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Affine Submanifolds of Rank Two

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In the paper we study the complete connected affine submanifolds $f : M^n \to \mathbb{R}^{n+m}$ of rank two, i.e., the strongly $(n-2)$-parabolic submanifolds according to A. Borisenko. The structures of these submanifolds are described and the explicit parametrization is given for two partial cases.

Key words: affine immersion, ruled submanifold.

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Introduction

A submanifold M^n in the Euclidean space E^N is called *strongly k-parabolic* [1, 3] if at any point the kernel of its vector valued second fundamental form has a dimension k. In the Euclidean case, the kernels of the second fundamental form and of the shape operator coincide because of metric structure. In contrast, in the case of the *affine* immersion $f : M^n \to \mathbb{R}^N,\ n \geq 3$, the affine fundamental form and the shape operator have *different* kernels in general. We propose the following definition of the strongly k-parabolic affine submanifold.

Definition. *The affine immersion $f : M^n \to \mathbb{R}^N,\ (n \geq 3)$ is called strongly k-parabolic if at any point the kernel of its vector valued affine fundamental form has a dimension k.*

For the strongly k-parabolic submanifold, the rank of the Gauss (Grassmann) map is $n-k$. That is why this type of submanifolds was called the *submanifolds of rank $(n-k)$* [6]. In this paper, we consider the simplest non-trivial case of the affine strongly $(n-2)$-parabolic submanifold in $\mathbb{R}^N$ or the submanifold of **rank two**, equivalently.

A subspace spanned by the second fundamental form is called the *first normal space* N_1. The rank two submanifold satisfies $\dim N_1 \leq 3$ at each point. In the Euclidean case, it is known that if $\dim N_1 = 1$, then the submanifold is a hypersurface; if $\dim N_1 = 3$, then it is either an Euclidean surface or a cone over a spherical surface up to the Euclidean factor; if $\dim N_1 = 2$, then it is of one of three types: *elliptic, hyperbolic and parabolic.* A complete parametric description of the elliptic submanifolds was given in [5]. Parabolic submanifolds of rank two were described by M. Dajczer and P. Morais in [6] and classified as ruled and non-ruled parabolic submanifolds. A submanifold is called ruled if M^n admits a hyperfoliation by the $\mathbb{R}^N$-totally geodesic submanifolds. M. Dajczer and P. Morais described parametrically a class of the ruled parabolic submanifolds and proved that only ruled parabolic submanifolds admit an isomeric immersion in the form of a hypersurface. They also gave the polar and bipolar parameterizations of the non-ruled parabolic submanifolds.

The main goal of this paper is to describe the affine submanifolds of rank two.

1. Preliminaries

Let $f : M^n \to (\mathbb{R}^{n+m},\ D)$ ($m < n$, D is a standard flat connection) be an affine immersion [8] with m-dimensional transversal differentiable distribution Q along f. For the arbitrary vector fields X and Y on M^n, we have

$$D_X Y = f_*(\nabla_X Y) + h(X,\ Y),\ h(X,\ Y) \in Q,$$

where ∇ is the induced torsion-free connection on M^n, h is the affine fundamental form. Let $\xi_1, \ldots, \xi_m$ be a basis of the transversal distribution Q. Then we have the affine analogs of the Gauss and Weingarten decompositions

$$D_X f_*(Y) = f_*(\nabla_X Y) + h^\alpha(X,\ Y)\xi_\alpha, \tag{1}$$

$$D_X \xi_\alpha = -f_*(S_\alpha X) + \tau_\alpha^\beta(X)\xi_\beta. \tag{2}$$

Let $\bar{Q} = \operatorname{span}\{\bar{\xi}_1, \ldots,\ \bar{\xi}_m\}$ be a transformation of Q by

$$\bar{\xi}_\alpha = \Phi_\alpha^\beta \xi_\beta + Z_\alpha, \tag{3}$$

where Z_α are the tangent vector fields on M^n, $\Phi = [\Phi_\alpha^\beta]_{m\times m}$ is a non-degenerate matrix of smooth functions. It is easy to prove that

$$\bar{h}^\alpha(X,\ Y) = [\Phi^{-1}]_\beta^\alpha h^\beta(X,\ Y), \tag{4}$$

$$\bar{\nabla}_X Y = \nabla_X Y - [\Phi^{-1}]_\beta^\alpha h^\beta(X,\ Y) Z_\alpha. \tag{5}$$

From (4), we can see that neither the kernel nor the rank of $h(X,\ Y) : T_xM \times T_xM \to Q_x$ depends on the choice of a transversal distribution.

A rank of the affine fundamental form is called *a pointwise codimension* of an affine immersion. Similarly to the Euclidean case, it is equal to the dimension of the first normal space.

