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Rate of Decay of the Bernstein Numbers

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We show that if a Banach space X contains uniformly complemented ℓ_2^n's then there exists a universal constant $b = b(X) > 0$ such that for each Banach space Y, and any sequence $d_n \downarrow 0$ there is a bounded linear operator $T : X \to Y$ with the Bernstein numbers $b_n(T)$ of T satisfying $b^{-1}d_n \le b_n(T) \le bd_n$ for all n.

Key words: B-convex space, Bernstein numbers, Bernstein pair, uniformly complemented ℓ_2^n, superstrictly singular operator.

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To the memory of M.I. Kadets

1. Introduction and Main Result

Let X, Y be Banach spaces and let $\mathcal{L}(X,Y)$ be the space of all bounded linear operators from X to Y. Notationally, all spaces are infinite dimensional real Banach spaces unless otherwise specified.

Definition 1. *An operator $T \in \mathcal{L}(X,Y)$ is called* superstrictly singular (SSS *for short;* finitely strictly singular *in other terminology*) *if there are no number $c > 0$ and no sequence of subspaces $E_n \subset X$, $\dim E_n = n$, such that*

$$\|Tx\| \ge c\|x\| \quad \textit{for all } x \textit{ in } \cup_n E_n\,. \tag{1}$$

Put for an operator T

$$b_n(T) = \sup \min_{x \in S_E} \|Tx\|\,, \tag{2}$$

where supremum is taken over all n-dimensional subspaces $E \subset X$ and S_E is the unit sphere of E. Evidently,

$$\|T\| = b_1(T) \ge b_2(T) \ge \cdots \ge 0\,,$$

T is SSS if and only if

$$b_n(T) \to 0 \quad \text{as} \quad n \to \infty$$

and the greatest constant c for which (1) is satisfied, is equal to $\lim_{n\to\infty} b_n(T)$.

Obviously, every compact operator is SSS and T has finite rank if and only if $b_n(T) = 0$ beginning with some integer n. Observe, that if T has infinite rank then for each n the set $\mathcal{I}_n(T)$ of all n-dimensional subspaces E such that $T|_E$ are injective, is dense in the set of all n-dimensional subspaces. Then the formula (2) turns into the following one

$$b_n(T) = \sup_{E \in \mathcal{I}_n(T)} \frac{1}{\|(T|_E)^{-1}\|}. \tag{3}$$

The $b_n(T)$, which are called the *Bernstein numbers*, were considered in Approximation and Operator Theory. The constants $b_n(T)$ show how small is the T-image of the unit sphere S_X. For a compact operator T in a Hilbert space H they coincide with *s-numbers* which are defined as eigenvalues of the operator $(T^*T)^{1/2}$. There are several generalizations of s-numbers to Banach spaces (see below for details).

The Bernstein numbers take origin (see Whitley [26]) in the following classical inequalities:

If p_n is a polynomial of degree at most n, then for its derivative

$$\|p_n'\| \le n^2 \|p_n\|,$$

the norm being the supremum norm on $[-1, 1]$ (Markov [13]).

If q_n is a complex trigonometric polynomial of degree at most n, then

$$\|q_n'\| \le n\|q_n\|,$$

the norm being the supremum norm on the unite circle (Bernstein [2]).

Both of these inequalities have the same form: A Banach space, a derivation operator D and an $(n+1)$-dimensional subspace F are given. The conclusion estimates the value of $\|D|_F\|$. From this point of view it is natural to ask to what extent the norm depend on F. In particular, what improvement is possible, i.e. what is the best possible constant

$$\inf\{\|D|_F\| : \dim F = n\}\,?$$

It appears that this constant is equal to n [26]. Considering the inverse of D we arrive to the notion of the Bernstein numbers. We find $b_n(T)$ as far as in (Krein/Krasnoselskiĭ/Milman [11]). After (Mitiagin/Henkin [16]), SSS operators were introduced implicitly by Mitiagin and Pełczyński [17] and explicitly, under the name "operators of the class C_0^*", by Milman [15].

