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Global Weak Solutions of the Navier–Stokes/Fokker–Planck/Poisson Linked Equations

O. Anoshchenko

Department of Mechanics and Mathematics, V.N. Karazin Kharkiv National University
4 Svobody Sq., Kharkiv 61077, Ukraine

E-mail: anoshchenko@univer.kharkov.ua

S. Iegorov

EPAM Systems
63 Kolomens'ka Str., Kharkiv 61166, Ukraine

E-mail: sergii_iegorov@epam.com

E. Khruslov

B. Verkin Institute for Low Temperature Physics and Engineering
National Academy of Sciences of Ukraine
47 Lenin Ave., Kharkiv 61103, Ukraine

E-mail: khruslov@ilt.kharkov.ua

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We consider the initial boundary value problem for the linked Navier–Stokes/Fokker–Planck/Poisson equations describing the flow of a viscous incompressible fluid with highly dispersed infusion of solid charged particles which are subjected to a random impact from thermal motion of the fluid molecules. We prove the existence of global weak solutions for the problem and study some properties of these solutions.

Key words: Navier–Stokes equation, Fokker–Planck equation, Poisson equation, global weak solution, modified Galerkin method, fixed point Schauder theorem, compactness of approximations.

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In the paper, we consider a system of the linked Navier–Stokes/Fokker–Planck/Poisson equations which describes the flow of viscous incompressible fluid with highly dispersed infusion of charged particles. These mixtures of fluid (or gas) and solid dispersive phase can be found both in nature (aerosols) and technical appliances (electrostatic precipitators).

In the flows of these mixtures solid particles are subjected to hydrodynamic (Stokes), gravitational and electrostatic forces. They are also subjected to a random impact from the thermal motion of fluid molecules. Speeds of solid particles in the flow differ significantly (the local speed distribution is close to the Maxwell one). Moreover, the motion of the solid phase fractions with different particle sizes is different. Therefore the solid phase of the mixture should be described with a distribution function of the particles over the coordinate, speed and size.

In the paper, we assume that solid particles are spheres and their radii r_ε lay in the range $(0, \varepsilon)$, where ε is a small parameter which characterizes the sizes of particles and the average distance $d_\varepsilon = O(\varepsilon^{\frac{1}{3}})$ between the neighboring particles. We also assume that the charges q_ε are of the same sign and proportional to some power of radii r_ε: $q_\varepsilon \sim q r_\varepsilon^\kappa$ $(1 \leq \kappa \leq 2)$ (this matches experimental data for highly dispersed aerosols [1]).

In this case, the system of equations which describes suspension motion has the form

$$\frac{\partial u}{\partial t} + (u \cdot \nabla_x) u - \nu \Delta_x u + \alpha \int_0^1 \int_{R_3} r(u - v) f dv dr - \nabla p = g; \quad x \in \Omega,\ t > 0, \tag{0.1}$$

$$\operatorname{div}_x u = 0, \tag{0.2}$$

$$-\Delta_x \phi = q \int_0^1 \int_{R_3} r f dv\, dr, \quad x \in \Omega, \tag{0.3}$$

$$\frac{\partial f}{\partial t} + (v \cdot \nabla_x) f + \operatorname{div}_v [\Gamma_r(x, v, t) f] = \sigma_r \Delta_v f \quad x \in \Omega,\ v \in R_3,\ t > 0, \tag{0.4}$$

$$\Gamma_r = \frac{\beta}{r^2}[u(x,t) - v] - \frac{\gamma}{r^{3-\kappa}} \nabla_x \phi(x,t) + g_1, \quad \sigma_r = \sigma r^{-5}. \tag{0.5}$$

Here: $u = u(x,t)$ and $p = p(x,t)$ are the velocity and the pressure of the fluid; $f = f(x, v, r, t)$ is a normalized solid particle distribution function with respect to the coordinates $x \in \Omega$, velocities $v \in R_3$ and reduced radii $r = \frac{r_\varepsilon}{\varepsilon} \in (0; 1]$ (where ε is the maximum particle radius); g, $(1 - \frac{\rho_0}{\rho_1}) g$ are the vectors of gravitational and Archimedean forces; Δ_x and Δ_v are notations for the Laplace operators over the variables $x \in R_3$ and $v \in R_3$, respectively; ∇_x is the gradient operator; the scalar product in R_3 is denoted by $\cdot$: $u \cdot v = \sum_{i=1}^3 u_i v_i$, $u \cdot \nabla_x = \sum_{i=1}^3 u_i \frac{\partial}{\partial x_i}$.

The numeric parameters α, β and γ are expressed in terms of mixture characteristics

$$\alpha = 6\pi\nu, \quad \beta = \frac{9\nu\rho_0}{2\rho_1\varepsilon^2}, \quad \gamma = \frac{3q}{4\pi\rho_1\varepsilon^{3-\kappa}},$$

where $\nu = \frac{\mu}{\rho_0}$ is the kinematic viscosity of the fluid, μ is the molecular viscosity, ρ_0, ρ_1 are the densities of the fluid and solid phases ($\rho_0 \ll \rho_1$); σ_r is the diffusion coefficient caused by the thermal movement of the particles. By the Einstein formula [2, 3],

$$\sigma_r = kT\frac{6\pi\mu r_\varepsilon}{m_\varepsilon^2} = \frac{\sigma}{r^5},$$

where k is the Boltzmann constant, T is the absolute temperature, m_ε is the solid particle mass, r_ε is its radius; $r = \frac{r_\varepsilon}{\varepsilon}$, $\sigma = kT\frac{27\mu}{8\pi\rho_1^2\varepsilon^5}$.

The perturbed system of the Navier–Stokes equations (0.1)–(0.2) and the Poisson equation (0.3) are considered in a bounded space domain $\Omega \subseteq R_3$ ($x \in \Omega$), while the Fokker–Planck equation (0.4), which depends on the parameter $r \in (0,1]$, is considered in the phase space of $R_3 \times R_3$ ($(x,v) \in \Omega \times R_3$).

We assume that for the velocity vector $u(x,t)$, the electric field potential $\phi(x,t)$ and the particle distribution function $f(x,v,r,t)$, the following homogeneous boundary-value conditions are fulfilled :

$$u(x,t) = 0, \quad x \in \partial\Omega,\ t \geq 0, \tag{0.6}$$

$$\phi(x,t) = 0, \quad x \in \partial\Omega,\ t \geq 0, \tag{0.7}$$

$$f(x,v,r,t) = 0, \quad (x,v) \in \Sigma^-,\ t \geq 0,\ r \in (0,1], \tag{0.8}$$

where $\Sigma^- = \{(x,v) \in \partial\Omega \times R_3 : (n(x),v) < 0\}$, and $n(x)$ is the vector of outer normal to $\partial\Omega$ at the point $x \in \Omega$. Condition (0.8) means that the particles do not enter the domain Ω from outside and if the particles reach the boundary from inside they stick to it.

We complement equations (0.1)–(0.5) and boundary conditions (0.6)–(0.8) with the initial conditions:

$$u(x,0) = u_0(x), \quad x \in \Omega, \tag{0.9}$$

$$f(x,v,r,0) = f_0(x,v,r), \quad (x,v) \in \Omega \times R_3,\ r \in (0,1], \tag{0.10}$$

where $u_0(x) \in H_0^1(\Omega)$, $f_0(x,v,r)$ are given initial field of fluid speeds and initial particle distribution function, respectively. Moreover, $\mathrm{div} u_0 = 0$, $f_0(x,v,r) \geq 0$ and $f_0(x,v,r) = 0$ when $(x,v) \in \Sigma^-$.

The goal of the paper is to prove the existence of weak solutions of the problem (0.1)–(0.10).

R e m a r k. The problem is considered with homogeneous boundary conditions. The inhomogeneous case with the boundary conditions $u(x,t) = U(x)$, $\phi(x,t) = \Phi(x)$ when $x \in \partial\Omega$, $t \geq 0$; $f(x,v,r,t) = F(x,v,r)$, where $(x,v) \in \Sigma^-$, $r \in [0,1]$ $(U(x), \Phi(x) \in C^2(\partial\Omega)$, $\int_{\partial\Omega} U dS = 0$, $F(x,v,r) \in C^2(\partial\Omega \times R_3 \times [0,1]))$ can be reduced to the homogeneous case.

The solvability of the initial-boundary value problems for the coupled kinetic (Fokker–Planck, Vlasov) and hydrodynamic (Navier–Stokes, Stokes) equations was studied in [4, 5] for the case of monodispersible (solid phase with particles of the same radius) and in [6, 7], for the case of polydispersible solid phase. Numerous papers are dedicated to studying the solutions of the initial-boundary problems for the coupled Vlasov/Poisson and Fokker–Planck/Poisson equations [8–14]. The system of the linked Navier–Stokes/Vlasov/Poisson equations which describes the flow of a polydispersible suspension of charged particles was considered in [6]. In the paper, the existence of global weak solutions of the initial-boundary value problem in a convex domain Ω and with the normalized radii of solid particles bounded from zero $(r \geq a > 0)$ was proved.

In the present paper, we prove the existence of global weak solutions for the system (0.1)–(0.4), i.e., for the polydispersible suspension of charged particles in an arbitrary domain Ω without lower bound for the particle radii $(0 < r \leq 1)$. The outline of the paper is as follows. In Sec. 1, we define a weak solution for the problem (0.1)–(0.10) and formulate the main result. In Sec. 2, we regularize the system (0.1)–(0.4) by cutting the force of interaction between the particles and the fluid, limiting the particle velocity, and define weak solutions for the regularized system. Then we construct the finite-dimensional approximations of the solution by using the Galerkin approach for the Navier–Stokes system and solving the regularized problem for the Fokker–Planck equation. Subsequently we apply the Schauder fixed point theorem. In Sec. 4, we prove the compactness for the approximations constructed in Sec. 3. Finally, in Sec. 5 we pass to the limit in the dimension and in the cutting parameter in the approximate integral identities. As a result, we get the required integral identities for the weak solution for the problem (0.1)–(0.10).

Generally, the scheme of the proof is the same as that described in [6], but there are difficulties caused by the diffusion term in the Fokker–Planck equation and the absence of the lower bound for particle radii. In order to get over these difficulties, we construct different approximating functions by using the methods developed in [7].

1. Definition of Weak Solution for Problem (0.1)–(0.10) and Formulation of Main Result

Let Ω be a bounded domain in R_3 with a smooth boundary $\partial\Omega$. We use the following notations: $G = \Omega \times R_3$ $(x \in \Omega, v \in R_3)$; $\langle\cdot,\cdot\rangle_\Omega$, $\langle\cdot,\cdot\rangle_G$, $\|\cdot\|_\Omega$, $\|\cdot\|_G$ are the scalar products and the norms in $L_2(\Omega)$ and $L_2(G)$, respectively; $Q = \Omega \times (0,1]$, $D = G \times (0,1)$, $r \in (0;1]$; $\Sigma = \partial\Omega \times R_3, \Sigma^\pm = \{x, v, \in \Omega;\ \pm n(x) \cdot v > 0\}$; $n(x)$ is the outer normal to $\partial\Omega$ at the point $x \in \Omega$; $H_0^1(\Omega)$ is a Sobolev space of the functions equal to zero at $\partial\Omega$; $J = J(\Omega)$, $J_0^1 = J_0^1(\Omega)$ are the closures of divergence-free vector functions from $C_0^1(\Omega)$ in $L_2(\Omega)$ and $H_0^1(\Omega)$, respectively; $H_0^1(R_3)$ is a closure of the set of functions $\psi(v) \in C^1(R_3)$ having a compact support by the norm $\|\psi\|_1 = \|\nabla\psi\|_{L_2(R_3)}$; $L_{2\sigma_r}(Q \times [0,T], H_0^1(R_3))$ is a space of functions with values in $H_0^1(R_3)$ defined in $Q \times [0,T]$ and having a finite L_2-norm with the weight σ_r:

$$\|f\|^2 = \int_0^T \int_Q \|f\|_1^2 \sigma_r dxdrdt.$$

We assume that initial data for the problem (0.1)–(0.10) fulfill the following conditions:

$$u_0(x) \in J_0^1(\Omega), \quad f_0(x,v,r) \geq 0, \quad f_0(x,v,r) \in L_\infty(D). \tag{1.1}$$

Moreover, $\exists\kappa > 0$, $a \geq 2$ (which depend on $f_0 \in L_\infty(D)$) such that

$$\sup_D [f_0(x,v,r)\exp(\frac{\kappa}{r^a})] \leq A < \infty \tag{1.2}$$

and

$$\int_D (r^{-9} + r^3|v|^2) f_0(x,v,r) dxdvdr \leq A_1 < \infty. \tag{1.3}$$

It is clear that the set of these functions $f_0(x,v,r)$ is dense in $L_1(D)$.

