-spherical surface. Notice that the Bianchi transformation possesses some exceptional features in terms of geodesic congruences which may be used to suggest a synthetic definition of the Bianchi transformation equivalent to (1).

A direct generalization of the classical theory of Bianchi transformations to the case of n-dimensional pseudo-spherical submanifolds in the $(2n-1)$-dimensional Euclidean space was suggested and described by Yu. Aminov in [4], see also [1, 2, 5].

On the other hand, the question of how to extend the concept of the Bianchi transformation to the case of n-dimensional pseudo-spherical submanifolds in N-dimensional Euclidean spaces with arbitrary $n \geq 2$, $N \geq 2n$ remains unsolved. This open problem was supplied by Yu. Aminov and A. Sym in [6], and this is just what motivated our results in this note.

In the simplest non-trivial case of $n = 2$, $N = 4$, if one asks to extend the Bianchi transformation to the case of two-dimensional surfaces in the four-dimensional Euclidean space E^4, a reasonable way is to accept the same formula (1) in order to construct a new surface F^* from a given pseudo-spherical surface $F \subset E^4$. Naturally, F^* is called a Bianchi transform of F provided that F^* is pseudo-spherical. However, it turns out that generically F^* is not pseudo-spherical and thus a generic pseudo-spherical surface in E^4 does not admit Bianchi transforms [6].

Pseudo-spherical surfaces in E^4 admitting Bianchi transforms were described in [7] in terms of solutions of some particular system of partial differential equations GCR, which may be viewed as a generalization of the sine-Gordon equation. The description deals with the fundamental forms of surfaces. However, no parametric representations for such particular pseudo-spherical surfaces were derived and no one concrete example was presented. Our note is just aimed to remove this gap.

First, in Sec. 1 we recall the classical construction of the Bianchi trasformation for pseudo-spherical surfaces in E^3. Next, in Sec. 2 we describe a constructive method for producing a pseudo-spherical surface in E^4 from a given pseudo-spherical surface in E^3, such surfaces in E^4 will be referred to as *stretched*. It is proved that an arbitrary stretched pseudo-spherical surface in E^4 admits a Bianchi transform and this Bianchi transform is stretched too. Relations between the stretched pseudo-spherical surfaces in E^4 and the solutions of the mentioned GCR-system of [7] are analyzed in Sec. 3. As consequence, it is shown that there exist pseudo-spherical surfaces in E^4, which are not stretched but admit Bianchi transforms (it should be quite interesting to find an explicit representation for these surfaces). Finally, in Sec. 4 we describe the stretched pseudo-spherical surfaces in E^4 produced from the standard pseudo-sphere (Beltrami surface) in E^3.

1. Classical Theory of Bianchi Transformation

Let $\tilde{F}$ be a regular two-dimensional surface of the constant negative Gauss curvature $K \equiv -k^2$ in E^3. Locally $\tilde{F}$ is parameterized by the horocyclic coordinates $(\varphi,\, v)$ so that its metric form reads $d\tilde{s}^2 = \frac{1}{k^2}d\varphi^2 + e^{2\varphi}dv^2$. From the intrinsic point of view, the coordinate curves $v = \text{const}$ are parallel geodesics, whereas $\varphi = \text{const}$ are horocircles in $\tilde{F}$.

Generically, given a horocyclic coordinate system $(\varphi,\, v)$, one can locally parameterize $\tilde{F}$ by another local coordinate system $(u,\, v)$ so that the coordinate curves $u = \text{const}$ and $v = \text{const}$ form a conjugate net in $\tilde{F}$. Then the metric form reads

$$d\tilde{s}^2 = \frac{1}{k^2}d\varphi(u,v)^2 + e^{2\varphi(u,v)}dv^2, \tag{2}$$

whereas the second fundamental form is diagonalized, $\tilde{b} = \tilde{b}_{11}du^2 + \tilde{b}_{22}dv^2$. Applying the fundamental Codazzi equations, it is easy to show that

$$\tilde{b}_{11} = e^{-\varphi}\partial_u\varphi, \quad \tilde{b}_{22} = -e^{3\varphi}\partial_u\varphi \tag{3}$$

after some rescaling $u \to f(u)$. Moreover, the fundamental Gauss equation reads

$$\partial_{uu}e^{2\varphi} + \partial_{vv}e^{-2\varphi} + 2k^2 = 0. \tag{4}$$

Thus, generically any pseudo-spherical surface in E^3 generates a solution of the nonlinear pde (4). In its turn, due to the classical Bonnet theorem, any solution of (4) generates via (2), (3) a pseudo-spherical surface in E^3 parameterized by conjugate coordinates, whose one family of the coordinate curves is parallel geodesics.