The important role has been played by Pietsch's paper [19] where systematic theory of abstract s-numbers in Banach spaces was developed (see also [20]). In particular, Pietsch noted the importance of duality and of the principle of local reflexivity. The term "superstrictly singular operator" was introduced in (Hinrichs/Pietsch [7]), where this class was investigated by machinery of superideals, and by Mascioni [14]. For further progress in the theory of SSS operators in general Banach spaces see e.g. (Plichko [24]) and (Flores/Hernández/Raynaud [6]).

As we noted, an operator T is SSS if and only if $b_n(T) \downarrow 0$. One can pose an "inverse" problem. Let X, Y be Banach spaces and $d_n \downarrow 0$. Does there exist $T \in \mathcal{L}(X,Y)$ such that $b_n(T) = d_n$ for every n? We have a little chance to obtain a positive answer. So, we will consider a weaker question which is natural in a more general setting.

According to Pietsch [21], a map s which assigns to each bounded linear operator T between Banach spaces a unique sequence $(s_n(T))$, is called an *s-function* if for all Banach spaces W, X, Y, Z:

1. $\|T\| = s_1(T) \geq s_2(T) \geq \cdots \geq 0$ for all $T \in \mathcal{L}(X,Y)$.
2. $s_n(S+T) \leq s_n(S) + \|T\|$ for all $S, T \in \mathcal{L}(X,Y)$ and all n.
3. $s_n(RST) \leq \|R\| s_n(S) \|T\|$ for all $T \in \mathcal{L}(X,Y)$, $S \in \mathcal{L}(Y,Z)$ and $R \in \mathcal{L}(Z,W)$.
4. If $T \in \mathcal{L}(X,Y)$ and rank $T < n$, then $s_n(T) = 0$.
5. $s_n(I) = 1$ for all n, where I is the identity map of ℓ_2^n.

The scalar $s_n(T)$ is called the *nth s-number* of the operator T. The Bernstein numbers are s-numbers. Another example of s-numbers are the *approximation numbers* defined by the formula

$$a_n(T) = \inf\{\|T - L\| : L \in \mathcal{L}(X,Y),\ \text{rank } L < n\}.$$

These numbers are connected with the well known approximation property of Banach spaces and characterize the ideal of approximable operators: $a_n(T) \to 0$ if and only if T is approximable. The approximation numbers are the largest s-numbers [19].

Aksoy and Lewicki [1] have introduced the following general concept.

Definition 2. *Banach spaces X and Y are said to form a* Bernstein pair with respect to s-numbers s_n *if for any sequence $d_n \downarrow 0$, there exists $T \in \mathcal{L}(X,Y)$ such* *that* $(s_n(T))$ *is equivalent to* (d_n)*, i.e. there is a constant* b *depending only on* T *such that for every* n

$$b^{-1}d_n \le s_n(T) \le bd_n\,.$$

This definition was motivated by well known Bernstein's "lethargy" theorem [4] and is a generalization of Bernstein pair with respect to the approximation numbers (see Hutton/Morell/Retherforsd [8, 9]). Note that Hutton, Morell and Retherforsd implicitly refereed Bernstein's lethargy theorem to [3]. In [8, 9] it was proved that many pairs of classical Banach spaces form the Bernstein pair with respect to the approximation numbers. The authors advanced a hypothesis that all couples of Banach spaces form Bernstein pairs (with respect to approximation numbers). Aksoy and Lewicki [1] showed that many classical Banach spaces form Bernstein pairs with respect to all s-numbers. Detailed investigations of "rate of decay" of many s-numbers (Kolmogorov, Gelfand, Weyl, Hilbert,... numbers) was carried out by Oikhberg [18]. We consider a similar question for the Bernstein numbers. Ideal properties of the Bernstein numbers was considered by Samarskiĭ [25] and Pietsch [21].

First, we present simple examples of pairs (X, Y) which are not Bernstein with respect to the Bernstein numbers. They are, in fact, well known (see e.g. Mitiagin/Pełczyński [17]).