We will be looking for weak solutions for the problem (0.1)–(0.10) in the following classes of functions $\forall T > 0$:

$$u(x,t) \in U_T(\Omega) \equiv L_\infty(0,T; J(\Omega)) \cap L_2(0,T; J_0^1(\Omega));$$

$$\phi(x,t) \in \Phi_T(\Omega) \equiv L_2(0,T; H_0^1(\Omega));$$

$$f(x,v,r,t) \in F_T(D) \equiv L_{2\sigma_r}(Q \times [0,T]; H_0^1(R_3)) \cap L_\infty(D \times [0,T]).$$

Definition 1.1. *The triple of functions $(u, \phi, f) \in U_T(\Omega) \times \Phi_T(\Omega) \times F_T(D)$ is a weak solution of the problem (0.1)–(0.10) if the following identities are satisfied:*

$$\int_0^T \{\langle u, \zeta_t + u \cdot \nabla_x \zeta\rangle_\Omega - \nu\langle\nabla_x u, \nabla_x \zeta\rangle_\Omega - \alpha\langle \int_0^1 \int_{R_3} r(u-v) f dvdr, \zeta\rangle_\Omega$$

$$+\langle g, \zeta\rangle_\Omega\}dt + \langle u_0, \zeta(0)\rangle_\Omega = 0, \tag{1.4}$$

$$\int_0^T \{\langle \nabla_x \phi, \nabla_x \eta\rangle_\Omega - g\langle \int_0^1 \int_{R_3} r f dv dr, \eta\rangle_\Omega\}dt = 0, \tag{1.5}$$

$$\int_0^T \int_0^1 \{\langle f, \xi + v\cdot\nabla_x\xi + \Gamma_r\cdot\nabla_v\xi\rangle_G - \sigma_r\langle\nabla_v f, \nabla_v\xi\rangle_G\}drdt$$

$$+\int_0^1 \langle f_0, \xi(0)\rangle_G dr = 0 \tag{1.6}$$

for any vector function $\zeta(x,t)$ and the functions $\eta(x,t)$, $\xi(x,v,r,t)$ which satisfy the following conditions:

$$\zeta \in U_T(\Omega)\cap L_\infty(\Omega\times[0,T]), \quad \zeta_t \in L_2(\Omega\times[0,T]), \quad \zeta(x,T) = 0;$$

$$\eta \in \Phi_T(\Omega);$$

$$\xi \in F_T(D), \quad \xi(x,v,r,T) = 0, \quad \xi|_{\Sigma^+_{1T}} = 0 \quad (\Sigma^\pm_{1T} = \Sigma^\pm\times(0,1]\times[0,T])$$

$$\xi_t,\ r^{-\frac{5}{2}}\nabla_x\xi,\ r^{-\frac{5}{2}}\nabla_v\xi \ \in\ L_1(D\times[0,T])\cap L_\infty(D\times[0,T]), \tag{1.7}$$

where $\xi(x,v,r,t)$ has a compact support with respect to $v\in R_3$.

If the above conditions are satisfied for any $T>0$, then the solution (u,ϕ,f) is called global.

The main result of this paper is the following.

Theorem 1.2. *Let the initial data $u_0(x)$ and $f_0(x,v,r)$ satisfy conditions* (1.1)–(1.3). *Then if in condition* (1.2) $\sup a > 2$, *then there exists a global solution (u,ϕ,f) for the problem* (0.1)–(0.10). *In case where* $\sup a = 2$, *there exists a weak solution $(u,\phi,f)\in U_T(\Omega)\cap\Phi_T(\Omega)\cap F_T(D)$ when $T < \sup\kappa(3\beta)^{-1}$.*

Theorem (1.2) is proved in Secs. 3–5.

The next theorem describes some properties of the weak solution for the problem (0.1)–(0.10).

Theorem 1.3. *The weak solution $\{u(x,t),\phi(x,t),f(x,v,r,t)\}$ has the properties:*

(i) *the function $f(x,v,r,t)$ is continuous with respect to t in the topology $L_1(D)$;*

(ii) $f(x,v,r,t) > 0$*;*

(iii) $\int_D f(x,v,r,t)dxdvdr \le \int_D f_0(x,v,r)dxdvdr$;

(iv) *the vector function* $u(x,t)$ *and the function* $\phi(x,t)$ *are continuous with respect to* t *in the weak topology* $L_2(\Omega)$;

(v) *the estimate*

$$\max_{0\le t\le T}(\|u\|^2_\Omega + \|\nabla\phi\|^2_\Omega + \int\limits_D r^3|v|^2 f dxdvdr$$

$$+\int\limits_0^T(\|\nabla_x u\|^2_\Omega + \int\limits_D r(u-v)^2 f dxdvdr)dt < C$$

is valid, where C *depends on* u_0 *and* f_0.

2. Initial Boundary Value Problem for the Fokker–Planck Equation

In this section we consider a special (regularized) initial-boundary problem for the Fokker–Planck equation (0.4)–(0.5) and establish some properties of its solution. These properties are used to construct approximations for the main problem (0.1)–(0.10) solution from Sec. 3.

Let V_R be a ball in R_3 with the radius R; ∂V_R denote its boundary: $V_R = \{v \in R_3 : |v| < R\}$, $\partial V_R = \{v \in R_3 : |v| = R\}$.

Consider the initial-boundary problem in the domain $\Omega \times V_R \times [0,T]$:

$$\frac{\partial f}{\partial t} + v\cdot\nabla_x f + \mathrm{div}_v[\Gamma^R_r(x,v,t)f] - \sigma_r\Delta_v f = h, \quad (x,v,t) \in \Omega\times V_R\times[0,T], \tag{2.1}$$

$$f(x,v,t) = 0, \quad (x,v,t) \in \Omega\times\partial V_R\times[0,T], \tag{2.2}$$

$$f(x,v,t) = 0, \quad (x,v) \in \partial\Omega\times V_R,\ v\cdot n(x) \le 0,\ t\in[0,T], \tag{2.3}$$

$$f(x,v,0) = f_0(x,v), \quad (x,v) \in \Omega\times V_R. \tag{2.4}$$

The vector function $\Gamma^R_r(x,v,t)$ is defined by the equality

$$\Gamma^R_r(x,v,t) = \beta_r(u(x,t)-v)\Theta_R(|u-v|^2) - \gamma_r\nabla_x\phi(x,t) + g_1, \tag{2.5}$$

where $\Theta_R(s) = \Theta(\frac{s}{R})$, $\Theta(s)$ is a $C^2(0,\infty)$ function such that $\Theta(s) = 1$ when $s < \frac{1}{2}$, $\Theta(s) = 0$ when $s > 1$, $\frac{\partial\Theta}{\partial s} \le 0$, $\beta_r = \beta r^{-2}$, $\gamma_r = \gamma r^{-3+\kappa}$, $\sigma_r = \sigma r^{-5}$, $r \in (0,1]$; $f^R_0(x,v,r) = f_0(x,v,r)\Theta_R(|v|)$; $f_0(x,v)$ and $h = h(x,v,t)$ are given functions.

A function $f(x,v,t) \in L_2(\Omega \times [0,T]; H_0^1(V_R))$ is a weak solution for the problem (2.1)–(2.4) if it satisfies the equality

$$\int_0^T \int_{V_R} \int_\Omega \{f(\frac{\partial \xi}{\partial t} + v \cdot \nabla \xi) + (\Gamma_r^R f - \sigma_r \nabla_v f) \cdot \nabla_v \xi + h \cdot \xi\} dx dv dt$$

$$= \int_{V_R} \int_\Omega f_0^R \xi(x,v,0) dx dv \tag{2.6}$$

for any function $\xi(x,v,t) \in H^1(\Omega \times V_R \times [0,T])$ such that $\xi(x,v,T) = 0$, $\xi|_{S_{RT}} = 0$,$\xi|_{\Sigma^+_{RT}} = 0$, where $S_{RT} = \{(x,v,t) \in \Omega \times V_R \times [0,T]\}$, $\Sigma^+_{RT} = \{(x,v,t) \in \partial\Omega \times V_R \times [0,T],\, n(x) \cdot v > 0\}$.

The following theorem is true.

Theorem 2.1. *Let $u(x,t) \in L_\infty(\Omega \times [0,T])$, $\nabla_x \phi(x,t) \in L_\infty(\Omega \times [0,T])$, $h(x,v,t) \in L_2(\Omega \times [0,T]; H^{-1}(V_R))$, $f_0(x,v) \in L_2(\Omega \times V_R)$.*

Then for all r, R $(0 < r \leq 1,\ R > 2)$ there exists a weak solution for the problem (2.1)–(2.5) *in the class*

$$Y = \{f \in L_2(\Omega \times [0,T];\, H_0^1(V_R)),\, \frac{\partial f}{\partial t} + v \cdot \nabla_x f \in L_2(\Omega \times [0,T];\, H^{-1}(V_R))\}.$$

We formulate the properties and estimates for the solution $f(x,v,t)$ we will require for further proof. We use the following notations: $|\cdot|_\infty$, $|\cdot|_1$, $|\cdot|_2$ are norms in the spaces $L_\infty(\Omega \times V_R)$, $L_1(\Omega \times V_R)$, $L_2(\Omega \times V_R)$; $\|\cdot\|_\infty$, $\|\cdot\|_2$ are norms in the spaces $L_\infty(\Omega \times [0,T])$ and $L_2(\Omega \times V_R \times [0,T])$, respectively.

(j) Positivity: if $f_0 \geq 0$ and $h \geq 0$, then $f \geq 0$;

(jj) L_∞ estimate: if $f_0 \in L_\infty(\Omega \times V_R)$ and $h \in L_1(0,T; L_\infty(\Omega \times V_R))$, then $f \in L_\infty(\Omega \times V_R \times [0,T])$ and the following estimate is true:

$$|f(t)|_\infty \leq |f_0|_\infty e^{3\beta_r t} + \int_0^t e^{3\beta_r (t-s)} |h(s)|_\infty ds;$$

(jjj) L_1 estimate: if $f_0 \in L_1(\Omega \times V_R)$ and $h \in L_1(\Omega \times V_R \times [0,T])$, then $f \in L_\infty(0,T;\, L_1(\Omega \times V_R))$ and

$$|f(t)|_1 \leq |f_0|_1 + \int_0^t |h(s)|_1 ds;$$

(jv) L_2 estimate: if $f_0 \in L_2(\Omega \times V_R)$ and $h \in L_2(\Omega \times V_R \times [0,T])$, then $f \in L_\infty(0,T;\, L_2(\Omega \times V_R))$ and

$$|f(t)|_2^2 + 2\sigma_r \int_0^t |\nabla_v f(s)|^2 ds \le |f_0|_2^2 e^{(3\beta_r+\delta)t} + \frac{2}{\delta}\int_0^t e^{(3\beta_r+\delta)(t-s)}|h(s)|_2^2 ds;$$

(v) Continuous dependency on the initial conditions and coefficients. Assume that $f_i(x,v,t)$ is a solution for the problem (2.1)–(2.5) which corresponds to $\{f_{0i}, u_i, \phi_i, h_i\}$, $(i = 1,2)$, moreover, $f_{0i} \in L_2(\Omega \times V_R)$, $u_i \in L_\infty(\Omega \times [0,T])$, $\nabla\phi \in L_\infty(\Omega \times [0,T])$, $h_i \in L_2(\Omega \times V_R \times [0,T])$. Then for $\forall\delta > 0$, the following inequality is true:

$$\max_{0<t\le T} |[f(t)]|_2^2 + \sigma_r \|\nabla_v [f]\|_2^2 \le (|[f_0]|_2^2 + \frac{2}{\delta}\|[h]\|_2^2)e^{(3\beta_r+\delta)T}$$

$$+(\beta_r^2\|[u]\|_\infty^2 + \gamma_r^2\|[\nabla_x\phi]\|_\infty^2)(\frac{1}{2\sigma_r}|f_{02}|_2^2 + \frac{1}{\delta\sigma_r}\|h_2\|_2^2)e^{(3\beta_r+\delta)T},$$

where $[\cdot]$ denotes the difference $[u] = u_1 - u_2$.