Let $f : M^n \to \mathbb{R}^{n+m}$ be an affine immersion of rank two with the transversal distribution Q and pointwise codimension two. A. Borisenko classified the points of multidimensional submanifold up to the affine transformation based on the affine type of the osculating paraboloid [2, 3]. Particularly, if F^2 is a submanifold in E^N of pointwise codimension two, then there are three classes of points with the osculating paraboloid:

$$(1)\ x^3 = 2x^1x^2,\ x^4 = (x^1)^2 - (x^2)^2,$$

$$(2)\ x^3 = (x^1)^2,\ x^4 = (x^2)^2,$$

$$(3)\ x^3 = 2x^1x^2,\ x^4 = (x^2)^2$$

up to the degenerate affine transformation of the ambient space.

Later on, the same classes of points on the submanifolds of this type were introduced by M. Dajczer and P. Morais [5, 6] by using another criterion. They called the points *elliptic*, *hyperbolic* and *parabolic*, respectively.

Remark that elliptic and parabolic submanifolds belong to a class of saddle submanifolds according to S. Shefel [10]. In [4] V. Glazyrin gave an equivalent definition of the saddle submanifold in the Euclidean space which can be adapted to the affine case. Thus, we can give the following definition of the affine k-saddle submanifold.

Definition. *The affine immersion $f : M^n \to \mathbb{R}^N$, $(n \geq 3)$ is called k-saddle if at any point of the immersed submanifold the affine fundamental form with respect to arbitrary transversal vector has not greater than k coefficients of the same sign after reduction to the diagonal form.*

Evidently, the elliptic and parabolic affine submanifolds are 1-saddle.

Obviously, the point classification of affine rank-two immersion of pointwise codimension 2 coincides with the point classification of affine immersion $M^2 \to \mathbb{R}^4$ of pointwise codimension 2. For affine metric of affine surface in R^4 use the following symmetric bilinear form [9]

$$G_\upsilon(X, Y) = \frac{1}{2}(\det(e_1, e_2, D_{e_1}X, D_{e_2}Y) + \det(e_1, e_2, D_{e_1}Y, D_{e_2}X)),$$

where $\upsilon = \{e_1, e_2\}$ is local differentiable frame on a neighbourhood of a point $x \in M^2$. Let $X = x^1e_1 + x^2e_2$, $Y = y^1e_1 + y^2e_2$, $\{\xi_1, \xi_2\}$ is transversal frame and $\omega = \det(e_1, e_2, \xi_1, \xi_2)$, then

$$G_\upsilon(X, Y) = ((h^1_{11}h^2_{12} - h^2_{11}h^1_{12})x^1y^1 + \frac{1}{2}(h^1_{11}h^2_{22} - h^2_{11}h^1_{22})(x^1y^2 + x^2y^1) + (h^1_{12}h^2_{22} - h^2_{12}h^1_{22})x^2y^2)\omega$$

The quadratic form $G_v(X,Y)$ at x can be definite, indefinite, 1-degenerate. There are parameterization of immersion and transversal frame [9] such that

1) if G_v positive definite, then
$G_v(X,Y)=(x^1y^1+x^2y^2)\omega,\ h^1=\begin{pmatrix}1&0\\0&-1\end{pmatrix},\ h^2=\begin{pmatrix}0&1\\1&0\end{pmatrix}$
and the point is *elliptic*;

2) if G_v indefinite, then
$G_v(X,Y)=(x^1y^1-x^2y^2)\omega,\ h^1=\begin{pmatrix}1&0\\0&1\end{pmatrix},\ h^2=\begin{pmatrix}0&1\\1&0\end{pmatrix}$ or
$G_v(X,Y)=\frac{1}{2}(x^1y^2+x^2y^1)\omega,\ h^1=\begin{pmatrix}1&0\\0&0\end{pmatrix},\ h^2=\begin{pmatrix}0&0\\0&1\end{pmatrix}$
and the point is *hyperbolic*;

3) if G_v is 1-degenerate, then
$G_v(X,Y)=x^1y^1\omega,\ h^1=\begin{pmatrix}1&0\\0&0\end{pmatrix},\ h^2=\begin{pmatrix}0&1\\1&0\end{pmatrix}$
and the point is *parabolic*.