Let $\rho(u,v)$ be the corresponding position vector of $\tilde{F}$. Consider a new surface $\tilde{F}^*$ in E^3 represented by the position vector

$$\rho^* = \rho - \partial_\varphi\rho = \rho - \frac{1}{\partial_u\varphi}\partial_u\rho. \tag{5}$$

It is easy to get that the metric form of $\tilde{F}^*$ reads $d\tilde{s}^{*2} = e^{-2\varphi}du^2 + \frac{1}{k^2}d\varphi^2$. Hence, if $\partial_v\varphi \neq 0$, then the surface $\tilde{F}^*$ is regular and has the constant negative Gauss curvature $K = -k^2$. Thus, $\tilde{F}^*$ is pseudo-spherical and it is called *a Bianchi transform* of $\tilde{F}$.

The described Bianchi transformation of the pseudo-spherical surfaces in E^3 has a number of remarkable geometric properties [3]. From the analytical point of view, it corresponds to the involuting transformation $\varphi(u,v) \to \varphi^*(u,v) = -\varphi(v,u)$ for the solutions of (4). Moreover, it may be interpreted as a particular transformation for the solutions of the sin-Gordon equation.

2. Stretched Pseudo-Spherical Surfaces and Bianchi Transformation

Now, view E^3 as a horizontal hyperplane $x^4 = 0$ in E^4 and hence consider the above surface $\tilde{F}$ as a surface in E^4. Given $\tilde{F} \subset E^3 \subset E^4$, define a new two-dimensional surface F in E^4 by

$$r(u, v) = (\rho(u, v), A\varphi(u, v) + B), \tag{6}$$

where $A \neq 0$, B are constant. Because of (2), the metric form of F is

$$ds^2 = d\tilde{s}^2 + A^2 d\varphi^2 = \left(A^2 + \frac{1}{k^2}\right) d\varphi^2 + e^{2\varphi(u,v)} dv^2. \tag{7}$$

It is easy to show that the Gauss curvature of F is $K = -\dfrac{k}{\sqrt{A^2k^2 + 1}}$. Hence F is pseudo-spherical, and the local coordinates $(\varphi,\ v)$ in F are horocyclic. It should be natural to say that the pseudo-spherical surface $F \subset E^4$ is obtained by *stretching* the pseudo-spherical surface $\tilde{F} \subset E^3 \subset E^4$. Thereby F is referred to as *stretched*, whereas $\tilde{F}$ is called *the base* of F. Evidently, the stretched pseudo-spherical surfaces form a particular class of the pseudo-spherical surfaces in E^4.

Let us apply to F the transformation

$$r^* = r - \partial_\varphi r = r - \frac{1}{\partial_u \varphi} \partial_u r. \tag{8}$$

The vector function r^* represents a new surface F^* in E^4.

Proposition 1. *F^* is a stretched pseudo-spherical surface. Moreover, the base of F^* is the Bianchi transform $\tilde{F}^*$ of the base $\tilde{F}$ of F.*

P r o o f. Due to (6), we have

$$r^* = \left(\rho - \frac{1}{\partial_u \varphi} \partial_u \rho, A\varphi + B - A\right). \tag{9}$$

In view of (5), $\rho^* = \rho - \frac{1}{\partial_u \varphi} \partial_u \rho$ represents exactly the Bianchi transform $\tilde{F}^*$ of $\tilde{F}$. Moreover, the metric form of $\tilde{F}^*$ is $d\tilde{s}^{*2} = e^{-2\varphi} du^2 + \frac{1}{k^2} d\varphi^2$, hence $\varphi^*(u, v) = -\varphi(v, u)$. Therefore, (9) may be rewritten as follows:

$$r^* = (\rho^*, A^*\varphi^* + B^*), \tag{10}$$

where $A^* = -A, B^* = B - A$. Comparing (6) with (10), one can easily conclude that F^* is a stretched pseudo-spherical surface whose base surface is $\tilde{F}^*$. Notice that the Gauss curvature of F^* is still the same, $K = -\dfrac{k}{\sqrt{A^2k^2 + 1}}$, q.e.d.