The proof of Theorem 2.1 and properties (j)–(v) for the case $\phi \equiv 0$ are given in [7]. The proof is completely similar to that for $\phi \ne 0$ and we do not give it here.

3. Regularization and Construction of Approximate Solutions for Problem (0.1)–(0.10)

Consider the following regularization for the problem (0.1)–(0.10):

$$\frac{\partial u}{\partial t} + u \cdot \nabla_x u - \nu\Delta_x u + \alpha \int_0^1 \int_{V_R} r\Theta_R(|u-v|^2)(u-v) f\, dv dr - \nabla_x p = g, \quad (3.1)$$

$$\mathrm{div} u = 0, \quad (x,t) \in \Omega \times [0,T],$$

$$u = 0, \quad (x,t) \in \partial\Omega \times [0,T], \quad (3.2)$$

$$u(x,0) = u_0(x), \quad x \in \Omega, \quad (3.3)$$

$$\varepsilon\Delta_x^2\phi - \Delta_x\phi = q\int_0^1 \int_{V_R} r\Theta_R(|v|) f\, dv dr, \quad x \in \Omega, \quad (3.4)$$

$$\phi = \frac{\partial\phi}{\partial n} = 0, \quad x \in \partial\Omega, \quad (3.5)$$

$$\frac{\partial f}{\partial t} v \cdot \nabla_x f + \mathrm{div}_v[\Gamma_r^R(x,v,t)f] = \sigma_r \Delta_v f, \quad (x,v,t) \in \Omega \times V_R \times [0,T], \tag{3.6}$$

$$f = 0, \quad (x,v,t) \in \Omega \times \partial V_R \times [0,T], \tag{3.7}$$

$$f = 0, \quad (x,v,t) \in \partial\Omega \times V_R \times [0,T],\ v \cdot n(x) < 0, \tag{3.8}$$

$$f(x,v,0) = f_0^R(x,v), \quad (x,v) \in \Omega \times V_R, \tag{3.9}$$

where $\varepsilon > 0$, V_R, $\Theta_R(s)$, $\Gamma_r^R(x,v,t)$ and f_0^R are the same as in Sec. 2.

The weak solution for this problem is introduced in the same way as in definition (1.4)–(1.6): $(u,\phi,f) \in U_T(\Omega) \times \Phi_T(\Omega) \times F_T(D_R)$ $(D_R = \Omega \times V_R \times (0,1])$, and the following integral equalities are satisfied:

$$\int_0^T \{\langle u, \zeta_t + u \cdot \nabla_x \zeta\rangle_\Omega - \nu\langle \nabla_x u, \nabla_x \xi\rangle_\Omega - \alpha\langle \int_0^1 \int_{V_R} r\Theta_R(|u-v|^2)(u-v) \times f dv dr, \zeta\rangle_\Omega + \langle g, \zeta\rangle_\Omega\} dt + \langle u_0, \zeta(0)\rangle_\Omega = 0, \tag{3.10}$$

$$\int_0^T \{\varepsilon\langle \Delta_x \phi, \Delta_x \eta\rangle_\Omega + \langle \nabla_x \phi, \nabla_x \eta\rangle_\Omega - q\langle \int_0^1 \int_{V_R} r\Theta_R f dv dr, \eta\rangle\} dt = 0, \tag{3.11}$$

$$\int_0^T \int_0^1 \{\langle f, \xi_t + v \cdot \nabla_x \xi + \Gamma_r \cdot \nabla_v \xi\rangle_G - \sigma_r \langle \nabla_v f, \nabla_v \xi\rangle_G\} dr dt + \int_0^1 \langle f_0^R, \xi(0)\rangle_G dr = 0 \tag{3.12}$$

for any vector function $\zeta(x,t) \in U_T(\Omega)$ and the functions $\eta(x,t) \in \Phi_T(\Omega)$, $\xi \in F_T(D_R)$ which satisfy conditions (1.7) and the condition $\xi(x,v,r,t) = 0$ when $(x,v,r,t) \in \Omega \times \partial V_R \times [0,1] \times [0,T]$.

We construct the approximate solutions $\{u^{(n)}, \phi^{(n)}, f^{(n)}\}$ for the problem (3.1)–(3.9) by using the Galerkin approximations for $u^{(n)}(x,t)$. Let $\{\psi^k(x)\}_{k=1}^\infty$ be an orthonormal basis in $L_2(\Omega)$ which consists of eigenfunctions of the problem

$$-\Delta\psi^{(k)}(x) + \nabla p^{(k)} = \lambda_k(\psi^{(k)}), \quad \mathrm{div}\psi^{(k)}(x) = 0, \quad x \in \Omega$$

$$\psi^{(k)}(x) = 0, \quad x \in \partial\Omega\,.$$

Let

$$u^{(n)}(x,t) = \sum_{k=1}^n C_k^{(n)}(t)\psi^{(k)}(x), \tag{3.13}$$

where $C_k^{(n)}(t) \in C^1[0,T]$ are unknown functions which satisfy $C_k^{(n)}(0) = \int_\Omega u_0(x)\psi^k(x)dx = C_{0k}$. We calculate the corresponding approximations $\phi^{(n)}$ and $f^{(n)}$ as the solutions for the problems (3.4)–(3.5) and (3.6)–(3.9), respectively. In these problems, $\phi(x,t) = \phi^{(n)}(x,t)$, $f(x,v,r,t) = f^{(n)}(x,v,r,t)$ and

$$\Gamma_r^R(x,v,t) = \beta_r\Theta_R(|u^{(n)} - v|^2)(u^{(n)}(x,t) - v) - \gamma_r\nabla_x\phi^{(n)}(x,t) + g_1. \tag{3.14}$$

To find $C_k^{(n)}(t)$, we require that equality (3.10) be true for $u = u^{(n)}$, $\phi = \phi^{(n)}$, $f = f^{(n)}$ for all vector functions $\zeta(x,t) = h(t)\psi^k$ $(k = 1,2\ldots n)$ where $h(t) \in C^1(0,T)$, $h(T) = 0$. This results in the relation

$$\cdot\nabla_x u^{(n)} + \alpha\int_0^1\int_{V_R} r\Theta_R(|u^{(n)} - v|^2)(u^{(n)} - v)f^{(n)}dxdr, \psi^k\rangle_\Omega$$
$$+\nu\langle\nabla_x u^{(n)}, \nabla_x\psi^k\rangle_\Omega = \langle g, \psi^k\rangle_\Omega, \quad k = 1,\ldots 2n, \tag{3.15}$$

which is a system of differential-functional equations for the coefficients $C_k^{(n)}(t)$,

$$\frac{dC_k^{(n)}}{dt} + \sum_{l,m=1}^n \hat\psi_{klm}C_l^{(n)}C_m^{(n)} + \sum_{l=1}^n \hat\psi_{kl}C_l^{(n)} + \alpha\langle\int_0^1\int_{V_R} r\Theta_R(|\sum_{k=1}^n C_k^{(n)}\psi^k - v|^2)$$
$$\times(\sum_{k=1}^n C_k^{(n)}\psi^k - v)f^{(n)}dvdr,\ \psi^k\rangle_\Omega = \hat g_k, \tag{3.16}$$

with the initial condition

$$C_k^{(n)}(0) = C_{0k}, \quad k = 1\ldots n. \tag{3.17}$$

Here $\hat\psi_{klm} = \hat\psi_{kml}$, $\hat\psi_{lm} = \hat\psi_{ml}$ and $\hat g_k$ are defined by the equalities: $\hat\psi_{klm} = \langle\psi^l\cdot\nabla_x\psi^m,\ \psi^k\rangle_\Omega$, $\hat\psi_{lm} = \nu\langle\nabla_x\psi^k,\ \nabla_x\psi^l\rangle_\Omega$, $\hat g_k = \langle g,\ \psi^k\rangle_\Omega$.

Lemma 3.1. *For all n, R and $\varepsilon > 0$ there exists a solution $\{u^{(n)}, \phi^{(n)}, f^{(n)}\}$ for the problem* (3.16), (3.17), (3.4)–(3.9), *where $u^{(n)}$ is defined by* (3.11) *and Γ_r^R is defined by* (3.12).

P r o o f. We denote by $w = \{e(t), \phi(x,t)\}$ the elements of the space $B = (C[0,T])^n \bigotimes L_2[0,T;\ C^2(\Omega)]$, where $e(t) = \{e_1(t)..e_n(t)\}$ is an n-component vector function from $(C(0,T))^n$, $\phi(x,t) \in L_2(0,T;\ C^2(\Omega))$. The norm in B has the form

$$\|w\| = \max_{1\le t\le T}[\sum_{i=1}^n e_i^2(t)]^{\frac12} + (\int_0^T \|\phi\|^2_{C^2(\Omega)}dt)^{\frac12}.$$

Let K be a bounded closed convex set in W:

$$K = \{w \in B : \ \|w\| \le C(R, \varepsilon, T); \ e_i, (0) = c_{0i} \ i = 1 \dots n\}. \tag{3.18}$$

The constant $C(R, \varepsilon, T)$ will be chosen further; c_{0i} are defined by equalities (3.17).

Let $w^0 = (e_1^0(t) \dots e_n^0(t); \ \phi^0(x,t))$ be an arbitrary element in K. Assume that

$$u^0(x,t) = \sum_{k=1}^{n} e_k^0(t) \psi^k(x). \tag{3.19}$$

After solving the problem (2.1)–(2.5) for $u(x,t) = u^0(x,t)$ and $\phi(x,t) = \phi^0(x,t)$, we can get its solution $f^0(x,v,r,t)$ defined for all $x \in \Omega$, $v \in V_R$, $r \in (0,1]$, $t \in [0,T]$. If T is defined as in Theorem 1.2, then $\sup |f^0(x,v,r,t)| < A_0$ which follows from (1.2) and property (jj) of the solution for the problem (2.1)–(2.5) (see Sec. 2).

The solution $f^0(x,v,r,t)$ being defined, we can find the vector function $e^1(t) = \{e_1^1(t) \dots e_n^1(t)\}$ as a solution for the linearized system (3.16) of the form

$$\frac{de_k^1}{dt} + \sum_{l,m=1}^{n} \hat{\psi}_{klm} e_l^0 e_m^1 + \sum_{l=1}^{n} \hat{\psi}_k l e_l^1$$

$$+ \alpha \langle \int_0^1 \int_{V_R} r \Theta_R(|\sum_{l=1}^{n} C_l^0 \psi^l - v|^2)(\sum_{l=1}^{n} e_l^0 \psi^l - v) f^0 dv dr, \ \psi^k \rangle_\Omega = \hat{g}_k \tag{3.20}$$

with the initial condition

$$e_k^1(0) = C_{0k}. \tag{3.21}$$

This Cauchy problem for the linear system of equations has a unique solution. Then, for given $f^0(x,v,r,t)$ we solve the boundary problem (3.4)–(3.5), where $f = f^0(x,v,r,t)$, and find $\phi^1(x,v,r,t)$. Using the well-known estimates for the solutions of boundary problems for elliptic equations [16] and embedding theorem, we get $\phi^1(x,v,r,t) \in W_2^4(\Omega) \subset C_0^2(\Omega)$. Therefore the operator Λ is defined: $\Lambda : \ K \to (C(0,T))^n \bigotimes L_2(0,T; \ C_0^2(\Omega))$. Taking into account the theorems on the continuous dependency of the solution on the coefficients and the right-hand side for the problems (2.1)–(2.4), (3.17), (3.18) and (3.4), (3.5), we can conclude that Λ is a continuous operator.