For affine immersion $f: M^n\to\mathbb{R}^{n+m}$ of rank two we can choose tangential $\{e_1,\ldots,\ e_n\}$ and transversal $\{\xi_1,\ldots,\ \xi_m\}$ frames in such a way that $e_j\in\ker h,\ j=\overline{3,n}$ and

$$h^1=\begin{pmatrix}1&0&0&\ldots&0\\0&h^1_{22}&0&\ldots&0\\0&0&0&\ldots&0\\\vdots&\vdots&\vdots&\ddots&\vdots\\0&0&0&\ldots&0\end{pmatrix},\ h^2=\begin{pmatrix}0&1&0&\ldots&0\\1&h^2_{22}&0&\ldots&0\\0&0&0&\ldots&0\\\vdots&\vdots&\vdots&\ddots&\vdots\\0&0&0&\ldots&0\end{pmatrix},\ h^k=O_{n\times n},$$

where h^i is a matrix of the affine fundamental form with respect to ξ_i, $k=\overline{3,m}$. Therefore for surface in $\mathbb{R}^4$ with the same h^1 and h^2, we have

$$G_v(X,Y)=(x^1y^1+\frac{1}{2}h^2_{22}(x^1y^2+x^2y^1)-h^1_{22}x^2y^2)\omega,$$

$$\det G_v=(-h^1_{22}-\frac{1}{4}(h^2_{22})^2)\omega^2.$$

Obviously, the sign of $(h^2_{22})^2+4h^1_{22}$ defines a *pointwise class* of immersion as follows:

- if $(h^2_{22})^2+4h^1_{22}<0$, then the point is *elliptic* (case (1));
- if $(h^2_{22})^2+4h^1_{22}>0$, then the point is *hyperbolic* (case (2));
- if $(h^2_{22})^2+4h^1_{22}=0$, then the point is *parabolic* (case (3)).

2. Main Theorem

We consider a *generalized affine ruled submanifold*. The submanifold $f(M^n)$ in the affine space $\mathbb{R}^N$ is called $(n-l)$ ruled if it is foliated by the $(n-l)$ affine subspaces of $\mathbb{R}^N$. These subspaces are called *rulings*. A transversal to the foliation l-dimensional submanifold is called *a base* of the ruled submanifold.

Theorem. *Let $f : M^n \to \mathbb{R}^{n+m}$ $(m < n)$ be a complete connected C^3-smooth affine immersion of rank two of the pointwise codimension two with the points of the same class. Then*

1) *a hyperbolic submanifold is an affine cylinder with $(n-2)$-dimensional rulings over a two-dimensional hyperbolic base;*

2) *an elliptic submanifold is a ruled submanifold with $(n-2)$-dimensional rulings over a two-dimensional elliptic base;*

3) *a parabolic submanifold minus a closed subset is a union of open subsets of the following types of submanifolds:*

   a) *a ruled submanifold with $(n-1)$-dimensional rulings over a curve;*
   b) *a cylinder with $(n-2)$-dimensional rulings over a two-dimensional parabolic base.*

P r o o f. The structure of the affine strongly parabolic submanifolds was described in [11]. Particularly, *if $f : (M^n, \nabla) \to (\mathbb{R}^{n+k}, D)$ is the affine immersion such that* $\dim\ker h = \text{const} \neq 0$, *then*

- *the nullity distribution $\mathcal{N} = \ker h$ is integrable, the leaves are totally geodesic in $\mathbb{R}^{n+k}$;*

- *there exists a transversal distribution which is stationary along the leaves of the foliation $\mathcal{FN}$;*

- *if (M^n, ∇) is complete, then each leaf of the foliation $\mathcal{FN}$ is complete.*

Let $f : M^n \to \mathbb{R}^{n+m}$ be the complete connected affine immersion of rank two and pointwise codimension two with the points of the same class. Since $\dim\ker h = n-2$, the submanifold is foliated by the $(n-2)$-dimensional affine subspaces.

Similarly to the Euclidean case [3], we can parameterize $f(M^n)$ locally as

$$\vec{r}(u^1, u^2, v^1, \ldots, v^{n-2}) = \vec{\rho}(u^1, u^2) + \sum_{s=1}^{n-2} \vec{a}_s(u^1, u^2) v^s, \tag{6}$$

$$\text{where } \vec{\rho}(u^1,u^2) = \begin{pmatrix} \rho^1 \\ \vdots \\ \rho^{m+2} \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \qquad \vec{a}_s(u^1,u^2) = \begin{pmatrix} a^1 \\ \vdots \\ a^{m+2} \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}.$$

Here 1 is at the $(m+2+s)$- th coordinate place of $\vec{a}_s$, and $v^s \in (-\infty, +\infty)$ because of completeness.

The transversal to $f(M^n)$ vector fields $\xi_1, \ldots, \xi_m$ are also transversal to a surface F^2 with a position-vector $\vec{\rho}$. Denote

$$h_{22}^2 = p, \; h_{22}^1 = q.$$

The Gauss decomposition (1) for F^2, after suitable change of transversal distribution (3), takes the form

$$\frac{\partial^2 \vec{\rho}}{\partial u^1 \partial u^1} = \bar{\xi}_1, \quad \frac{\partial^2 \vec{\rho}}{\partial u^1 \partial u^2} = \bar{\xi}_2, \quad \frac{\partial^2 \vec{\rho}}{\partial u^2 \partial u^2} = \mu^1 \frac{\partial \vec{\rho}}{\partial u^1} + \mu^2 \frac{\partial \vec{\rho}}{\partial u^2} + q\bar{\xi}_1 + p\bar{\xi}_2, \tag{7}$$

where $\mu^1 = \Gamma_{22}^1, \; \mu^2 = \Gamma_{22}^2$.