For a subspace E of a Banach space X denote by $\lambda(E, X)$ the *relative projection constant*

$$\lambda(E, X) = \inf \|P\|\,,$$

where inf is taken over all projections P of X onto E. Given a Banach space X put

$$p_n(X) = \inf\{\lambda(E, X) : E \subset X\,,\ \dim E = n\}.$$

Note that one can take infimum here only over a dense subset of all n-dimensional subspaces.

Proposition 1. *Let* $T \in \mathcal{L}(X, H)$*, where* H *is a Hilbert space and* $\dim T(X) = \infty$*. Then for every* n

$$b_n(T) \le \frac{1}{p_n(X)}\|T\|\,.$$

P r o o f. Let $b > 1$ and $E_b \in \mathcal{I}_n(T)$ be such that $\|(T|_{E_b})^{-1}\| < bb_n(T)$ (see (3)). Take the orthogonal projection Q of H onto $T(E_b)$. Then $P = (T|_{E_b})^{-1}QT$ is a projection of X onto E_b. So

$$\lambda(E_b, X) \le \|P\| \le \|(T|_{E_b})^{-1}\| \cdot \|Q\| \cdot \|T\| < bb_n(T)^{-1}\|T\|.$$

Hence

$$p_n(X) = \inf_{\dim E = n} \lambda(E, X) \le \lambda(E_b, X) \le bb_n(T)^{-1}\|T\|.$$

Since $b > 1$ is arbitrary, this implies Proposition 1. ∎

Corollary 1. *Let an operator $T \in \mathcal{L}(X,Y)$ can be factored through a Hilbert space H : $T = RS$, $R \in \mathcal{L}(X,H)$, $S \in \mathcal{L}(H,Y)$ and $\dim T(X) = \infty$. Then for every n*

$$b_n(T) \leq \frac{1}{p_n(X)} \|R\| \|S\| .$$

P r o o f. Indeed, by Proposition 1,

$$b_n(T) \leq b_n(R)\|S\| \leq \frac{1}{p_n(X)} \|R\| \|S\| . \qquad \blacksquare$$

For operators, factored through Hilbert spaces see [12].

Definition 3. *We say that a Banach space X* contains no uniformly complemented finite-dimensional subspaces *if $p_n(X) \to \infty$ as $n \to \infty$.*

The well known Pisier space $\mathcal{P}$ [22, 23] contains no uniformly complemented finite-dimensional subspaces. Moreover, there exists $\lambda > 0$ such that $p_n(\mathcal{P}) \geq \lambda\sqrt{n}$ for all n.

Corollary 2. *Every operator from a Banach space X, containing no uniformly complemented finite-dimensional subspaces, into a Hilbert space H is SSS.*

Corollary 3. *There is $\lambda > 0$ such that for every operator $T \in \mathcal{L}(\mathcal{P}, H)$ and every n*

$$b_n(T) \leq \frac{1}{\lambda\sqrt{n}} \|T\| .$$

R e m a r k 1. Since every n-dimensional subspace $E \subset X$ is a range of a projection $P : X \to E$ with $\|P\| \leq \sqrt{n}$ (Kadets/Snobar [10]), one cannot obtain a better estimation of $b_n(T)$ with using of projections. A similar estimation for operators from $C(K)$ into H, but with constants $\sqrt[4]{n}$ instead of $\sqrt{n}$, was noted in [17].

Proposition 1 implies

Corollary 4. *Assume X contains no uniformly complemented finite-dimensional subspaces and H is a Hilbert space. Then the pair (X, H) is not Bernstein with respect to the Bernstein numbers.*

P r o o f. Indeed, by Proposition 1, for every $T \in \mathcal{L}(X,H)$ the sequence $b_n(T)$ cannot go to 0 "more slowly" than $1/p_n(c)$. $\blacksquare$

An n-dimensional normed space $(E, \| \ \|)$ is said to be *a-isomorphic to* ℓ_2^n (write $E \overset{a}{\sim} \ell_2^n$), $a > 1$, if there exists an Euclidean norm $\| \ \|_2$ on E such that for every $e \in E$

$$a^{-1}\|e\| \le \|e\|_2 \le a\|e\|.$$

If in this definition the constants a and n are inessential, we say simply about *almost Euclidean* subspaces.