We now show that $C(R, \varepsilon, T)$ can be chosen such that L maps K into itself,

$$\|w^0\| \le C(R, \varepsilon, T) \Rightarrow \|w^1\| \le C(R, \varepsilon, T).$$

We rewrite the problem (3.17)–(3.18) in terms of the vector function $\phi^1(x,t) = \sum_{k=1}^{n} e_k^1(t) \psi^k(x)$ in the following way:

$$\langle \frac{du^1}{dt} + u^0 \nabla_x u^1 + \alpha \int_0^1 \int_{V_R} r \Theta_R(|u^0 - v|^2)(u^0 - v) f^0 dv dr, \ \psi^k \rangle_\Omega$$

$$+\nu\langle\nabla u^1,\,\nabla\psi^k\rangle_\Omega=\langle g,\,\psi^k\rangle_\Omega\,,\quad k=1,2\dots n, \tag{3.22}$$

$$u^1(x,0)=u_0(x), \tag{3.23}$$

where $u^0(x,t)=\sum_{k=1}^n e_k^0(t)\psi^k(x)$.

After multiplying the k-th equation by e_k^1 and summing all equations for $k=1\dots n$, we will obtain

$$\frac{1}{2}\frac{d}{dt}\|u^1\|_\Omega^2+\nu\|\nabla u^1\|_\Omega=\langle g,\,u^1\rangle_\Omega-\alpha\langle\int\limits_0^1\int\limits_{V_R} r\Theta_R(|u^0-v|^2)(u^0-v)f^0dvdr,\,u^1\rangle_\Omega. \tag{3.24}$$

As $\forall t\, u(x,t)\in H_0^1(\Omega)$, the first term on the right of (3.24) can be estimated in the following way:

$$|\langle g,\,u^1\rangle_\Omega|\le\frac{\nu}{4}\|\nabla u^1\|_\Omega^2+\frac{1}{\nu\lambda}\|g\|_\Omega^2, \tag{3.25}$$

where λ is the smallest eigenvalue of the operator Δ in Ω with zero boundary conditions.

Similarly, by taking into account properties (jj) and (jjj) of the solution f^0 for the problem (2.1)–(2.5), we can estimate the second term:

$$|\alpha\langle\int\limits_0^1\int\limits_{V_R} r\Theta_R(|u^0-v|^2)(u^0-v)f^0dvdr,\,u^1\rangle_\Omega|$$

$$\le\frac{\nu}{4}\|\nabla u^1\|_\Omega^2+\frac{\alpha^2R^2|V_R|}{\nu\lambda}\max_{D_{1R}}(e^{3\gamma_rT}f_0)\int\limits_{D_{1R}} f_0dxdvdr$$

$$\le\frac{\nu}{4}\|\nabla u^1\|_\Omega^2+C_0(R,T), \tag{3.26}$$

where the constant $C_0(R,T)$ depends on the initial function $f_0(x,v,r)$, and due to its properties (1.1), (1.3), $C_0(R,T)<\infty$ for all $T>0$ if $a>2$. If $a=2$, then $C_0(R,T)<\infty$ for $T<\kappa(3\gamma)^{-1}$.

From (3.24)–(3.26) we get

$$\max_{0\le t\le T}\|u^1\|_\Omega^2+\nu\int\limits_0^T\|\nabla u^1\|_\Omega^2dt\le\|u_0\|_\Omega^2+C(\|g\|_\Omega^2T$$

$$+C_0(R,T)T)=C_1(R,T). \tag{3.27}$$

The following estimate is true for the solution $\phi^1(x,t)$ for the problem (3.4)–(3.5) [16]:

$$\|\phi^1\|_{W_2^4(\Omega)}^2\le C(\varepsilon)\|Q\|_{L_2(\Omega)}^2\quad\forall t,$$

where $C(\varepsilon) > 0$ does not depend on t and

$$Q(x,t) = q\int_0^1\int_{V_R} r\Theta_R(|v|)f^0(x,v,r,t)dvdr. \qquad (3.28)$$

We estimate the norm of Q in $L_2(\Omega)$ similarly to (3.23), i.e., taking into account properties (jj) and (jjj) of the solution f^0 for the problem (2.1)–(2.5) and properties (1.3)–(1.4) of the initial function $f_0(x,v,r)$. Then we use the embedding theorem and obtain the inequality

$$\int_0^T \|\phi^1\|^2_{C^2(\Omega)}dt \le C\int_0^T \|\phi^1\|^2_{W_2^4(\Omega)}dt \le C_1(\varepsilon). \qquad (3.29)$$

Due to the Parseval identity, (3.28) and (3.29) result in

$$\max_{0\le t\le T}\sum_{k=1}^n (e_k^1(t))^2 + \int_0^T \|\phi^1\|^2_{C^2}dx \le C_1(R,T) + C_1(\varepsilon).$$

Choosing the constant $C(R,T,\varepsilon) = C_1^{\frac{1}{2}}(R,T) + C_1^{\frac{1}{2}}(\varepsilon)$ in the definition of K, we notice that $w^1 = \{e^1(t), \phi^1(x,t)\} \in K$ if $w^0 \in K$. This means that the operator Λ maps K into itself. Now we show that its image ΛK is compact in $(C(t))^n \times L_2(0,T,C^2(\Omega))$. To this end, we estimate the derivative $\frac{dw^1}{dt}$. We multiply the k-th equation from (3.22) by $\frac{de^1}{dt}$ and sum all equations from 1 to n:

$$\|u_t^1\|^2_\Omega + \frac{\nu}{2}\frac{d}{dt}\|\nabla u^1\|^2_\Omega = \langle g,\, u_t^1\rangle_\Omega - \langle u^0\cdot\nabla_x u^1,\, u_t^1\rangle_\Omega$$

$$-\alpha\langle\int_0^1 r\Theta_R(|u^0-v|^2)(u^0-v)f^0dvdr,\, u_t^1\rangle_\Omega.$$

Then we estimate the terms on the right-hand side by using the Young inequality to obtain

$$\frac{1}{4}\|u_t^1\|^2_\Omega + \frac{\nu}{2}\frac{d}{dt}\|\nabla u^1\|^2_\Omega \le \|g\|^2_\Omega + |u^0|^2_{C(\Omega)}\|\nabla u^1\|^2_\Omega + C_0(R,T),$$

where the constant $C_0(R,T)$ is the same as in (3.26). Integrating this inequality by $t \in [0,T]$, we get

$$\frac{1}{4}\int_0^T \|u_t^1\|^2_\Omega dt + \frac{\nu}{2}\|\nabla u^1(T)\|^2_\Omega \le T\|g\|_\Omega + \|u^0\|^2_\infty\int_0^T \|\nabla u^1\|^2_\Omega dt$$

$$+C_0(R,T)+\frac{\nu}{2}\|\nabla u^1(0)\|^2_\Omega. \tag{3.30}$$

Due to the eigenfunction properties $\psi^k(x) \in H^2(\Omega)$, the norms $\|u^0\|_\infty$ and $\|\nabla u^1(0)\|_\Omega$ are finite (but depend on n and R). Therefore, (3.30) results in

$$\int_0^T \|u^1_t\|^2_\Omega dt \le C(u,R,T)$$

and then, according to the Parseval identity, we get

$$\int_0^T \sum_{k=1}^n \left(\frac{\partial e^1_k}{\partial t}\right)^2 dt \le C(u,R,T).$$

As a result, the vector function $e^1(t)$ is in the space $W^1_2[0,T]$ which is compactly embedded into $C[0,T]$. To finish the proof of the compactness of the operator Λ, we will use the following lemma [17].

Lemma 3.2. *Let B_0,B and B_1 be Banach spaces such that $B_0 \subset B \subset B_1$, B_0 and B_1 are reflexive, and the embedding of B_0 in B is compact. Consider the Banach space*

$$W=\{v:\ v\in L_{p_0}(0,T;\,B_0),\, v_t=\frac{dv}{dt}\in L_{p_1}(0,T;\,B_1)\},$$

where $0<T<\infty$ and $1<p_i<\infty$, $i=0,1$. The norm in the space W is defined as a sum

$$\|v\|_{L_{p_0}(0,T;\,B_0)}+\|v_t\|_{L_{p_0}(0,T;\,B_1)}.$$

Then the embedding of W in $L_{p_0}(0,T;\,B)$ is compact.

According to this lemma, we introduce the Banach spaces $B_0 = W^4_2(\Omega)$, $B = C^2(\Omega)$, $B_1 = L_2(\Omega)$ and

$$W=\{\phi(x,t):\ \phi(x,t)\in L_2(0,T;\,W^4_2(\Omega));\ \phi'_t\in L_2(0,T;\,L_2(\Omega))\}. \tag{3.31}$$

Then all the conditions of Lemma 3.2 are satisfied, and if we prove that $\frac{\partial\phi^1}{\partial t} \in L_2(0,T;\,L_2(\Omega))$, then we will prove the compactness of embedding of space (3.31) in $L_2(0,T;\,C^2(\Omega))$. By the definition of the function $\phi^1(x,t)$, its derivative $\phi^1_t(x,t)=\frac{\partial\phi^1}{\partial t}$ is a solution of the boundary problem

$$\varepsilon\Delta^2\phi^1_t-\Delta\phi^1_t=q\int_0^1\int_{V_R} r\Theta_R(|v|)\frac{\partial f^0}{\partial t}(x,v,r,t)dvdr, \quad x\in\Omega\times[0,T], \tag{3.32}$$

$$\phi_t^1(x,t) = \frac{\partial \phi_t^1}{\partial n}(x,t) = 0, \quad x \in \partial\Omega \times [0,T]. \tag{3.33}$$

Taking into account (2.1) for the function $f^0(x,v,r,t)$, (3.32) can be rewritten in the form

$$\varepsilon\Delta^2\phi_t^1 - \Delta\phi_t^1 = -q\int_0^1\int_{V_R} rv\cdot\nabla_x f^0\Theta_R(|v|)dvdr - q\int_0^1\int_{V_R} r\Theta_R(|v|)$$

$$\times\mathrm{div}_v(\Gamma_r^R(x,v,t)f^0)dvdr + q\int_0^1\int_{V_R}\sigma_r\Delta_v f^0\Theta_R(|v|)dvdr. \tag{3.34}$$

Here $\Gamma_r^R(x,v,t)$ is defined by (2.5), where $u(x,t) = u^0(x,t)$ and $\phi(x,t) = \phi^0(x,t)$. Multiply (3.34) by $\phi_t^1(x,t)$ and integrate it with respect to $x \in \Omega$. Then, after integration by parts with respect to v and x, we get

$$\varepsilon\|\Delta\phi_t^1\|_\Omega^2 + \|\nabla\phi_t^1\|_\Omega^2 = q\int_{D_{1R}} rv\cdot\nabla_x\phi_t^1 f^0\Theta_R(|v|)dxdvdr$$

$$+q\int_{D_{1R}}\frac{\beta}{r}\Theta_R(|u^0-v|^2)(u^0-v)\cdot\nabla_v\Theta_R(|v|)f^0\phi_t^1 dxdvdr$$

$$-q\int_{D_{1R}}\frac{\gamma}{r^{2-k}}\nabla_x\phi^0\cdot\nabla_v\Theta_R(|v|)f^0\phi_t^1 dxdvdr + q\int_{D_{1R}} g_1\cdot\nabla_v\Theta_R(|v|)f^0\phi_t^1 dxdvdr$$

$$+q\int_{D_{1R}}\frac{\sigma}{r^4}\Delta_v\Theta_R(|v|)f^0\cdot\phi_t^1 dxdvdr = F(t), \tag{3.35}$$

where $D_{1R} = \Omega\times V_R\times(0,1]$.

Now we apply the Friedrichs inequality

$$\|\phi_t^1\|_\Omega^2 \le C\|\nabla\phi_t^1\|_\Omega^2 \tag{3.36}$$

and estimate the right part of (3.35) with the account of properties (jj) and (jjj) of the solution $f^0(x,v,r,t)$,

$$|F(t)| \le \frac{1}{2}\|\nabla\phi_t^1\|_\Omega^2 + C(R)(1+\|\phi^0(t)\|_{C^1(\Omega)})\max_{D_{1R}}(f_0^R e^{\frac{3\beta}{r}t})\int_{D_{1R}} r^{-4}f_0^R dxdvdr, \tag{3.37}$$

where f_0^R is the initial function for the problem (2.1)–(2.5).

Since $w^0 = (c^0(t), \phi^0(x,t)) \in K$ and therefore

$$\int\limits_0^T |\phi^0|^2_{C^2(\Omega)} dx < C(R, \varepsilon, T),$$

from (3.35)–(3.37) we can conclude that

$$\int\limits_0^T \|\phi^1_t\|^2_\Omega dt \leq \hat{C}(R, \varepsilon, T)$$

and T is chosen according to Theorem 1.2.

