Journal of Mathematical Physics, Analysis, Geometry
2017, Vol. 13, No. 2, pp. 173–194
doi: 10.15407/mag13.02.173

On One Class of Non-Dissipative Operators

V.N. Levchuk

Poltava National Technical Yuri Kondratyuk University
24 Pershotravnevyi Ave., Poltava 36011, Ukraine
E-mail: lentina88@mail.ru

Received October 19, 2015, revised November 10, 2016

The non-dissipative operator of integration is studied in the weight space. Its similarity to the operator of integration in the space without weight is proved. The functional model for this operator is obtained.

Key words: non-dissipative operator, characteristic function, similarity, functional model.

Mathematics Subject Classification 2010: 47A45.

The study of the basicity of systems of functions, as a rule, is based on the study of some properties of linear operators. The studying of the so-called class of quasi-exponentials, which provoked special interest, was started by B.S. Pavlov [1] and then developed and continued by S.V. Hruščev, N.K. Nikolsky, B.S. Pavlov [2]. An approach suggested by G.M. Gubreyev [3] is an important method of studying problems of basicity in this realm of analysis. He succeeded in harmonic combination of deep problems of spectral analysis of non-selfadjoint operators and delicate analytical results of the theory of functions. This paper is a development of the ideas of G.M. Gubreyev and V.N. Levchuk [4] by which the study of Dunkl kernels is based on the analysis of a non-selfadjoint operator with two-dimensional imaginary component. (The function $d_\alpha(\lambda) = 2^\alpha\Gamma(\alpha+1)\lambda^{-\alpha}\left(J_\alpha(\lambda) + iJ_{\alpha+1}(\lambda)\right)$ is said to be a Dunkl kernel, where $J_\alpha(\lambda)$ is a Bessel function.) In contrast to [4], here the power dependence of the weight function is not supposed. The paper is dedicated to the study of one class of Volterra non-dissipative operators and to the construction of model representations for them. It turns out that many statements from [4] are general and can be obtained for "arbitrary" weight functions $\varphi(x)$. In Section 1, general properties of the operator B are studied ant its characteristic function is calculated. Calculation of this characteristic function is based on the solution of the equation of the second order which, depending on the choice of the weight $\varphi(x)$, turns into a Bessel equation, a Mathieu equation, or a Lamé equation. Similarity of the studied non-dissipative operator to the operator of integration in the space of quadratically summed functions on $[-a, a]$ is proved. In Section 2, a functional model of a non-dissipative operator in the L. de Branges space is listed, and it is shown that in the special case, when $\varphi(x) = x^\nu$, the Dunkl kernels "coincide" with $E(\lambda)$.

1. Properties of the Operator B

I. Let $\varphi(x)$ be a real function on $[-a, a]$ $(0 < a \leqslant \infty)$ such that

$$\varphi(x) \geqslant 0 \quad (x \in [0, a])\,; \quad \varphi(-x) = (-1)^\nu \varphi(x) \quad (\nu \in \mathbb{R}); \quad \varphi \in C^1(0, a). \tag{1}$$

Define the Hilbert space

$$L^2_\varphi(-a, a) \stackrel{\text{def}}{=} \left\{ f(x) : \int\limits_{-a}^{a} |f(x)|^2 \, |\varphi(x)| \, dx < \infty \right\}. \tag{2}$$

The decomposition

$$L^2_\varphi(-a, a) = L_+ \oplus L_- \tag{3}$$

is true where

$$L_\pm \stackrel{\text{def}}{=} \left\{ f_\pm(x) = \frac{1}{2} \left(f(x) \pm f(-x) \right);\ f(x) \in L^2_\varphi(-a, a) \right\}. \tag{4}$$

Specify the linear operator

$$(\mathrm{B} f)(x) \stackrel{\text{def}}{=} i \int\limits_0^x f_-(t) dt + \frac{i}{\varphi(x)} \int\limits_0^x f_+(t) \varphi(t) dt \tag{5}$$

in $L^2_\varphi(-a, a)$.

Lemma 1. *If* (1) *takes place for* $\varphi(x)$*, then the functions*

$$F_\pm(x, \varphi) = \frac{1}{\varphi(x)} \int\limits_0^x f_\pm(t) \varphi(t) dt \tag{6}$$

satisfy the relations

$$F_\pm(-x, \varphi) = \mp F_\pm(x, \varphi). \tag{7}$$

The proof of the statement follows from (6) after the substitution $t \to -t$ under the integral sign

$$F_+(-x,\varphi) = -\frac{1}{\varphi(x)} \int\limits_0^{-x} f_+(t)(-1)^\nu \varphi(t)dt = -F_+(x,\varphi)$$

in virtue of (1). For $F_-(x,\varphi)$, the proof of (7) is similar. The statement of the lemma is true for $\varphi(x) \equiv 1$ ($\forall x \in (-a,a)$) also.