R e m a r k 2. If $E \overset{a}{\sim} \ell_2^n$ then for every subspace $F \subset E$ there is a projection $P : E \to F$ with $\|P\| \le a^2$.

If $E \overset{a}{\sim} \ell_2^n$ then it have an a-orthonormal basis, i.e. a system $(e_i)_1^n$ such that $\|e_i\| = 1$ for all i and for all scalars (a_i)

$$a^{-1}\left(\sum\nolimits_1^n a_i^2\right)^{1/2} \le \left\|\sum\nolimits_1^n a_i e_i\right\| \le a\left(\sum\nolimits_1^n a_i^2\right)^{1/2}.$$

R e m a r k 3. For an a-orthonormal basis, the norm of each projection P_i, $i < n$, of E onto $\mathrm{lin}(e_j)_1^i$ along to $\mathrm{lin}(e_j)_{i+1}^n$ is not greater than a^2.

Definition 4. (see e.g. [22, p. 215]). *A Banach space X contains* uniformly complemented ℓ_2^n's *if there is a constant d such that for every $\varepsilon > 0$ and for each n there is a subspace $E \subset X$ and a projection $P : X \to E$ such that $E \overset{1+\varepsilon}{\sim} \ell_2^n$ and $\|P\| < d$.*

Note that by Dvoretzky's theorem, if this holds for *some* ε, then it automatically holds for *all* ε.

We will show that uniformly complemented almost Euclidean subspaces play a crucial role in constructing of Bernstein pairs.

Theorem 1. *If a Banach space X contains uniformly complemented ℓ_2^n's then there exists a universal constant $b = b(X) > 0$ such that for each Banach space Y, and any sequence $d_n \downarrow 0$ there exist a bounded linear operator $T : X \to Y$ such that for all n*

$$b^{-1}d_n \le b_n(T) \le bd_n.$$

Corollary 5. *Let a Banach space X contain uniformly complemented ℓ_2^n's. Then for every Banach space Y the pair (X, Y) is Bernstein with respect to the Bernstein numbers.*

A Banach space X is *B-convex* if it does not contain ℓ_1^n's uniformly. Since every B-convex Banach space contains uniformly complemented ℓ_2^n's (see e.g. [22, pp. 208, 215]), we have

Corollary 6. *Let X be a B-convex Banach space. Then for every Banach space Y the pair (X, Y) is Bernstein with respect to the Bernstein numbers.*

This corollary recalls us the well known Davis–Johnson compact non-nuclear operator in a B-convex Banach space [5].

Problem. Does (X, X) form a Bernstein pair with respect to the Bernstein numbers for every Banach space X?

2. Proof of the Main Result

To prove Theorem 1 we construct a "bounded minimal system" consisting of almost Euclidean subspaces of arbitrary large dimensions in an arbitrary Banach space containing uniformly complemented ℓ_2^n's.

Lemma 1. *Let X contain uniformly complemented ℓ_2^n's, with corresponding ε and d and let $d' > (1+\varepsilon)^4 d$. Then for each finite codimensional subspace $X' \subset X$, each finite dimensional subspace $E \subset X$ and each m there exists a subspace $E' \subset X'$, $E' \overset{1+\varepsilon}{\sim} \ell_2^m$ and a projection $P' : X \to E'$ with $\|P'\| < d'$ and* $\ker P' \supset E$.

P r o o f. By definition, one can find an almost Euclidean subspace $E_0 \subset X$, $\dim E_0 > m + \dim E + \dim X/X_0$ and a projection $P_0 : X \to E_0$ with $\|P_0\| < d$. Since E_0 is almost Euclidean, by Remark 2, there exists a projection $Q_0 : E_0 \to E_1 := E_0 \cap X_0$ with $\|Q_0\| \le (1+\varepsilon)^2$. Obviously, $\dim E_1 \ge m + \dim E$. Put $P_1 = Q_0 P_0$. Then P_1 is a projection of X onto E_1 and $\|P_1\| \le (1+\varepsilon)^2 d$.