Thus, any stretched pseudo-spherical surface in E^4 admits a Bianchi transform which is a stretched pseudo-spherical surface too. Besides, the Bianchi transformation of the stretched pseudo-spherical surfaces in E^4 is generated by the classical Bianchi transformation of their base surfaces in E^3.

R e m a r k. The same stretching procedure was applied in [8] to produce two-dimensional pseudo-spherical surfaces, which admit Bianchi transforms, in Riemannian products $M^n \times R^1$, where M^n is the sphere S^n or the Lobachevski space H^n. It turns out that a pseudo-spherical surface in $M^3 \times R^1$ admits a Bianchi transform if and *only if* it is either a stretched surface or a hypersurface in a horizontal slice $M^3 \times \{h_0\} \subset M^3 \times R^1$. As we will see in the next section, this is not true for the case of $R^3 \times R^1$, i.e. if $M^3 = E^3$.

3. Stretched Pseudo-Spherical Surfaces and Solutions of the GCR-System

Pseudo-spherical surfaces in E^4 admitting Bianchi transforms were described in [7]. Roughly speaking, a pseudo-spherical surface with $K \equiv -1$ in E^4, which is not a hypersurface in any hyperplane $E^3 \subset E^4$, admits a Bianchi transform if and only if it can be parameterized in such a way that its fundamental forms read

$$ds^2 = d\varphi^2 + e^{2\varphi} dv^2, \tag{11}$$

$$II^1 = e^{-\varphi}\partial_u\varphi du^2 - e^{3\varphi}\partial_u\varphi dv^2, \quad II^2 = e^{\varphi} P dv^2, \tag{12}$$

$$\mu_{12} = Q du, \tag{13}$$

where the functions $\varphi(u,v)$, $P(u,v)$ and $Q(u,v)$ satisfy the Gauss–Codazzi–Ricci equations

$$\partial_{uu} e^{2\varphi} + \partial_{vv} e^{-2\varphi} + 2(PQ+1) = 0, \tag{14}$$

$$\partial_u P - Q e^{2\varphi}\partial_u\varphi = 0, \tag{15}$$

$$\partial_v Q + P e^{-2\varphi}\partial_v\varphi = 0 \tag{16}$$

and the regularity conditions

$$\partial_u\varphi \neq 0, \quad \partial_v\varphi \neq 0, \quad P \neq 0, \quad Q \neq 0. \tag{17}$$

Due to the classical Bonnet theorem, any solution $\{\varphi, P, Q\}$ of the GCR-system (14)–(17) generates a pseudo-spherical surface with $K \equiv -1$ in E^4 which admits a Bianchi transform.

Since the stretched pseudo-spherical surfaces in E^4 admit Bianchi transforms, they correspond to some particular solutions of (14)–(17).

Proposition 2. *The stretched pseudo-spherical surface F in E^4, represented by (6) with $A = \frac{\sqrt{k^2-1}}{k}$, has the following fundamental forms:*

$$ds^2 = d\varphi^2 + e^{2\varphi} dv^2, \tag{18}$$

$$II^1 = e^{-\varphi} \partial_u \varphi \, du^2 - e^{3\varphi} \partial_u \varphi \, dv^2, \quad II^2 = e^{2\varphi} \sqrt{k^2-1} \, dv^2, \tag{19}$$

$$\mu_{12} = e^{-\varphi} \sqrt{k^2-1} \, du. \tag{20}$$

P r o o f. Set $A = \frac{\sqrt{k^2-1}}{k}$ in (6). Then (7) implies (18).

Differentiate (6) and write the vectors tangent to F

$$\partial_u r = \left(\partial_u \rho, \frac{\sqrt{k^2-1}}{k} \partial_u \varphi \right), \quad \partial_v r = \left(\partial_v \rho, \frac{\sqrt{k^2-1}}{k} \partial_v \varphi \right). \tag{21}$$

The normal plane to $F \subset E^4$ is spanned by the following unit vectors:

$$N_1 = (n, 0)\,, \quad N_2 = \left(-\frac{\sqrt{k^2-1}}{\partial_u \varphi} \partial_u \rho, \frac{1}{k} \right), \tag{22}$$

where n is the unit vector normal to the base surface $\tilde{F} \subset E^3 \subset E^4$. Differentiate (21) and find the second fundamental forms $II^\sigma = \langle d^2 r, N_\sigma \rangle$ of F with respect to the normal frame (22); this yields (19).