Here and below we assume that the indices $\{i, \; j, \; k\}$ and $\{s, \; l\}$ run over the ranges $\{1, \; 2\}$ and $\{1, \ldots, n-2\}$, respectively, unless otherwise stated. Now we can find

$$\frac{\partial \vec{r}}{\partial u^i} = \frac{\partial \vec{\rho}}{\partial u^i} + v^s \frac{\partial \vec{a}_s}{\partial u^i}, \qquad \frac{\partial \vec{r}}{\partial v^s} = \vec{a}_s$$

$$\frac{\partial^2 \vec{r}}{\partial u^i \partial u^j} = \frac{\partial^2 \vec{\rho}}{\partial u^i \partial u^j} + v^s \frac{\partial^2 \vec{a}_s}{\partial u^i \partial u^j}, \qquad \frac{\partial^2 \vec{r}}{\partial u^i \partial v^s} = \frac{\partial \vec{a}_s}{\partial u^i}, \qquad \frac{\partial^2 \vec{r}}{\partial v^l \partial v^s} = 0.$$

From the definition of a kernel of the affine fundamental form and the choice of a frame, it follows that

$$\frac{\partial^2 \vec{r}}{\partial u^i \partial v^s} = \frac{\partial \vec{a}_s}{\partial u^i} = \lambda_{si}^k \frac{\partial \vec{\rho}}{\partial u^k}. \tag{8}$$

Hence,

$$\frac{\partial^2 \vec{r}}{\partial u^i \partial u^j} = \frac{\partial^2 \vec{\rho}}{\partial u^i \partial u^j} + v^s \left(\frac{\partial \lambda_{si}^k}{\partial u^j} \frac{\partial \vec{\rho}}{\partial u^k} + \lambda_{si}^k \frac{\partial^2 \vec{\rho}}{\partial u^k \partial u^j} \right).$$

As the mixed derivatives are equal, we have

$$\frac{\partial \lambda_{s1}^k}{\partial u^2} \frac{\partial \vec{\rho}}{\partial u^k} + \lambda_{s1}^k \frac{\partial^2 \vec{\rho}}{\partial u^k \partial u^2} = \frac{\partial \lambda_{s2}^k}{\partial u^1} \frac{\partial \vec{\rho}}{\partial u^k} + \lambda_{s2}^k \frac{\partial^2 \vec{\rho}}{\partial u^k \partial u^1}.$$

Denote $\dfrac{\partial\vec{\rho}}{\partial u^k} = \vec{\rho}_k$ and substitute it into the Gauss decomposition (7),

$$\frac{\partial\lambda^1_{s1}}{\partial u^2}\vec{\rho}_1 + \frac{\partial\lambda^2_{s1}}{\partial u^2}\vec{\rho}_2 + \lambda^1_{s1}\bar{\xi}_2 + \lambda^2_{s1}(\mu^1\vec{\rho}_1 + \mu^2\vec{\rho}_2 + q\bar{\xi}_1 + p\bar{\xi}_2) =$$

$$\frac{\partial\lambda^1_{s2}}{\partial u^1}\vec{\rho}_1 + \frac{\partial\lambda^2_{s2}}{\partial u^1}\vec{\rho}_2 + \lambda^1_{s2}\bar{\xi}_1 + \lambda^2_{s2}\bar{\xi}_2.$$

Comparing the tangent components, we obtain

$$\frac{\partial\lambda^1_{s1}}{\partial u^2} + \lambda^2_{s1}\mu^1 = \frac{\partial\lambda^1_{s2}}{\partial u^1}, \quad \frac{\partial\lambda^2_{s1}}{\partial u^2} + \lambda^2_{s1}\mu^2 = \frac{\partial\lambda^2_{s2}}{\partial u^1}. \tag{9}$$

Comparing the transversal components, we obtain

$$q\lambda^2_{s1} = \lambda^1_{s2}, \quad \lambda^1_{s1} + p\lambda^2_{s1} = \lambda^2_{s2}. \tag{10}$$

The tangent frame for (6) consists of

$$\vec{r}_1 = \vec{\rho}_1 + \lambda^k_{s1}v^s\vec{\rho}_k, \quad \vec{r}_2 = \vec{\rho}_2 + \lambda^k_{s2}v^s\vec{\rho}_k, \quad \vec{r}_s = \vec{a}_s.$$