Since E_1 is almost Euclidean, by Remark 2, there exists a subspace $E' \subset E_1$, $\dim E' = m$, and a projection $Q_1 : E_1 \to E'$ with $\|Q_1\| \le (1+\varepsilon)^2$ and $\ker Q_1 \supset P(E)$. Then $P' = Q_1 P_1$ is the desired projection. ■

Lemma 2. *Let X contain uniformly complemented ℓ_2^n's, with corresponding ε and d. Then for any subsequence $(m_k)_{k=1}^\infty$ of integers there are subspaces $E_k \subset X$, each $E_k \overset{1+\varepsilon}{\sim} \ell_2^{m_k}$, with projections $P_k : X \to E_k$, $\|P_k\| \le d$, such that each E_i, $i \ne k$, belongs to* $\ker P_k$.

P r o o f. Of course, one must write $\|P_k\| \le d'$, where d' is from the previous lemma, but the exact value of the constant d is non-essential here. We present a construction only.

Take, by definition, a subspace $E_1 \subset X$, $E_1 \overset{1+\varepsilon}{\sim} \ell_2^{m_1}$, and a projection $P_1 : X \to E_1$ with $\|P_1\| \le d$.

Then take, by Lemma 1, a subspace $E_2 \subset \ker P_1$, $E_2 \overset{1+\varepsilon}{\sim} \ell_2^{m_2}$, and a projection $P_2 : X \to E_2$ with $\|P_2\| \le d$ and $\ker P_2 \supset E_1$.

Next take, by Lemma 1, a subspace $E_3 \subset \ker P_1 \cap \ker P_2$, $E_3 \overset{1+\varepsilon}{\sim} \ell_2^{m_3}$, and a projection $P_3 : X \to E_3$ with $\|P_3\| \le d$ and $\ker P_3 \supset (E_1 \cup E_2)$, and so on. ■

R e m a r k 4. Let (E_k) be subspaces from Lemma 2. Then for every $k \ge 1$

$$X = E_1 \oplus E_2 \oplus \cdots E_k \oplus (\cap_{i=1}^k \ker P_i).$$

Next, using the Dvoretzky theorem, we construct in an arbitrary Banach space a subspace with "bounded minimal system" consisting of almost Euclidean subspaces of arbitrary large dimensions. Denote by $[A]$ the closed linear span of the set A.

Lemma 3. *Let Y be a Banach space, $\varepsilon > 0$, and $(m_k)_{k=1}^\infty$ be a sequence of integers. Then there exist subspaces $F_k \subset Y$, each $F_k \overset{1+\varepsilon}{\sim} \ell_2^{m_k}$, and projections $Q_k : [F_i]_1^\infty \to \mathrm{lin}(F_i)_1^k$ along $[F_i]_{k+1}^\infty$ with $\|Q_k\| \le 1 + \varepsilon$.*

P r o o f. Lemma 3 is a standard combination of the Dvoretzky and Mazur theorems. We present a construction only. Recall that a subset $\Phi \subset Y^*$ λ-*norms* a subspace $F \subset Y$ if for every $y \in S_F$ there is $\varphi \in \Phi$ such that $\varphi(y) \ge \lambda$. For each finite-dimensional subspace $F \subset Y$ and $0 < \lambda < 1$ there is a finite set $\Phi \subset S_{Y^*}$ which λ-norms F.

So, take a subspace $F_1 \subset Y$, $F_1 \overset{1+\varepsilon}{\sim} \ell_2^{m_1}$, and a finite subset $\Phi_1 \subset S_{X^*}$ which $(1+\varepsilon)^{-1}$-norms F_1.

Then take a subspace

$$F_2 \subset \Phi_1^\top := \{y \in Y : \varphi(y) = 0 \text{ for all } \varphi \in \Phi_1\},$$

$F_2 \overset{1+\varepsilon}{\sim} \ell_2^{m_2}$, and a finite subset $\Phi_2 \subset S_{X^*}$ which $(1+\varepsilon)^{-1}$-norms $F_1 + F_2$.