Finally, differentiate (22) and find $\mu_{12} = \langle dN_1, N_2 \rangle$; this proves (20), q.e.d.

Comparing (18)–(20) with (11)–(13), we can see that the stretched pseudo-spherical surface F corresponds to the solution

$$\left\{ \varphi, \quad P = \sqrt{k^2-1} e^{\varphi}, \quad Q = \sqrt{k^2-1} e^{-\varphi} \right\}, \tag{23}$$

whereas $\varphi(u, v)$ is determined by the base $\tilde{F}$ and solves equation (11) which reduces to (4). This solution was presented in the formula (32) of [7], where one has to set $c_1 = 0$, $c_2 = \sqrt{k^2-1}$.

Notice that (11), (12) has other solutions different from (23), see [7]. It means that there are pseudo-spherical surfaces in E^4 which are neither stretched surfaces nor hypersurfaces in $E^3 \subset E^4$ but admit Bianchi transforms.

4. An Example: Stretched Pseudo-Spheres in E^4

Let $\tilde{F} \subset E^3$ be a pseudo-sphere represented by the position vector

$$\rho(\varphi, v) = (e^{\varphi} \cos v, e^{\varphi} \sin v, \Psi)\,,$$

where $\Psi(\varphi)$ satisfies $(\Psi')^2 + e^{2\varphi} = \frac{1}{k^2}$, hence

$$\Psi = \pm\frac{1}{k}\left(\sqrt{1-k^2e^{2\varphi}} + \frac{1}{2}\ln(1-\sqrt{1-k^2e^{2\varphi}}) - \frac{1}{2}\ln(1+\sqrt{1-k^2e^{2\varphi}})\right). \quad (24)$$

The local coordinates (φ, v) in $\tilde{F}$ are horospherical since $d\tilde{s}^2 = \frac{1}{k^2}d\varphi^2 + e^{2\varphi}dv^2$. However, if we apply the Bianchi transformation (1), then the transformed surface $\tilde{F}^*$ degenerates to a curve (the axis of rotation of $\tilde{F}$). So we need some other horocyclic coordinates in F. Such coordinates are given by

$$\varphi = -\ln\left(\frac{2e^{-\hat{\varphi}}}{k^2\hat{v}^2 + e^{-2\hat{\varphi}}}\right), \quad v = \frac{2\hat{v}}{k^2\hat{v}^2 + e^{-2\hat{\varphi}}}. \quad (25)$$

In fact, it is easy to verify that the metric form of $\tilde{F}$ reads $d\tilde{s}^2 = \frac{1}{k^2}d\hat{\varphi}^2 + e^{2\hat{\varphi}}d\hat{v}^2$, so the local coordinates $(\hat{\varphi}, \hat{v})$ in $\tilde{F}$ are horocyclic.

Taking $\tilde{F}$ as the base, a stretched pseudo-spherical surface F in E^4 is represented by the position vector

$$r(\hat{\varphi}, \hat{v}) = \left(e^{\varphi}\cos v, e^{\varphi}\sin v, \Psi, A\hat{v} + B\right),$$

where $\varphi(\hat{\varphi}, \hat{v})$, $v(\hat{\varphi}, \hat{v})$, $\Psi(\varphi(\hat{\varphi}, \hat{v}))$ are explicitly given by (24), (25), and $A \neq 0$, B are arbitrary constants. In terms of the original coordinates (φ, v), the stretched surface F is represented by

$$r(\varphi, v) = \left(e^{\varphi}\cos v, e^{\varphi}\sin v, \Psi(\varphi), A\ln\left(\frac{1+v^2e^{2\varphi}k^2}{2}\right) + B\right).$$

This surface in E^4 should be called a stretched pseudo-sphere (a stretched Beltrami surface). Applying the Bianchi transformation, one may obtain a new sequence of the stretched pseudo-spherical surfaces in E^4.

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