The regularity condition implies that $\{\vec{\rho}_1,\ \vec{\rho}_2\}$ and $\{\vec{r}_1,\ \vec{r}_2\}$ are linearly independent, that is, the matrix

$$C = \begin{pmatrix} 1 + v^s\lambda^1_{s1} & v^s\lambda^1_{s2} \\ v^s\lambda^2_{s1} & 1 + v^s\lambda^2_{s2} \end{pmatrix}$$

is non-degenerate. Consider

$$\det C = 1 + v^s(\lambda^1_{s1} + \lambda^2_{s2}) + v^s v^l(\lambda^1_{s1}\lambda^2_{l2} - \lambda^2_{s1}\lambda^1_{l2}).$$

By plugging (10) and $q = -\dfrac{p^2}{4}$, we get

$$\det C = 1 + v^s(2\lambda^1_{s1} + p\lambda^2_{s1}) + v^s v^l(\lambda^1_{s1}(\lambda^1_{l1} + p\lambda^2_{l1}) - q\lambda^2_{s1}\lambda^2_{l1}) =$$

$$= (1 + v^s(\lambda^1_{s1} + \frac{p}{2}\lambda^2_{s1}))^2.$$

A. In the elliptic case $-q > \dfrac{p^2}{4}$. Then $\det C > (1 + v^s(\lambda^1_{s1} + \frac{p}{2}\lambda^2_{s1}))^2$. Hence $\det C \neq 0$ for all v^s, and the regularity condition is fulfilled for the arbitrary functions λ^i_{sj}.

B. In the parabolic case $q = -\dfrac{p^2}{4}$. The regularity condition ($\det C \neq 0$ for all v^s) implies

$$\lambda^1_{s1} = -\frac{p}{2}\lambda^2_{s1}, \quad \lambda^1_{s2} = -\frac{p^2}{4}\lambda^2_{s1}, \quad \lambda^2_{s2} = \frac{p}{2}\lambda^2_{s1}. \tag{11}$$

C. In the hyperbolic case $-q < \dfrac{p^2}{4}$, the solution for $\det C = 0$ exists for every set of nonzero functions λ^i_{sj}. Hence, the regularity condition is fulfilled only in the case of $\lambda^i_{sj} \equiv 0$.

Now we can prove the theorem.

1. If the submanifold $f(M^n)$ is hyperbolic, then (C) implies $\vec{a}_s(u^1, u^2)$ are constant vectors, and the submanifold (6) is a cylinder with $(n-2)$-dimensional rulings over the two-dimensional hyperbolic surface F^2.

2. If the submanifold $f(M^n)$ is elliptic, then the regularity condition is fulfilled for an arbitrary choice of λ^i_{sj}. Substituting (10) into (9), we have

$$\begin{array}{rcl}\dfrac{\partial\lambda^1_{s1}}{\partial u^2} + \lambda^2_{s1}\mu^1 & = & \dfrac{\partial q}{\partial u^1}\lambda^2_{s1} + q\dfrac{\partial\lambda^2_{s1}}{\partial u^1}, \\ \dfrac{\partial\lambda^2_{s1}}{\partial u^2} + \lambda^2_{s1}\mu^2 & = & \dfrac{\partial\lambda^1_{s1}}{\partial u^1} + \dfrac{\partial p}{\partial u^1}\lambda^2_{s1} + p\dfrac{\partial\lambda^2_{s1}}{\partial u^1}.\end{array}$$

Therefore,

$$\begin{array}{rcl}\dfrac{\partial\lambda^1_{s1}}{\partial u^2} & = & q\dfrac{\partial\lambda^2_{s1}}{\partial u^1} + \dfrac{\partial q}{\partial u^1}\lambda^2_{s1} - \lambda^2_{s1}\mu^1, \\ \dfrac{\partial\lambda^1_{s1}}{\partial u^1} & = & \dfrac{\partial\lambda^2_{s1}}{\partial u^2} - p\dfrac{\partial\lambda^2_{s1}}{\partial u^1} - \dfrac{\partial p}{\partial u^1}\lambda^2_{s1} + \lambda^2_{s1}\mu^2.\end{array} \tag{12}$$

Equating the mixed derivatives of λ^1_{s1}, we obtain the equation

$$\begin{array}{c} q\dfrac{\partial^2\lambda^2_{s1}}{(\partial u^1)^2} + p\dfrac{\partial^2\lambda^2_{s1}}{\partial u^1\partial u^2} - \dfrac{\partial^2\lambda^2_{s1}}{(\partial u^2)^2} + \dfrac{\partial\lambda^2_{s1}}{\partial u^1}\left(\dfrac{\partial p}{\partial u^2} + 2\dfrac{\partial q}{\partial u^1} - \mu^1\right) \\ + \dfrac{\partial\lambda^2_{s1}}{\partial u^2}\left(\dfrac{\partial p}{\partial u^1} - \mu^2\right) + \lambda^2_{s1}\left(\dfrac{\partial^2 p}{\partial u^1\partial u^2} + \dfrac{\partial^2 q}{(\partial u^1)^2} - \dfrac{\partial\mu^1}{\partial u^1} - \dfrac{\partial\mu^2}{\partial u^2}\right) = 0.\end{array}$$

This is a partial differential equation of elliptic type. It has a solution which depends on the functions p, q, μ^1, μ^2. Thus, the elliptic submanifold is a ruled one and, in particular, it can be a cylinder.