Thus, we have proved that the operator Λ continuously maps a closed bounded convex set $K \subset (C[0,T])^n \times L_2(0,T;\, C^2(\Omega))$ into itself and its image $\Lambda(K)$ is compact in $(C(0,T))^n \times L_2(0,T;\, C^2(\Omega))$. According to the Schauder theorem, the map Λ has a fixed point $w \in (e_1(t)..e_n(t), \phi(x,t)) \in K$. The sequence $w^i = \Lambda^i w^0$ ($w^i \to w$ in $B_0 = (C(0,T)^n \times L_2(0,T;\, C^2_0(\Omega)))$ when $i \to \infty$) corresponds to the sequence of the solutions f^i for the problem (2.1)–(2.5) which converges in $L_\infty(D_R \times [0,t])$ and in $L_{2\sigma_r}(Q \times [0,T], H^1_0(V_R))$ due to property (v) (Sec. 2). Taking the above into account as well as (3.15), (3.16), (3.20), (3.4)–(3.5) and (3.6)–(3.9), we can conclude that the limit functions $u^{(n)}_{R,\varepsilon}(x,t) = \sum e^{(n)}_k(t)\phi_k(x)$, $\phi^{(n)}_{R,\varepsilon}(x,t)$ and $f^{(n)}_{R,\varepsilon}(x,v,r,t)$ satisfy the identities

$$\int\limits_0^T \{\langle u^{(n)}_{R,\varepsilon},\, \zeta^{(m)}_t + u^{(n)}_{R,\varepsilon} \cdot \nabla_x \zeta^{(m)}\rangle_\Omega - \nu\langle \nabla_x u^{(n)}_{R,\varepsilon},\, \nabla_x \zeta^{(m)}\rangle_\Omega$$

$$-\alpha\langle \int\limits_0^1 \int\limits_{V_R} r\Theta_R(|u^{(n)} - v|^2)(u^{(n)}_{R,\varepsilon} - v) f^{(n)}_{R,\varepsilon} dv dr,\, \zeta^{(m)}\rangle_\Omega + \langle g,\, \zeta\rangle_\Omega\} dt$$

$$+\langle u_0, \zeta^{(m)}(0)\rangle_\Omega = 0, \tag{3.38}$$

$$\int\limits_0^T \{\varepsilon\langle \Delta_x \phi^{(n)}_{R,\varepsilon},\, \Delta_x \eta\rangle_\Omega + \langle \nabla_x \phi^{(n)}_{R,\varepsilon},\, \nabla_x \eta\rangle_\Omega$$

$$-q\langle \int\limits_0^1 \int\limits_{V_R} r\Theta_R(|v|) f^{(n)}_{R,\varepsilon} dv dr,\, \eta\rangle_\Omega\} dt = 0, \tag{3.39}$$

$$\int\limits_0^T \int\limits_0^1 \{\langle f^{(n)}_{R,\varepsilon},\, \xi_t + v \cdot \nabla_x \xi + \Gamma^R_r(u^{(n)}_{R,\varepsilon}, \nabla\phi^{(n)}_{R,\varepsilon}, v) \cdot \nabla_v \xi\rangle_G + \sigma_r \langle \nabla_v f^{(n)}_{R,\varepsilon},\, \nabla_v \xi\rangle_G\} dr dt$$

$$+\int_0^1 \langle f_0^R,\, \xi(0)\rangle_G dr = 0 \tag{3.40}$$

for any vector function $\zeta^{(m)}(x,t)$,

$$\zeta^{(m)}(x,t) = \sum_{k=1}^{m} h_k^{(m)}(t)\psi^k(x), \quad h_k^{(m)}(x) \in C^1[0,T],$$

$$h^{(m)}(T) = 0 \quad \forall m \le n,$$

and arbitrary functions $\xi(x,v,r,t)$ which satisfy (1.7) and $\eta(x,t) \in L_2(0,T; W_2^2(\Omega))$.

If now we pass to the limit in these identities for $n \to \infty$, $R \to \infty$ and $\varepsilon \to 0$, we will obtain (1.4)–(1.6) which define the solution for (0.1)–(0.10). To this end, we have to study the compactness properties for the set of approximations $\{u_{R,\varepsilon}^{(n)}, \phi_{R,\varepsilon}^{(n)}, f_{R,\varepsilon}^{(n)};\ n = 1,2,\ldots, \varepsilon > 0, R > 0\}$.

4. Compactness of Approximations

Lemma 4.1. *The following uniform (with respect to R, ε) inequalities are true:*

$$0 \le f_{R,\varepsilon}^{(n)}(x,v,r,t) \le A_1 \quad (0 \le t < T), \tag{4.1}$$

$$\int_G f_{R,\varepsilon}^{(n)}(x,v,r,t)dxdv \le \int_G f_0(x,v,r)dxdv \quad \forall r > 0, \tag{4.2}$$

where $G_R = \Omega \times V_R$ and the constant A_1 only depends on the initial function $f_0(x,v,r)$; the time T is defined in the same way as in Theorem 1.2.

P r o o f. These inequalities follow from properties (j), (jj) and (jjj) of the solution for the problem (2.1)–(2.5) and properties (1.1)–(1.3) of the initial function $f_0(x,v,r)$.

Lemma 4.2. *There exists a function $\hat{R}_T(\varepsilon) : (0,\infty) \to (0,\infty)$ such that for all $n = 1,2\ldots$, $\varepsilon > 0$ and $R \ge \hat{R}_T(\varepsilon)$ the inequality*

$$\max_{0\le t\le T}\{\|u_{R,\varepsilon}^{(n)}\|_\Omega^2 + \int_{D_{1R}} r^3|v|^2 f_{R,\varepsilon}^{(n)} dxdvdr + \varepsilon\|\Delta\phi_{R,\varepsilon}^{(n)}\|_\Omega^2 + \|\nabla\phi_{R,\varepsilon}^{(n)}\|_\Omega^2\}$$

$$+\int_0^T \|\nabla u_{R,\varepsilon}^{(n)}\|_\Omega^2 dt + \int_0^T\int_{D_{1R}} r\Theta_R(|u_{R,\varepsilon}^{(n)} - v|^2)|u_{R,\varepsilon}^{(n)} - v|^2 f_{R,\varepsilon}^{(n)} dxdvdrdt \le A_3$$

is true, where $D_{1R} = \Omega \times V_R \times (0,1]$, and the constant A_3 depends on u_0, f_0,T and the parameters α, β, γ, σ; T is defined in Theorem 1.2.

P r o o f. It follows from (3.16) that $e_k^{(n)}(t) \in C^1[0,T]$, and therefore due to (3.11) and the properties of eigenfunctions, $u_{R,\varepsilon}^{(n)}(x,t) \in C^1(\Omega \times [0,T]) \cap C(0,T;\, C^2(\Omega))$ and $f_{R,\varepsilon}^{(n)}(x,v,r,t) \in L_\infty(D_R \times [0,T]) \cap L_{2\sigma_r}(Q \times [0,T];\, H_0^1(V_R))$. Thus, if the initial function $f_0^R(x,v,r)$ is smooth enough ($\forall r,\ f_0^R(x,v,r) \in C_0^1(G_R)$), then the weak solution $f_{R,\varepsilon}^{(n)}(x,v,r,t)$ for the problem (2.1)–(2.5) is strong, i.e., $f_{R,\varepsilon}^{(n)} \in L_2(\Omega \times [0,T],\, W_2^2(V_R) \cap W_2^1(G_R \times [0,T]))$. Similarly, the weak solution $\phi_{R,\varepsilon}^{(n)}$ for the problem (3.4)–(3.5) is strong: $\phi_{R,\varepsilon}^{(n)} \in W_2^4(\Omega) \cap \mathring{W}_2^2(\Omega)$. We multiply the k-th equality (3.22) by $c_k^{(n)}(t)$ and then sum all equalities from 1 to n. Taking into account (3.12), we come to the equality

$$\frac{1}{2}\frac{d}{dt}\|u^{(n)}\|_\Omega^2 + \nu\|u^{(n)}\|_\Omega^2$$
$$+\alpha \int\limits_{D_{1R}} r\Theta_R(|u^{(n)} - v|^2)(u^{(n)} - v)\cdot u^{(n)} f^{(n)} dxdvdr = \langle g,\, u^{(n)}\rangle_\Omega. \tag{4.3}$$

Here and further the subscripts R and ε are temporarily omitted for simplicity.

Now we obtain an estimate for $\phi^{(n)}(x,t)$. According to (3.4)–(3.5), the derivative $\phi_t^{(n)} = \frac{\partial \phi^{(n)}}{\partial t}$ is the solution for the boundary problem

$$\varepsilon\Delta^2\phi_t^{(n)} - \Delta\phi_t^{(n)} = q\int\limits_0^t r\Theta_R(|v|)\frac{\partial f^{(n)}}{\partial t} dvdr \quad x \in \Omega$$
$$\phi_t = \frac{\partial \phi^{(n)}}{\partial n_x} = 0 \quad x \in \partial\Omega.$$

Multiplying (3.4) for $f^{(n)}$ by $\phi^{(n)}(x,t)$, integrating by parts with the account of (3.5), we obtain

$$\begin{aligned}
\frac{\varepsilon^2}{2}\frac{d}{dt}\|\Delta\phi^{(n)}\|_\Omega^2 + \frac{1}{2}\frac{d}{dt}\|\nabla\phi^{(n)}\|_\Omega^2 &= q\int\limits_{D_{1R}} r\Theta_R(|v|)\nabla_x\phi^{(n)}\cdot v f^{(n)} dxdvdr \\
&+q\beta\int\limits_{D_{1R}} r^{-1}\Theta_R(|u^{(n)} - v|^2)(u^{(n)} - v)\cdot\nabla_v\Theta_R(|v|)\phi^{(n)} f^{(n)} dxdvdr \\
-q\gamma\int\limits_{D_{1R}} r^{-1}\nabla_x\phi^{(n)}\cdot\nabla_v\Theta_R(|v|)\phi^{(n)}&\cdot f^{(n)} dxdvdr + q\int\limits_{D_{1R}} rq_1\cdot\nabla_v\Theta_R(|v|) \\
&\times\phi^{(n)} f^{(n)} dxdvdr + q\sigma\int\limits_{D_{1R}} r^{-4}\Delta_v\Theta_R(|v|)\phi^{(n)} f^{(n)} dxdvdr.
\end{aligned}\tag{4.4}$$

Now we multiply (2.1) by $r^3|v|^2$ and integrate with respect to $(x, v, r) \in D_{1R}$ and with the account of (2.5). Then, after integrating by parts with respect to $x \in \Omega$ and $v \in V_R$, we obtain

$$\frac{d}{dt}\int\limits_{D_{1R}} r^3|v|^2 f^{(n)} dxdvdr - 2\beta \int\limits_{D_{1R}} r\Theta_R(|u^{(n)} - v|^2)(u^{(n)} - v) \cdot v f^{(n)} dxdvdr$$
$$+2\gamma \int\limits_{D_{1R}} r\nabla_x \phi^{(n)} \cdot v f^{(n)} dxdvdr - 2\int\limits_{D_{1R}} r^3 q_1 \cdot v f^{(n)} dxdvdr - 6\sigma \int\limits_{D_{1R}} r^{-2}$$
$$\times f^{(n)} dxdvdr = -\int\limits_{\Sigma^+_{1R}} r^3|v|^2 n_x \cdot v f^{(n)} dS_x dvdr + \sigma \int\limits_{\Gamma_{1R}} r^{-2} R^2 \frac{\partial f^{(n)}}{\partial n_v} dxdvdr, \quad (4.5)$$

where $\Sigma^+_{1R} = \{(x, v, r) \in \partial\Omega \times V_R \times (0, 1] : n_x \cdot v > 0\}$, $\Gamma_{1R} = \Omega \times \partial V_R \times (0, 1]$, n_x is the outer normal to $\partial\Omega$, n_v is the outer normal to ∂V_R.