Next, take a subspace $F_3 \subset \Phi_2^\top$, $F_3 \overset{1+\varepsilon}{\sim} \ell_2^{m_3}$, and a finite subset $\Phi_3 \subset S_{X^*}$ which $(1+\varepsilon)^{-1}$-norms $F_1 + F_2 + F_3$, and so on. ■

In the proof we use diagonal operators in Euclidean spaces whose Bernstein numbers are well known.

Definition 5. *Let E and F be linear spaces with bases $(e_n)_1^m$ and $(f_n)_1^m$. Let $(d_n)_1^m$ be scalars. A map*

$$D\left(\sum\nolimits_1^m a_n e_n\right) = \sum\nolimits_1^m d_n a_n f_n$$

is called the diagonal operator *corresponding to (e_n), (f_n) and (d_n).*

Proposition 2. (sf. [19, Th. 7.1]). *Let $(e_n)_1^m$ be the standard basis of ℓ_2^m, $d_1 \geq d_2 \geq \cdots \geq d_m \geq 0$ and D be the diagonal operator in ℓ_2^m corresponding to $(e_n)_1^m$ and $(d_n)_1^m$. Then for all $n \leq m$*

$$\min\{\|Dx\| : x \in \mathrm{lin}(e_j)_1^n\,,\ \|x\| = 1\} = d_n \quad \textit{and}$$
$$\max\{\|Dx\| : x \in \mathrm{lin}(e_j)_n^m\,,\ \|x\| = 1\} = d_n\,.$$

Corollary 7. *Assume m-dimensional normed spaces E and F have a-orthonormal bases $(e_n)_1^m$ and $(f_n)_1^m$, $d_1 \geq d_2 \geq \cdots \geq d_m \geq 0$ and D is the diagonal operator corresponding to (e_n), (f_n), (d_n). Then there is $c > 1$, depending only on a, such that for all $n \leq m$*

$$\min\{\|Dx\| : x \in \mathrm{lin}(e_j)_1^n\,,\ \|x\| = 1\} \geq \frac{d_n}{c} \quad \textit{and}$$
$$\max\{\|Dx\| : x \in \mathrm{lin}(e_j)_n^m\,,\ \|x\| = 1\} \leq c d_n\,.$$

P r o o f of Theorem 1. Let $d_n \downarrow 0$. Take a subsequence $(n_k)_{k=1}^\infty$ of integers which approach to ∞ so quickly that for all $k \geq 1$

$$d_{n_{k+1}} < \frac{1}{4}\, d_{n_k}. \tag{4}$$

Hence, for every $k \geq 1$

$$\sum_{i=k+1}^{\infty} d_{n_i} < \frac{1}{2} d_{n_k}. \tag{5}$$

Let $0 < \varepsilon < 1$, E_k be subspaces from Lemma 2 and F_k, $k \geq 1$, be subspaces from Lemma 3 with $m_k := n_k - n_{k-1}$ (and $n_0 = 0$). Take in each E_k and each F_k some $(1+\varepsilon)$-orthonormal bases. Rearrange these bases in the natural way, putting first the basis $e_1, \ldots, e_{n_1}$ of E_1, then the basis $e_{n_1+1}, \ldots, e_{n_2}$ of E_2 and so on; and similarly for Y. We obtain systems $(e_n)_1^\infty$ in X and $(f_n)_1^\infty$ in Y.

Put $N_k = \{n : \ n_{k-1} < n \leq n_k\}$. Using Corollary 7, (with c from this corollary) we construct for every $k \geq 1$ the diagonal operator $D_k : E_k \to F_k$ corresponding to the bases (e_n), (f_n) and scalars (d_n), $n \in N_k$, such that for all $n \in N_k$

$$\min\Big\{\|D_k x\| : x \in [e_j]_{n_{k-1}+1}^n\,,\ \|x\| = 1\Big\} \geq \frac{d_n}{c} \quad \text{and} \tag{6}$$
$$\max\{\|D_k x\| : x \in [e_j]_n^{n_k}\,,\ \|x\| = 1\} \leq c d_n\,. \tag{7}$$

Let P_k be the projections from Lemma 2. For every $x \in X$ put

$$Tx = \sum_{i=1}^{\infty} D_i P_i x \tag{8}$$

(bellow we will show that the series (8) converges for each $x \in X$).