3. If the submanifold $f(M^n)$ is parabolic, then $q = -\dfrac{p^2}{4}$ and we can substitute (11) into (9)

$$\begin{array}{rcl}\dfrac{\partial}{\partial u^2}(-\dfrac{p}{2}\lambda^2_{s1}) + \lambda^2_{s1}\mu^1 & = & \dfrac{\partial}{\partial u^1}(-\dfrac{p^2}{4}\lambda^2_{s1}), \\ \dfrac{\partial\lambda^2_{s1}}{\partial u^2} + \lambda^2_{s1}\mu^2 & = & \dfrac{\partial}{\partial u^1}(\dfrac{p}{2}\lambda^2_{s1})\end{array}$$

to get

$$\begin{array}{rcl}-\dfrac{\partial p}{\partial u^2}\lambda^2_{s1} - p\dfrac{\partial\lambda^2_{s1}}{\partial u^2} + 2\lambda^2_{s1}\mu^1 & = & -p\dfrac{\partial p}{\partial u^1}\lambda^2_{s1} - \dfrac{p^2}{2}\dfrac{\partial\lambda^2_{s1}}{\partial u^1}, \\ \dfrac{\partial\lambda^2_{s1}}{\partial u^2} + \lambda^2_{s1}\mu^2 & = & \dfrac{1}{2}\dfrac{\partial p}{\partial u^1}\lambda^2_{s1} + \dfrac{p}{2}\dfrac{\partial\lambda^2_{s1}}{\partial u^1}.\end{array}$$

Therefore,

$$\begin{aligned} \lambda^2_{s1}\left(-\frac{\partial p}{\partial u^2}+\frac{p}{2}\frac{\partial p}{\partial u^1}+2\mu^1+p\mu^2\right) &= 0, \\ \frac{\partial \lambda^2_{s1}}{\partial u^2}-\frac{p}{2}\frac{\partial \lambda^2_{s1}}{\partial u^1}+\lambda^2_{s1}\mu^2-\frac{1}{2}\frac{\partial p}{\partial u^1}\lambda^2_{s1} &= 0. \end{aligned} \tag{13}$$

Denote

$$G(u^1,u^2)=-\frac{\partial p}{\partial u^2}+\frac{p}{2}\frac{\partial p}{\partial u^1}+2\mu^1+p\mu^2.$$

As the considered immersion is C^3-smooth, the function $G(u^1,u^2)$ is continuous. Set $D=\{(u^1,u^2):\ G(u^1,u^2)=0\}$. Denote by $\overset{\circ}{D}$ the interior of D. The range of definition G splits into a closed subset $\partial\overset{\circ}{D}$ and a union of open subsets, on each of which $G(u^1,u^2)$ is either a null function or a nonvanishing function, or it vanishes at the points of a null set.

If in a neighborhood of P_0 the function $G(u^1,u^2)$ is nonvanishing or it vanishes on a null set, then in a neighborhood of this point the system (13) has the unique solution $\lambda^2_{s1}=0$ (for all $s=\overline{1,n-2}$) since the function λ^2_{s1} is continuously differentiable. From (11) and (8), it follows that $\vec{a}_s(u^1,u^2)$ are constant vectors, and a considered submanifold (6) is a cylinder with $(n-2)$-dimensional rulings over the two-dimensional parabolic surface F^2.

If $G(u^1,u^2)$ is a null function in a neighborhood of P_0, namely,

$$-\frac{\partial p}{\partial u^2}+\frac{p}{2}\frac{\partial p}{\partial u^1}+2\mu^1+p\mu^2\equiv 0, \tag{14}$$

then system (13) has a nontrivial solution. Geometrically, (14) means the following. It is known [6, 2] that on a parabolic surface F^2 there is an asymptotic direction. With respect to the chosen frame (7), the asymptotic direction is $Z=\{-p/2,\ 1\}$ (we assume that $e_1,\ e_2$ correspond to $\vec{\rho}_1,\ \vec{\rho}_2$). Taking into account that on a surface F^2

$$\nabla_{e_1}e_1=0,\ \nabla_{e_1}e_2=0,\ \nabla_{e_2}e_2=\mu^1e_1+\mu^2e_2,$$

and $e_2=\dfrac{p}{2}e_1+Z$, we find

$$\begin{aligned} \nabla_Z Z=\nabla_{-\frac{p}{2}e_1+e_2}\left(-\frac{p}{2}e_1+e_2\right) &= \frac{p}{2}\frac{1}{2}\frac{\partial p}{\partial u^1}e_1-\frac{1}{2}\frac{\partial p}{\partial u^2}e_1+\mu^1e_1+\mu^2e_2 \\ &= \frac{1}{2}\left(\frac{p}{2}\frac{\partial p}{\partial u^1}-\frac{\partial p}{\partial u^2}+2\mu^1+p\mu^2\right)e_1+\mu^2 Z. \end{aligned}$$