According to the properties of the solution for the problem (2.1)–(2.5), $f^{(n)} \geq 0$ everywhere and $f^{(n)} = 0$ on Γ_{1R}, and then $\frac{\partial f}{\partial n_v} \leq 0$ on Γ_{1R}. It follows that the right-hand side of (4.5) is not positive. Taking this into account, we obtain the inequality from (4.3)–(4.5),

$$\frac{1}{2}\frac{d}{dt}(\|u^{(n)}\|^2_\Omega + \frac{\alpha\gamma}{\beta q}\|\nabla\phi^{(n)}\|^2_\Omega + \varepsilon\frac{\alpha\gamma}{\beta q}\|\Delta\phi^{(n)}\|^2_2 + \frac{\alpha}{\beta}\int\limits_{D_{1R}} r^3|v|^2 f^{(n)} dxdvdr)$$
$$+ \nu\|\nabla u^{(n)}\|^2_\Omega + \alpha\int\limits_{D_{1R}} r\Theta_R(|u^{(n)} - v|^2)|u^{(n)} - v|^2 f^{(n)} dxdvdr \leq \int\limits_\Omega g \cdot u^{(n)} dx$$
$$+ \frac{\alpha}{\beta}\int\limits_{D_{1R}} r^3 \cdot q_1 \cdot v f^{(n)} dxdvdr + \frac{3\alpha\sigma}{\beta}\int\limits_{D_{1R}} r^{-2} f^{(n)} dxdvdr$$
$$+ \alpha\gamma\int\limits_{D_{1R}} r^{-1}\Theta_R(|u^{(n)} - v|^2)(u^{(n)} - v) \cdot \nabla_v\Theta_R(|v|)\phi^{(n)} f^{(n)} dxdvdr$$
$$+ \frac{\alpha\gamma\sigma}{\beta}\int\limits_{D_{1R}} r^{-4}\Delta_v\Theta_R(|v|)\phi^{(n)} f^{(n)} dxdvdr$$
$$- \frac{\alpha\gamma^2}{\beta}\int\limits_{D_{1R}} r^{-1}\nabla_x\phi^{(n)} \cdot \nabla_v\Theta_R(|v|)\phi^{(n)} f^{(n)} dxdvdr$$
$$- \frac{\alpha\gamma}{\beta}\int\limits_{D_{1R}} r\nabla_x\phi^{(n)} \cdot v(1 - \Theta_R(|v|)) f^{(n)} dxdvdr = \sum_{k=1}^{7} I_k, \quad (4.6)$$

where $I_k\,(k = 1 \ldots 7)$ denote all summands of the right-hand side of the above inequality. The first three summands can be easily estimated by using the

Minkowski and the Friedrichs inequalities (for $u^{(n)} \in \overset{\circ}{J}{}^1(\Omega)$) with the account of Lemma 4.1 and properties (1)–(3) of the initial function $f_0(x, v, r)$,

$$|I_1| \le \frac{\nu}{2}\|\nabla u^{(n)}\|_\Omega + C_1,$$

$$|I_2| \le \frac{\alpha}{4\beta} \int\limits_{D_{1R}} r^3|v|^2 f^{(n)} dxdvdr + C_2, \tag{4.7}$$

$$|I_3| \le C_3,$$

where C_1, C_2, C_3 do not depend on n, R and $T > 0$ is arbitrary.

To estimate other summands, we use the following estimate for the Green function $G(x, y)$ of the problem (3.4)–(3.9):

$$\max_{x,y\in\Omega} |G(x, y)| + \max_{x,y,\in\Omega} |\nabla_x G(x, y)| \le C\varepsilon^{-1},$$

which can be obtained with the account to the form of the fundamental solution $\Gamma(x, y)$ for (3.4)

$$\Gamma(x, t) = \frac{1}{4\pi|x - y|}(e^{-\frac{|x-y|}{\sqrt{\varepsilon}}} - 1)$$

and the smoothness of $\partial\Omega$.

From this estimate and Lemma 4.1, it follows that the solution $\phi(x, t)$ for the problem (3.4)–(3.5) satisfies the inequality

$$\|\phi(\cdot, t)\|_{L_\infty(\Omega)} + \|\nabla_x\phi(\cdot, t)\|_{L_\infty(\Omega)} \le C\varepsilon^{-1} \int\limits_{D_{1R}} rf^{(n)}(x, v, r, t)dxdvdr$$
$$\le C_0\varepsilon^{-1} \quad (\forall t). \tag{4.8}$$

Taking into account the properties of the function $\Theta_R(|v|)$, Lemma 4.1 and properties (1.2), (1.3) of the initial function $f_0(x, v, r)$, we obtain

$$|I_4| \le \frac{\alpha}{2} \int\limits_{D_{1R}} r\Theta_R(|u^{(n)} - v|^2)|u^{(n)} - v|^2 f^{(n)} dxdvdr + \frac{C_4}{\varepsilon^2 R^2}\,,$$

$$|I_5| \le \frac{C_5}{\varepsilon R^2}\,, \quad |I_6| \le \frac{C_6}{\varepsilon^2 R}\,. \tag{4.9}$$

To estimate the integral I_7, we should notice that $\Theta_R(|v|) = 1$ when $|v| \le \frac{R}{2}$, and therefore we integrate only for $|v| > \frac{R}{2}$. Thus,

$$|I_7| \le \frac{\sqrt{2}\alpha\gamma}{\beta R^{\frac{1}{2}}} \int\limits_{D_{1R}} r|\nabla_x\phi^{(n)}| \cdot |v|^{\frac{3}{2}}(1 - \Theta_R(|v|))f^{(n)} dxdvdr$$

and, applying (4.8), we get

$$|I_7| \le \frac{\sqrt{2}\alpha\gamma C_0}{\beta R^{\frac{1}{2}}\varepsilon} \int\limits_{D_{1R}} r|v|^{\frac{3}{2}} f^{(n)} dxdvdr. \tag{4.10}$$

We now factorize $f = f^{\frac{3}{4}} \cdot f^{\frac{1}{4}}$ and $r = r^{\frac{9}{4}} \cdot r^{-\frac{5}{4}}$ and estimate the integral on the right-hand side of (4.9) by applying the Young inequality (for $q = \frac{4}{3}$ and $p = 4$)

$$\int\limits_{D_{1R}} r|v|^{\frac{3}{2}} f^{(n)} dxdvdr \le \frac{3}{4} \int\limits_{D_{1R}} r^3 |v|^2 f^{(n)} dxdvdr$$
$$+\frac{1}{4} \int\limits_{D_{1R}} r^{-5} f^{(n)} dxdvdr. \tag{4.11}$$

From (4.10), (4.11), Lemma 4.1 and properties (1.2),(1.3) of the initial function f_0, we get the estimate

$$|I_7| \le \frac{C_7}{\varepsilon R^{\frac{1}{2}}} \int\limits_{D_{1R}} r^3 |v|^2 f^{(n)} dxdvdr + \frac{C_8}{\varepsilon R^{\frac{1}{2}}}. \tag{4.12}$$

We now integrate inequality (4.6) with respect to t. Then, taking into account (4.7), (4.9) and (4.12), we obtain

$$\max_{0\le t\le T} (\|u^{(n)}\|^2_\Omega + \frac{\alpha\gamma}{\beta q} \|\nabla\phi^{(n)}\|^2_\Omega + \varepsilon\frac{\alpha\gamma}{\beta q} \|\Delta\phi^{(n)}\|^2_\Omega + \frac{\alpha}{\beta} \int\limits_{D_{1R}} r^3 |v|^2 f^{(n)} dxdvdr)$$
$$+ \frac{\nu}{2} \int\limits_0^T \|\nabla u^{(n)}\|^2_\Omega dt + \frac{\alpha}{2} \int\limits_0^T \int\limits_{D_{1R}} r\Theta_R(|u^{(n)} - v|^2) |u^{(n)} - v|^2 f^{(n)} dxdvdr$$
$$\le (C_1 + C_2 T + C_3)T + (\frac{C_4}{\varepsilon R^{\frac{3}{2}}} + \frac{C_5}{R^{\frac{3}{2}}} + \frac{C_6}{\varepsilon R^{\frac{1}{2}}} + C_8) \frac{T}{\varepsilon R^{\frac{1}{2}}}$$
$$+ (\frac{\alpha}{4\beta} + \frac{C_7 T}{\varepsilon R^{\frac{1}{2}}}) \max_{0\le t\le T} \int\limits_{D_{1R}} r^3 |v|^2 f^{(n)} dxdvdr + \|u_0^{(n)}\|^2_\Omega$$
$$+ \frac{\alpha}{\beta} \int\limits_{D_{1R}} r^3 |v|^2 f_0^{(n)} dxdvdr + \frac{\alpha\gamma}{\beta q} (\varepsilon \|\Delta\phi^{(n)}(0)\|^2_\Omega + \|\nabla\phi^{(n)}(0)\|^2_\Omega). \tag{4.13}$$

The constants $C_i\,(i = 1, \dots, 8)$ on the right-hand side of this inequality are independent of n, ε, R. We choose R such that $\frac{\alpha}{4\beta} + \frac{C_7 T}{\varepsilon R^{\frac{1}{2}}} < \frac{\alpha}{2\beta}$, i.e., $R \ge (\frac{4\beta}{\alpha\varepsilon} C_7 T)^2$, and set $\hat{R}_T(\varepsilon) = (\frac{4\beta}{\alpha\varepsilon} C_7 T)^2$. To prove Lemma 4.2, we have to prove that the last summand on the right-hand side of (4.13) does not depend on n, ε and R. To this end, we multiply (3.4) (for $t = 0$) by $\phi^{(n)}(x, 0)$ and integrate it over Ω. After integrating by parts and applying (3.5), we get

$$\varepsilon\|\Delta\phi^{(n)}(0)\|^2_\Omega + \|\nabla\phi^{(n)}(0)\|^2_\Omega = \int\limits_\Omega Q_R(x)\phi^{(n)}(x, 0)dx, \tag{4.14}$$

where

$$Q_R(x) = q\int\limits_0^1\int\limits_{V_R} r\Theta^2_R(|v|)f_0(x, v, r)dvdr.$$

We now show that $Q_R(x) \in L_{\frac{3}{2}}(\Omega)$ uniformly with respect to R. Since $0 \le \Theta_R(|v|) \le 1$, we have

$$\int\limits_\Omega Q^{\frac{3}{2}}(x)dx \le q^{\frac{3}{2}}\int\limits_\Omega(\int\limits_0^1\int\limits_{V_R} rf_0(x, v, r)dvdr)dx.$$

Multiplying and dividing the integrand by $r^{-2}(1 + r^6|v|^2)^{\frac{2}{3}}$ and then applying the Hölder inequality with the conjugates $\frac{3}{2}$ and 3, we obtain

$$\int\limits_\Omega Q^{\frac{3}{2}}(x)dx \le q^{\frac{3}{2}}\int\limits_\Omega(\int\limits_0^1\int\limits_{V_R} r^{-3}(1 + r^6|v|^2)f_0^{\frac{3}{2}}dvdr)$$
$$\times(\int\limits_0^1\int\limits_{R_3}\frac{r^9}{(1 + r^6|v|^2)^2}dvdr)^{\frac{1}{2}}dx. \tag{4.15}$$

Hence, taking into account the equality

$$\int\limits_0^1\int\limits_{R_3}\frac{r^9}{(1 + r^6|v|^2)^2}dvdr = \int\limits_{R_3}\frac{dw}{(1 + |w|^2)^2} = C < \infty \tag{4.16}$$

and properties (1.2), (1.3) of the initial function $f_0(x, v, r)$, we can conclude that

$$\int\limits_\Omega Q^{\frac{3}{2}}dx \le q^{\frac{3}{2}}|f_0|^{\frac{1}{2}}_\infty\int\limits_D(r^{-3} + r^3|v|^2)f_0(x, v, r)dxdvdr \le A.$$

Therefore, estimating the right-hand side of (4.14) with the Hölder inequality, we obtain

$$\varepsilon\|\Delta\phi^{(n)}(0)\|^2_\Omega + \|\nabla\phi^{(n)}(0)\|^2_\Omega \le \|Q_R\|_{L_{\frac{3}{2}}(\Omega)}\|\phi^{(n)}(0)\|_{L_3(\Omega)} \le A\|\nabla\phi^{(n)}(0)\|_\Omega.$$

Here we used the embedding theorem for $\overset{\circ}{W}{}_2^1(\Omega)$ and $L_3(\Omega)$. From the above inequality it follows that

$$\varepsilon\|\Delta\phi^{(n)}(0)\|_\Omega^2 + \|\nabla\phi^{(n)}(0)\|_\Omega^2 \le C,$$

where C does not depend on n, ε, R. Thus Theorem 4.2 is proved.