We make forth estimations. Let d be from Lemma 2 and c be from Corollary 7.

1. For every $k \ge 1$ and $x \in X$, $\|x\| = 1$,

$$\sum_{i=k+1}^{\infty} \|D_i P_i x\| < 2cdd_{n_k} .$$

Indeed, $P_i x \in E_i$ and $\|P_i x\| \le \|P_i\|\|x\| \le d$ for all i, so

$$\sum_{i=k+1}^{\infty} \|D_i P_i x\| \le \text{by } (7) \le cdd_{n_k+1} + \sum_{i=k+2}^{\infty} cdd_{n_i+1}$$
$$\le \text{by } (5) \le cdd_{n_k} + \frac{c}{2} dd_{n_{k+1}} < 2cdd_{n_k} .$$

In particular, this inequality shows that series (8) converges for each $x \in X$, so T is well defined.

2. For every $k \ge 1$ and $n \in N_k$

$$\sup \left\{ \|Tx\| : x \in [e_j]_n^{n_k} \oplus \cap_{i=1}^k \ker P_i \, , \ \|x\| = 1 \right\} \le 3cdd_n$$

(by Remark 4, the sum here is direct).

Indeed, take $x \in [e_j]_n^{n_k} \oplus \cap_{i=1}^k \ker P_i$, $\|x\| = 1$. Then, by definition of P_i, $Tx = \sum_{i=k}^{\infty} D_i P_i x$, so

$$\|Tx\| \le \|D_k P_k x\| + \sum_{i=k+1}^{\infty} \|D_i P_i x\| \le (\text{by } \mathbf{1}) \le \|D_k P_k x\| + 2cdd_{n_k}$$
$$\le (\text{since } \|P_k\| \le d, \text{ by } (7)) \le cdd_n + 2cdd_n = 3cdd_n .$$

3. For every $k \ge 1$ and $x \in \operatorname{lin}(E_i)_1^k$, $\|x\| = 1$,

$$\|Tx\| \ge \frac{1}{4(1+\varepsilon)} \cdot \frac{d_{n_k}}{c} .$$

We prove estimation **3** by induction. For $k = 1$, $Tx = D_1 x$, so **3** is followed from (6) if we take in (6) $n = n_1$. Suppose $k > 1$, estimation **3** is proved for $k-1$, and $x \in \operatorname{lin}(E_i)_1^k$, $\|x\| = 1$. Then

$$x = x_1 + x_2 \, , \quad x_1 \in \operatorname{lin}(E_i)_1^{k-1}, \quad x_2 \in E_k \, ,$$

and, by the construction of P_i,

$$Tx_1 = \sum_{i=1}^{k-1} D_i P_i x_1 \quad \text{and} \quad Tx_2 = D_k P_k x_2.$$

Hence, by the construction of D_i,

$$Tx_1 \in \operatorname{lin}(F_i)_1^{k-1} \quad \text{and} \quad Tx_2 \in F_k\,.$$

By the construction of projections Q_i from Lemma 3,

$$Q_{k-1}Tx = Q_{k-1}Tx_1 + Q_{k-1}Tx_2 = Tx_1$$

and

$$(Q_k - Q_{k-1})Tx = (Q_k - Q_{k-1})Tx_1 + (Q_k - Q_{k-1})Tx_2 = Tx_2\,.$$

Since $\|Q_i\| \le 1+\varepsilon$, hence $\|Q_i - Q_{i-1}\| \le 2(1+\varepsilon)$. So,

$$\|Tx\| \ge \frac{1}{1+\varepsilon}\|Q_{k-1}Tx\| = \frac{1}{1+\varepsilon}\|Tx_1\| \tag{9}$$

and

$$\|Tx\| \ge \frac{1}{2(1+\varepsilon)}\|(Q_k - Q_{k-1})Tx\| = \frac{1}{2(1+\varepsilon)}\|Tx_2\|. \tag{10}$$