We can see that if (14) holds, then the integral curve for Z is a pregeodesic and it becomes a geodesic after affine parameter rescaling. Hence, in this case, the asymptotic curve on F^2 is a straight line. Therefore, there exists a parameterization

$$\vec{\rho}(w^1, w^2) = \vec{\psi}(w^1) + w^2\vec{\varphi}(w^1),$$

where

$$\vec{\rho}_1 = \vec{\psi}'(w^1) + w^2\vec{\varphi}'(w^1), \qquad \vec{\rho}_2 = \vec{\varphi}(w^1),$$

$$\vec{\rho}_{11} = \vec{\psi}''(w^1) + w^2\vec{\varphi}''(w^1) = \hat{\xi}_1, \qquad \vec{\rho}_{12} = \vec{\varphi}'(w^1) = \hat{\xi}_2, \qquad \vec{\rho}_{22} = 0.$$

The regularity of a surface F^2 implies a linear independence of

$$\vec{\varphi}(w^1),\ \vec{\varphi}'(w^1),\ \vec{\psi}'(w^1),\ \vec{\psi}''(w^1).$$

Hence, $\vec{\varphi}(w^1) \neq \overrightarrow{\mathrm{const}}$, and the surface F^2 is a ruled one with the trivial kernel of the affine fundamental form.

Now we construct a ruled submanifold f over the basic surface F^2

$$\vec{r}(w^1, w^2, v) = \vec{\rho}(w^1, w^2) + v^s\vec{a}_s(w^1, w^2),\ \text{where}\ \frac{\partial\vec{a}_s}{\partial w^i} = \lambda^k_{si}\vec{\rho}_k.$$

From (11) with $p = 0$, we obtain $\lambda^1_{s1} = \lambda^1_{s2} = \lambda^2_{s2} = 0$. From (13), we get $\dfrac{\partial\lambda^2_{s1}}{\partial w^2} = 0$ and, hence, $\lambda^2_{s1} = \lambda_s(w^1)$. From

$$\frac{\partial\vec{a}_s}{\partial w^1} = \lambda^2_{s1}\vec{\rho}_2 = \lambda_s(w^1)\vec{\varphi}(w^1), \qquad \frac{\partial\vec{a}_s}{\partial w^2} = 0$$

we find

$$\vec{a_s} = \int \lambda_s(w^1)\vec{\varphi}(w^1)dw^1.$$

Thus, the local parameterization of parabolic submanifold (6) is

$$\vec{r}(w^1, w^2, v^1, \ldots, v^{n-2}) = \vec{\psi}(w^1) + w^2\vec{\varphi}(w^1) + \sum_{s=1}^{n-2} v^s \int \lambda_s(w^1)\vec{\varphi}(w^1)dw^1.$$

Taking into account the independence of $\vec{\varphi}(w^1), \vec{\varphi}'(w^1),\ \vec{\psi}'(w^1),\ \vec{\psi}''(w^1)$, we can see that in this case the submanifold is a ruled submanifold with $(n-1)$-dimensional rulings over a curve, i.e., a ruled submanifold in a classical meaning.

Example. *The elliptic submanifold of rank two.*

We construct the affine elliptic submanifold $M^4 \to \mathbb{R}^6$ with flat connection. The classification of flat affine surfaces in $\mathbb{R}^4$ with flat normal connection was given by M. Magid and L. Vrancken in [7]. A unique surface of elliptic type is a complex curve. Denote $u^1 = x,\ u^2 = y$ and take, for example, in (6)

$$\vec{\rho}(x, y) = \{x, y, x^2 - y^2, 2xy\}.$$

Then $\vec{\rho}'_1 = \{1, 0, 2x, 2y\}$, $\vec{\rho}'_2 = \{0, 1, -2y, 2x\}$. Gauss decomposition (7) yields

$$\vec{\rho}''_{11} = \{0, 0, 2, 0\} = \xi_1,\ \vec{\rho}''_{12} = \{0, 0, 0, 2\} = \xi_2,\ \vec{\rho}''_{22} = \{0, 0, -2, 0\} = -\xi_1.$$