We also require the lemma below.

Lemma 4.3. *For $0 < \delta < T$ the following estimate is true:*

$$\int\limits_0^{T-\delta} \|u_{R\varepsilon}^{(n)}(t+\delta) - u_{R\varepsilon}^{(n)}(t)\|_\Omega^2 dt \le C\delta^{\frac{1}{2}},$$

where C does not depend on n, ε, R.

The proof of this lemma can be found in [7] (Lemma 4.1). Consider the sets of the functions

$$U = \{u_{\varepsilon R}^{(n)}(x,t),\, x \in \Omega,\, t \in [0,T],\, n \in \mathbb{N},\, \varepsilon > 0,\, R \ge \hat{R}_T(\varepsilon)\},$$

$$\Phi = \{\phi_{\varepsilon R}^{(n)}(x,t),\, x \in \Omega,\, t \in [0,T],\, n \in \mathbb{N},\, \varepsilon > 0,\, R \ge \hat{R}_T(\varepsilon)\},$$

$$F = \{f_{\varepsilon R}^{(n)}(x,v,r,t),\, (x,v) \in \Omega \times R_3,\, r \in (0,1],\, t \in [0,T],\, n \in \mathbb{N},\, \varepsilon > 0,\, R \ge \hat{R}_T(\varepsilon)\},$$

where $f_{\varepsilon R}^{(n)}(x,v,r,t)$ is continued by zero for $|v| \ge R$, $\hat{R}_T(\varepsilon)$ is defined in Lemma 4.2, T is defined in Theorem 1.2.

Taking into account Lemmas 4.1–4.3, we conclude the following:

1) the set U is *-weakly compact in $L_\infty(0,T;\, J(\Omega))$ and compact in $L_2(\Omega\times[0,T])$;

2) the set Φ is *-weakly compact in $L_\infty(0,T;\, W_2^1(\Omega))$;

3) the set F is *-weakly compact in $L_\infty(D \times [0,T])$ and weakly compact in $L_{2\sigma_r}(Q \times [0,T];\, H_0^1(R_3))$.

These compactness properties are used for passing to the limit for $n \to \infty$, $\varepsilon \to 0$, $R \to \infty$ in identities (3.37)–(3.38) to obtain the required identities (1.4)–(1.6) for the weak solution (u, ϕ, f) for the problem (0.1)–(0.5).

5. Passage to the Limit in (3.38)–(3.40)

We assume that $\varepsilon = \frac{1}{n}$ and $R = Cn^2$, where $C = (\frac{4\beta C_7 T}{\alpha})^2$, T is defined in Theorem 1.2, and C_7 is defined in Lemma 4.2.

We consider the sequences of approximating functions $\{u^{(n)}(x,t)\}$, $\{\phi^{(n)}(x,t)\}$ and $\{f^{(n)}(x,v,r,t)\}$. By virtue of Lemmas 4.1–4.3, the sequences are in the sets U, Φ and F, respectively, and therefore we can chose the subsequences $\{u^{(n_k)}(x,t)\}$, $\{\phi^{(n_k)}(x,t)\}$ and $\{f^{(n_k)}(x,v,r,t)\}$ converging to some functions $u(x,t)$, $\phi(x,t)$, $f(x,v,r,t)$ by means of 1), 2) and 3). We keep the previous notations for these subsequences. We now show that the limit functions $u(x,t)$, $\phi(x,t)$ and $f(x,v,r,t)$ satisfy identities (1.4)–(1.6).

1. Taking into account that $u^{(n)}$ converges to u strongly in $L_2(\Omega\times[0,T])$ and *-weakly in $L_\infty(0,T;\, W_2^1(\Omega))$, and $f^{(n)}$ converges to f *-weakly in $L_\infty(\Omega\times[0,T])$, we pass to the limit in identity (3.38). This is done exactly in the same way as in [7]. As a result, we get (1.4) for $u(x,t)$ and $f(x,v,r,t)$.

2. To obtain (1.5), we pass to the limit in identity (3.39) for $\varepsilon = \frac{1}{n}$ and $n \to \infty$. The first term in it tends to zero. Indeed, as $\eta \in L_\infty(0,T;\, \mathring{W}_2^2(\Omega))$, according to Lemma 4.2,

$$|\frac{1}{n}\int_0^T \langle \Delta\phi^{(n)},\, \Delta\eta\rangle_\Omega dt| \le \frac{1}{\sqrt{n}}\int_0^T \frac{1}{\sqrt{n}}\|\Delta\phi^{(n)}\|_\Omega \|\Delta\eta\|_\Omega dt$$

$$\le \frac{\sqrt{A_3}}{\sqrt{n}}\int_0^T \|\Delta\eta\|_\Omega dt \to 0 \quad (n\to\infty). \tag{5.1}$$

Further, taking into account that $\phi^{(n)}(x,t) \to \phi(x,t)$ *-weakly in $L_\infty(0,T;\, W_2^1(\Omega))$ and $\eta \in \Phi_T(\Omega) \subset L_1(0,T;\, \mathring{W}_2^1(\Omega))$, we pass to the limit in the second summand

$$\lim_{n\to\infty}\int_0^T \langle \nabla_x\phi^{(n)},\, \nabla\eta\rangle_\Omega dt = \int_0^T \langle \nabla_x\phi,\, \nabla\eta\rangle dt. \tag{5.2}$$

In order to pass to the limit in the third term, we present it in the following way:

$$\int_0^T \langle \int_0^1 \int_{R_3} r\Theta^{(n)}(|v|) f^{(n)} dv dr,\, \eta\rangle_\Omega dt = \int_0^T \langle \int_0^1 \int_{|v|>R} r\Theta^{(n)} f^{(n)} dv dr,\, \eta\rangle_\Omega dt$$

$$+\int_0^T \int_{D_R} r\Theta^{(n)} f^{(n)} \eta\, dx dv dr dt = I_{1R}^{(n)} + I_{2R}^{(n)}, \tag{5.3}$$

where $D_R = D \cap \{|v| < R\}$, R is a big number chosen below.

Similarly to (4.15), we obtain

$$\int_{\Omega}(\int_0^1 \int_{|v|>R} r\Omega^{(n)} f^{(n)} dvdr)^{\frac{3}{2}} dx \leq (\int_0^1 \int_{|v|>R} \frac{r^9 dvdr}{(1+r^6|v|^2)^2})^{\frac{1}{2}}$$
$$\times \int_D (r^{-3} + r^3|v|^2) f_0 dxdvdr.$$

Since the integral (4.16) converges and the initial function f_0 satisfies conditions (1.3), we can make the right-hand side of this inequality be small enough by choosing R big enough. Thus, taking into account $\eta(x,t) \in \Phi_T(\Omega) \subset L_2(0,T; L_3(\Omega))$, we may conclude that for any $\delta > 0$ there exists $R(\delta)$ such that

$$|I_{1R}^{(n)}| \leq \delta \tag{5.4}$$

uniformly with respect to n if $R \geq R(\delta)$.

We choose a small δ and a corresponding $R(\delta)$. Since $\Theta^{(n)}(|v|) = 1$ in $D_{R(\delta)}$ for n big enough, the function $r \cdot \eta(x)$ (extended with 0 outside $D_{R(\delta)}$) belongs to $L_1(D \times [0,T])$. Then, taking into account the *-weak convergence of $f^{(n)}$ to f in $L_\infty(D \times [0,T])$, we can see that

$$\lim_{n\to\infty} I_{2R(\delta)}^{(n)} = \int_0^T \int_{D_{R(\delta)}} r f \eta \, dxdvdrdt. \tag{5.5}$$

In [7], it is proven that $f^{(n)}(x,v,r,t)$ converges to $f(x,v,r,t)$ in the weak topology $L_1(D)$ uniformly over t. Therefore, due to Lemma 4.1, $f(x,v,r,t) \in L_1(D \times [0,T])$. Taking into account (5.3)–(5.5), we pass to the limit for $n \to \infty$ and $\delta \to 0$ to obtain the third summand in (1.5). We now recall (5.1), (5.2) and obtain (1.5) for $\phi(x,t)$ and $f(x,v,r,t)$.

3. To obtain (1.6), we pass to the limit for $n \to \infty$ in identity (3.40). It can be done easily for the first, second and fourth summands. Indeed, from the *-weak convergence of $f^{(n)}$ to f in $L_\infty(D \times [0,T])$ and (1.7) it follows that

$$\int_0^T \int_0^1 \langle f^{(n)}, \xi_t \rangle_G drdt = \int_0^T \int_D f^{(n)} \cdot \xi_t dxdvdrdt \to \int_0^T \int_D f \cdot \xi_t dxdvdrdt$$
$$= \int_0^T \int_0^1 \langle f, \xi_t \rangle_G drdt. \tag{5.6}$$

The support of the function $f(x,v,r,t)$ with respect to v being compact, $v\cdot\nabla_x\xi \in L_1(D\times[0,T])$. Thus we get

$$\int_0^T\int_0^1 \langle f^{(n)},\, v\cdot\nabla_x\xi\rangle_G drdt = \int_0^T\int_D f^{(n)}v\cdot\nabla_x\xi dxdvdrdt$$
$$\to \int_0^T\int_D fv\cdot\nabla_x\xi dxdvdrdt = \int_0^T\int_0^1 \langle f, v\cdot\nabla_x\xi\rangle_G drdt. \tag{5.7}$$

Moreover, due to the weak convergence of $f^{(n)}$ to f in $L_{2\sigma_r}(Q\times[0,T];\, H_0^1(R_3))$ ($\sigma_r=\sigma r^{-5}$, $Q=\Omega\times[0,1)$) and (1.7), we have

$$\int_0^T\int_0^1 \sigma_r\langle\nabla_v f^{(n)},\, \nabla_v\xi\rangle_G drdt \to \int_0^T\int_0^1 \sigma_r\langle\nabla_v f,\, \nabla_v\xi\rangle_G drdt \tag{5.8}$$

for $n\to\infty$.

To pass to the limit in the third summand, we rewrite it in the form

$$\int_0^T\int_0^1 \langle\Gamma_r^{(n)},\, \nabla_v\xi\rangle_G drdt = \int_0^T\int_D \beta r^{-2}f^{(n)}\Theta^{(n)}(|u^{(n)}-v|^2)$$
$$\times(u^{(n)}-v)\cdot\nabla_v\xi dxdvdrdt + \int_0^T\int_D \gamma r^{-2}f^{(n)}\nabla_x\phi^{(n)}\cdot\nabla_v\xi dxdvdrdt$$
$$+\int_0^T\int_D f^{(n)}g_1\cdot\nabla_v\xi dxdvdrdt = I_1^{(n)}(\xi)+I_2^{(n)}(\xi)+I_3^{(n)}(\xi), \tag{5.9}$$

where $\Gamma_r^{(n)}=\Gamma_r^R$, $\Theta^{(n)}=\Theta_R$ for $R=Cn^2$.