Since $\|x\| = 1$, we have that

$$\text{either } \|x_1\| \ge \frac{1}{2} \text{ or } \|x_2\| \ge \frac{1}{2}\,.$$

If $\|x_1\| \ge \frac{1}{2}$, then by the induction assumption

$$\begin{aligned}\|Tx\| &\overset{\text{by (9)}}{\ge} \frac{1}{1+\varepsilon}\|Tx_1\| \ge \frac{1}{1+\varepsilon}\cdot\frac{1}{2}\cdot\frac{1}{4(1+\varepsilon)}\cdot\frac{d_{n_{k-1}}}{c}\\ &\overset{\text{by (4)}}{\ge} \frac{1}{2(1+\varepsilon)^2}\cdot\frac{1}{4}\cdot\frac{4d_{n_k}}{c} \overset{\text{since } \varepsilon<1}{\ge} \frac{1}{4(1+\varepsilon)}\cdot\frac{d_{n_k}}{c}\,.\end{aligned}$$

If $\|x_2\| \ge \frac{1}{2}$, then

$$\|Tx\| \overset{\text{by (10)}}{\ge} \frac{1}{2(1+\varepsilon)}\|Tx_2\| \overset{\text{by (6)}}{\ge} \frac{1}{2(1+\varepsilon)}\cdot\frac{1}{2}\cdot\frac{d_{n_k}}{c} = \frac{1}{4(1+\varepsilon)}\cdot\frac{d_{n_k}}{c}\,.$$

Therefore, **3** is proved.

4. For every $k \ge 1$ and $n \in N_k$

$$\min\{\|Tx\| : x \in \operatorname{lin}(e_j)_1^n\,,\ \|x\| = 1\} \ge \frac{1}{4(1+\varepsilon)}\cdot\frac{d_n}{c}\,.$$

Indeed, take $x \in \operatorname{lin}(e_j)_1^n$, $\|x\| = 1$, where $n \in N_k$. Then, as in **3**, $x = x_1 + x_2$, $x_1 \in \operatorname{lin}(E_i)_1^{k-1}$, $x_2 \in E_k$; either $\|x_1\| \ge \frac{1}{2}$ or $\|x_2\| \ge \frac{1}{2}$; $Tx_1 \in \operatorname{lin}(F_i)_1^{k-1}$, $Tx_2 \in F_k$, and the inequalities (9), (10) hold.

If $\|x_1\| \ge \frac{1}{2}$, then

$$\|Tx\| \ge \text{by } \mathbf{3} \ge \frac{1}{4(1+\varepsilon)} \cdot \frac{d_{n_{k-1}}}{c} \ge \frac{1}{4(1+\varepsilon)} \cdot \frac{d_n}{c}\,.$$

If $\|x_2\| \ge \frac{1}{2}$, then

$$\|Tx\| \overset{\text{by (10)}}{\ge} \frac{1}{2(1+\varepsilon)} \|Tx_2\| = \frac{1}{2(1+\varepsilon)} \|D_k x_2\| \overset{\text{by (6)}}{\ge} \frac{1}{4(1+\varepsilon)} \cdot \frac{d_n}{c}\,.$$

Therefore, **4** is proved.

Put $b = \max\{3cd,\, 4(1+\varepsilon)c\}$. Inequality **4** shows that for all n

$$b_n(T) \ge b^{-1} d_n.$$

Let $G \subset X$ be an n-dimensional subspace and $n \in N_k$. Then, by Remark 4,

$$G \cap \left([e_j]_n^{n_k} \oplus \cap_{i=1}^{k} \ker P_i \right) \ne 0.$$

So, the inequality **4** confirms that for all n

$$\min_{x \in S_G} \|Tx\| \le b d_n\,,$$

i.e.

$$b_n(T) \le b d_n\,.$$

■

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