Therefore, for our example $\mu_1 = \mu_2 = 0,\ q = -1,\ p = 0$. The system (12) is

$$\frac{\partial \lambda^1_{s1}}{\partial y} = -\frac{\partial \lambda^2_{s1}}{\partial x},\quad \frac{\partial \lambda^1_{s1}}{\partial x} = \frac{\partial \lambda^2_{s1}}{\partial y}.$$

Thus, the functions λ^1_{s1} and λ^2_{s1} are the real and the imaginary parts of the complex function f_{s1}. Put, for example, $\lambda^1_{11} = 0,\ \lambda^2_{11} = 1,\ \lambda^1_{21} = x,\ \lambda^2_{21} = y$. From (10), we obtain $\lambda^1_{12} = -1,\ \lambda^2_{12} = 0,\ \lambda^1_{22} = -y,\ \lambda^2_{22} = x$. To find $\vec{a}_s$, we have the equations:

$$\frac{\partial \vec{a}_1}{\partial x} = \vec{\rho}'_2,\quad \frac{\partial \vec{a}_1}{\partial y} = -\vec{\rho}'_1,\quad \frac{\partial \vec{a}_2}{\partial x} = x\vec{\rho}'_1 + y\vec{\rho}'_2,\quad \frac{\partial \vec{a}_2}{\partial y} = -y\vec{\rho}'_1 + x\vec{\rho}'_2.$$

Therefore,

$$\vec{a}_1 = \{-y, x, -2xy, x^2 - y^2\},\ \vec{a}_2 = \{\frac{1}{2}x^2 - \frac{1}{2}y^2, xy, \frac{2}{3}x^3 - 2xy^2, -\frac{2}{3}y^3 + 2x^2y\}.$$

Thus, submanifold (6) obtains the following parameterization:

$$\vec{r}(x, y, v^1, v^2) = \begin{pmatrix} x \\ y \\ x^2 - y^2 \\ 2xy \\ 0 \\ 0 \end{pmatrix} + v^1 \begin{pmatrix} -y \\ x \\ -2xy \\ x^2 - y^2 \\ 1 \\ 0 \end{pmatrix} + v^2 \begin{pmatrix} \frac{1}{2}x^2 - \frac{1}{2}y^2 \\ xy \\ \frac{2}{3}x^3 - 2xy^2 \\ -\frac{2}{3}y^3 + 2x^2y \\ 0 \\ 1 \end{pmatrix}$$

It is easy to see that if the base of the elliptic submanifold is the complex curve

$$\vec{\varphi} = \{g_1(z), g_2(z)\},$$

$z = u^1 + iu^2$, then the parameterization of this submanifold takes the form (6), where $\vec{\rho}(u^1, u^2)$ is a real form of the vector function $\vec{\varphi}(z)$, and the vector functions $\vec{a}_s(u^1, u^2)$ are the real forms of

$$\int f_s(z) d\vec{\varphi}(z).$$

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References

[1] *A.A. Borisenko*, On Complete Parabolic Surfaces. — *J. Soviet Math.* **48** (1990), No. 1, 6–14.

[2] *A.A. Borisenko*, Affine classification of Points of multidimensional surfaces. — *Syb. Math. J.* **31** (1990), No. 3, 379–387.

[3] *A.A. Borisenko*, Extrinsic Geometry of Strongly Parabolic Multidimensional Submanifolds. — *Russ. Math. Surv.* **52** (1997), 1141–1190.

[4] *V.V. Glazyrin*, Topological and Metric Structures of k-saddle Surfaces. — *Syb. Math. J.* **19** (1978), 389–395.

[5] *M. Dajczer and L. Florit*, A class of Austere Submanifolds. — *Illinois J. Math.* **45** (2001), 735–755.

[6] *M. Dajczer and P. Morais*, Parabolic Submanifolds of Rank Two. — *Matemática Contemporânea, Sociedade Brasileira de Matemática* **34** (2008), 195–232.

[7] *M. Magid and L. Vrancken*, Flat Affine Surfaces in $\mathbb{R}^4$ with Flat Normal Connection. — *Geometriae Dedicata* **81** (2000), 19–31.

[8] *K. Nomizu and T. Sasaki*, Affine Differential Geometry. Cambridge Univ. Press, Cambridge, 1994.

[9] *C. Scharlach and L. Vrancken*, Parallel Surfaces in Affine 4-space. — *Abh. Math. Sem. Univ. Hamburg* (2003), No. 73, 67–179.

[10] *S.Z. Shefel*, Two Classes of k-dimensional Surfaces in n-dimensional Euclidean Space. — *Syb. Math. J.* **10** (1969), No. 2, 328–333. (Russian)

[11] *O.O. Shugailo*, On Affine Immersions with Flat Connections. — *J. Math. Phys., Anal., Geom.* **8** (2012), No. 1, 90–105.