Due to the *-weak convergence of $f^{(n)}$ to f,

$$\lim_{n\to\infty} I_3^{(n)}(\xi) = \int_0^T\int_D fg_1\cdot\nabla_v\xi dxdvdrdt. \tag{5.10}$$

We now show that

$$\lim_{n\to\infty} I_1^{(n)}(\xi) = I_1(\xi) \equiv \int_0^T\int_D \beta r^{-2}f(u-v)\cdot\nabla_v\xi dxdvdrdt. \tag{5.11}$$

We present the difference $I_1^{(n)}(\xi) - I_1(\xi)$ as follows:

$$I_1^{(n)}(\xi) - I_1(\xi) = \sum_{i=1}^{4} B_i^{(n)}(\xi), \tag{5.12}$$

where

$$B_1^{(n)}(\xi) = \int_0^T \int_D \beta r^{-2}(f^{(n)} - f)(u - v) \cdot \nabla_v \xi dxdvdrdt,$$

$$B_2^{(n)}(\xi) = \int_0^T \int_D \beta r^{-2} f^{(n)}[\Theta^{(n)}(|u^{(n)} - v|^2) - \Theta^{(n)}(|u - v|^2)](u^{(n)} - v) \cdot \nabla_v \xi dxdvdrdt,$$

$$B_3^{(n)}(\xi) = \int_0^T \int_D \beta r^{-2} f^{(n)}[\Theta^{(n)}(|u - v|^2) - 1](u^{(n)} - v) \cdot \nabla_v \xi dxdvdrdt,$$

$$B_4^{(n)}(\xi) = \int_0^T \int_D \beta r^{-2} f^{(n)}(u^{(n)} - u) \cdot \nabla_v \xi dxdvdrdt.$$

According to (1.7), $\xi(x, v, r, t)$ has a compact support $\exists R_\xi\ \xi(x, v, r, t) = 0$ for $|v| > R_\xi$ and, moreover, $r^{-2}\nabla_v \xi \in L_2(D \times [0, T])$. Consequently, $r^{-2}(u(x, t) - v) \cdot \nabla_v \xi \in L_1(D \times [0, T])$. Therefore, due to the *-weak convergence of $f^{(n)}$ to f in $L_\infty(D \times [0, T])$,

$$B_1^{(n)}(\xi) \to 0 \quad \text{for} \quad n \to \infty,\ \forall \xi. \tag{5.13}$$

Further, taking into account that $f^{(n)} \in L_\infty(D \times [0, T])$ and $r^{-2}\nabla_v \xi \in L_\infty(D \times [0, T])$ uniformly with respect to n and using the Cauchy inequality, we obtain

$$|B_2^{(n)}(\xi)| \le C\{\int_0^T \int_{|v|<R_\xi} \int_\Omega |\Theta^{(n)}(|u^{(n)} - v|^2) - \Theta^{(n)}(|u - v|^2)|^2 dxdvdrdt\}$$
$$\times \{R_\xi^{\frac{3}{2}}(\int_0^T \|u^{(n)}\|_\Omega^2 dt)^{\frac{1}{2}} + R_\xi^{\frac{5}{2}} |\Omega|^{\frac{1}{2}} T^{\frac{1}{2}}\}. \tag{5.14}$$

Due to the convergence of $u^{(n)}(x, t)$ to $u(x, t)$ in $L_2(\Omega \times [0, T])$ and the properties of the functions $\Theta^{(n)}(s)$, we have [7]:

$$\lim_{n\to\infty} \int_0^T \int_{|v|\le R_\xi} \int_\Omega |\Theta^{(n)}(|u^{(n)} - v|^2) - \Theta^{(n)}(|u - v|^2)|^2 dxdvdrdt = 0.$$

Therefore, from (5.14) we can conclude that

$$B_2^{(n)}(\xi) \to 0 \quad \text{for } n \to \infty. \tag{5.15}$$

To estimate $B_3^{(n)}(\xi)$, we split the domain Ω into two parts:

$$\Omega_{1T}^A = \{(x,t) \in \Omega_T : |u(x,t)| < A\}, \qquad \Omega_{2T}^A = \Omega \setminus \Omega_{1T}^A.$$

Since $u(x,t) \in L_2(\Omega_T)$, $\text{mes}\Omega_{2T}^A \to 0$ for $A \to \infty$. We now present $B_3^{(n)}(\xi)$ in the form

$$B_3^{(n)}(\xi) = B_{31}^{(n)}(\xi) + B_{32}^{(n)}(\xi), \tag{5.16}$$

where

$$B_{3i}^{(n)}(\xi) = \int_0^T \int_0^1 \int_{|v| \le R_\xi} \int_{\Omega_{iT}^A} \beta r^{-2} f^{(n)} [\Theta^{(n)}(|u-v|^2) - 1](u-v) \cdot \nabla_v \xi dx dv dr dt (i = 1, 2).$$

Since $|u - v|^2 \le (A + R_\xi)^2$ for $x \in \Omega_{1T}^A$ for n big enough ($n \ge N(A, R_\xi)$), $\Theta^{(n)}(|u - v|^2) = 1$, and therefore

$$B_{31}^{(n)}(\xi) = 0 \quad \text{for} \quad n \ge N(A, R_\xi). \tag{5.17}$$

Taking into account that $(\Theta^{(n)} - 1)f^{(n)}$ is bounded uniformly with respect to n and using the Cauchy inequality, we obtain

$$|B_{32}^{(n)}(\xi)| \le C\{\int_0^T \int_0^1 \int_{|v|<R_3} \int_{\Omega_{2T}^A} \frac{|\nabla\xi|^2}{r^4} dx dv dr dt\}^{\frac{1}{2}} \{R_\xi^3 \int_0^T \|u^{(n)}\|_\Omega^2 dt + R_\xi^5 |\Omega_{2T}^A| T\}^{\frac{1}{2}},$$

where C is independent from n and A, $|\Omega_{2T}^A| = \text{mes}\Omega_{2T}^A$.

By virtue of properties (1.7) of the function $\xi(x, v, r, t)$, the first factor in the inequality above tends to zero when $\text{mes}\,\Omega_{2T}^A \to 0$, and therefore $B_{32}^{(n)}(\xi) \to 0$ when $A \to \infty$ uniformly with respect to n. It follows from (5.16), (5.17) that

$$B_3^{(n)}(\xi) \to 0 \quad \text{for} \quad n \to \infty \quad \forall\xi. \tag{5.18}$$

We now estimate the summand $B_4^{(n)}(\xi)$ using the Cauchy inequality and taking into account the boundedness for $f^{(n)}$ in $L_\infty(D \times [0, T])$,

$$|B_4^{(n)}(\xi)| \le C\{R_\xi^3 \int_0^T \|u^{(n)} - u\|_\Omega^2 dt\}^{\frac{1}{2}} \{\int_0^T \int_D \frac{|\nabla\xi|^2}{r^4} dx dv dr dt\}^{\frac{1}{2}}.$$

Due to the convergence of $u^{(n)}$ to u in $L_2(\Omega_T)$ it follows that

$$B_4^{(n)}(\xi) \to 0 \quad \text{for} \quad n \to \infty \quad \forall \xi. \tag{5.19}$$

Combining (5.12), (5.13), (5.15), (5.18) and (5.19), we obtain (5.11).

In order to pass to the limit in $I_2^{(n)}$, we will use the lemma proved in [18].

Lemma 5.1. *Let Ω be a bounded domain in R_3 with a smooth boundary $\partial\Omega$ and $\phi_\varepsilon(x)$ be the solution for the problem:*

$$\varepsilon \Delta^2 \phi_\varepsilon - \Delta \phi_\varepsilon = F \quad (x \in \Omega), \tag{5.20}$$

$$\phi_\varepsilon = 0, \quad \varepsilon \frac{\partial \phi_\varepsilon}{\partial n} = 0 \quad (x \in \partial\Omega), \tag{5.21}$$

where $\varepsilon \geq 0$, $F \in L_p(\Omega)$ $(p > \frac{6}{5})$.

Then

$$\int_\Omega |\nabla \phi_\varepsilon - \nabla \phi_0| dx \to 0 \textit{ for } \varepsilon \to 0$$

uniformly with respect to F such that $\|F\|_{L_p(\Omega)} \leq C$.

We introduce the notations:

$$F^{(n)}(x,t) = q \int_0^1 \int_{R_3} r f^{(n)}(x,v,r,t) \Theta^{(n)}(|v|) dv dr,$$

$$F(x,t) = q \int_0^1 \int_{R_3} r f(x,v,r,t) dv dr, \quad \phi(x,t) = \int_\Omega G(x,y) F(y,t) dy,$$

$$\phi^{(n)}(x,t) \int_\Omega G^{(n)}(x,y) F^{(n)}(y,t) dy, \quad \tilde{\phi}^{(n)}(x,t) = \int_\Omega G(x,y) F^{(n)}(y,t) dy,$$

where f is the *-weak limit of $f^{(n)}$ in $L_\infty(D \times [0,T])$, $G^{(n)}(x,y)$ is the Green function of the problem (5.20)–(5.21) for $\varepsilon = \frac{1}{n}$; $G(x,y)$ is its Green function for $\varepsilon = 0$.

It is clear that $\phi^{(n)}(x,t)$ is the solution for the problem (5.20)–(5.21) for $\varepsilon = \frac{1}{n}$ and $F = F^{(n)}(x,t)$; $\tilde{\phi}^{(n)}(x,t)$ is the solution for $\varepsilon = 0$, $F = F^{(n)}(x,t)$. As shown above (see (4.15), (4.16)), $F^{(n)}(x,t) \in L_{\frac{3}{2}}(\Omega)$ uniformly with respect to n and t. Thus, by Lemma 5.1,

$$\|\nabla \phi^{(n)} - \nabla \tilde{\phi}^{(n)}\|_{L_1(\Omega)} \to 0 \quad \text{for} \quad n \to \infty \tag{5.22}$$

uniformly with respect to t.

Taking into account the *-weak convergence of $f^{(n)}$ in $L_\infty(D\times[0,T])$, we can prove that $F^{(n)}(x,t)$ converges to $F(x,t)$ uniformly with respect to t in the weak topology $L_1(\Omega)$. This can be proved similarly to [7].

Since the integral operator with the kernel $\nabla_x G(x,y)$, which maps $L_1(\Omega)$ into itself, is compact (see [19]), it follows that

$$\|\nabla\tilde{\phi} - \nabla\phi\|_{L_1(\Omega)} \to 0 \quad \text{for} \quad n\to\infty \tag{5.23}$$

uniformly with respect to t.

Considering (5.22), (5.23), we notice that

$$\|\nabla_x\phi^{(n)} - \nabla_x\phi\|_{L_1(\Omega)} \to 0 \quad \text{for} \quad n\to\infty \tag{5.24}$$

uniformly with respect to t.

Now we rewrite the summand $I_2^{(n)}(\xi)$ from (5.9) in the form

$$I_2^{(n)}(\xi) = \int_0^T\int_D \gamma r^{-2} f^{(n)}\nabla_x\phi\cdot\nabla_v\xi dxdvdrdt$$

$$+\int_0^T\int_D \gamma r^{-2} f^{(n)}(\nabla_x\phi^{(n)} - \nabla_x\phi)\cdot\nabla_v\xi dxdvdrdt = I_{21}^{(n)}(\xi) + I_{22}^{(n)}(\xi). \tag{5.25}$$

By (1.7), the vector function $r^{-2}\nabla_v\xi \in L_\infty(D\times[0,T])$ has a compact support ($\xi(x,v,r,t) = 0$ for $|v| > R_\xi$) and $\nabla_x\phi \in L_1(\Omega)$ and hence $r^{-2}\nabla_x\phi\cdot\nabla_v\xi \in L_1(D\times[0,T])$. With the account of the *-weak convergence of $f^{(n)}$ to f,

$$\lim_{n\to\infty} I_{21}^{(n)}(\xi) = \int_0^T\int_D \gamma r^{-2} f\nabla_x\phi\cdot\nabla_v\xi dxdvdrdt. \tag{5.26}$$

We continue the function $\nabla_x\phi^{(n)}(x,t) - \nabla_x\phi(x,t)$ in $D\times[0,T]$ assuming that it does not depend on v and n for $|v| < R_\xi$ and is equal to zero for $|v| > R_\xi$. Then from (5.24) it follows that

$$\|\nabla_x\phi^{(n)} - \nabla_x\phi\|_{L_1(D)} \to 0 \quad \text{for} \quad n\to\infty$$

uniformly with respect to t.

Since $r^{-2}f^{(n)}\nabla_v\xi \in L_\infty(D\times[0,T])$ due to condition (1.7) and Lemma 4.1, we conclude that $I_{22}^{(n)} \to 0$ for $n\to\infty$.

Then according to (5.25), (5.26),

$$\lim_{n\to\infty} I_2^{(n)}(\xi) = \int_0^T \int_D \gamma r^{-2} f \nabla_x \phi \cdot \nabla_v \xi dx dv dr dt. \qquad (5.27)$$

Combining (5.9)–(5.11) and (5.27), we obtain the third term in identity (1.6) and thus prove (1.6). Theorem 1.2 is proved.

Theorem 1.3 can be proven similarly to that in [7] with the estimates of Lemmas 4.1–4.3 being taken into account.

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