Statement 1. *Let* $\varphi(x)$ *have the properties* (1) *and*

$$M = \int\limits_0^a \varphi(x) \int\limits_0^x \frac{dt}{\varphi(t)} dx < \infty, \ \widetilde{M} = \int\limits_0^a \frac{1}{\varphi(x)} \int\limits_0^x \varphi(t)dt < \infty, \tag{8}$$

then the operator B (5) *is bounded.*

P r o o f. Relations (3), (7) imply

$$\langle \mathrm{B}f, g\rangle = i \left\langle \int\limits_0^x f_-(t)dt, g_+(x) \right\rangle + i \left\langle \frac{1}{\varphi(x)} \int\limits_0^x f_+(t)\varphi(t)dt, g_-(x) \right\rangle = \mathrm{A}_1 + \mathrm{A}_2.$$

Using the Cauchy–Bunyakovsky inequality, we obtain

$$|\mathrm{A}_1| \leqslant 2 \int\limits_0^a \left\{ \int\limits_0^x |f_-(t)|^2 \varphi(t)dt \right\}^{\frac{1}{2}} \left\{ \int\limits_0^x \frac{dt}{\varphi(t)} \right\}^{\frac{1}{2}} |g_+(x)| \, \varphi(x)dx \leqslant \sqrt{2}\|f_-\|.$$

$$\left\{ \int\limits_0^a |g_+(x)|^2 \varphi(x)dx \right\}^{\frac{1}{2}} \left\{ \int\limits_0^a \varphi(x) \int\limits_0^x \frac{dt}{\varphi(t)} dx \right\}^{\frac{1}{2}} \leqslant \|f_-\| \, \|g_+\| \, \sqrt{\mathrm{M}}.$$

Analogously, $|\mathrm{A}_2| \leqslant \|f_+\| \, \|g_-\| \sqrt{\widetilde{\mathrm{M}}}$. And since $\|f\| \geqslant \|f_\pm\|$, $\|g\| \geqslant \|g_\pm\|$, then

$$|\langle \mathrm{B}f, g\rangle| \leqslant \left(\sqrt{\mathrm{M}} + \sqrt{\widetilde{\mathrm{M}}} \right) \|f\| \cdot \|g\|,$$

which proves the statement.

R e m a r k 1. Conditions (8) are also necessary for the boundedness of the operator B (5). After the partial integration, (8) implies

$$b = \int\limits_0^a \varphi(x)dx < \infty, \qquad \widetilde{b} = \int\limits_0^a \frac{dx}{\varphi(x)} < \infty. \tag{9}$$

If one of the integrals in (8) converges and (9) takes place, then the second integral in (8) converges also.

It is easy to see that

$$(\mathrm{B}^* f)(x) = -i\left(\int\limits_x^a f_-(t)dt + \frac{1}{\varphi(x)}\int\limits_x^a f_+(t)\varphi(t)dt\right)\chi_+(x)$$

$$+i\left(\int\limits_{-a}^x f_-(t)dt + \frac{1}{\varphi(x)}\int\limits_{-a}^x f_+(t)\varphi(t)dt\right)\chi_-(x),$$

where $\chi_\pm(x)$ are characteristic functions of the sets $[0, \pm a]$. Therefore,

$$\begin{aligned}\left(\tfrac{\mathrm{B}-\mathrm{B}^*}{i} f\right)(x) &= \left(\int\limits_0^a f_-(t)dt + \tfrac{1}{\varphi(x)}\int\limits_0^a f_+(t)\varphi(t)dt\right)\chi_+(x)\\ &-\left(\int\limits_{-a}^0 f_-(t)dt + \tfrac{1}{\varphi(x)}\int\limits_{-a}^0 f_+(t)\varphi(t)dt\right)\chi_-(x).\end{aligned} \tag{10}$$

Since

$$\int\limits_{-a}^0 f_-(t)dt = -\int\limits_0^a f_-(t), \qquad \frac{1}{\varphi(x)}\int\limits_{-a}^0 f_+(t)\varphi(t)dt = \frac{1}{\varphi(-x)}\int\limits_0^a f_+(t)\varphi(t)dt,$$

then

$$\left(\frac{\mathrm{B}-\mathrm{B}^*}{i}\right) L^2_\varphi(-a,a) = \operatorname{span}\{e_1(x), e_2(x)\}, \tag{11}$$

where $\langle e_k, e_s\rangle = \delta_{k_1 s}$, and

$$e_1(x) \stackrel{\text{def}}{=} \frac{1}{\sqrt{2b}}; \quad e_2(x) \stackrel{\text{def}}{=} \frac{1}{\sqrt{2\widetilde{b}}}\left\{\frac{\chi_+(x)}{\varphi(x)} - \frac{\chi_-(x)}{\varphi(-x)}\right\}. \tag{12}$$

Here b and $\widetilde{b}$ are given by (9). The matrix

$$\sqrt{b\widetilde{b}}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

corresponds to the operator $\frac{\mathrm{B}-\mathrm{B}^*}{i}$ in this basis. Thus the system

$$\Delta = \left(\mathrm{B}, L^2_\varphi(-a,a),\ \{g_k(x)\}_1^2,\ \mathrm{J_p} = \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix}\right) \tag{13}$$

is an operator complex [5] including

$$\frac{B-B^*}{i}f=\sum_{\alpha,\beta=1}^{2}\langle f,e_\alpha\rangle\,(\mathrm{J_p})_{\alpha,\beta}\mathrm{e}_\beta,$$

where

$$g_1(x)\overset{\text{def}}{=}\left(b\widetilde{b}\right)^{\frac{1}{4}}e_1(x),\quad g_2(x)\overset{\text{def}}{=}\left(b\widetilde{b}\right)^{\frac{1}{4}}e_2(x). \tag{14}$$

R e m a r k 2. The operator $\frac{\mathrm{B-B^*}}{i}$ can have rank more than two if the first condition in (1) does not hold. So, in the case when the continuous function has "l" changes of sign on $[0,a]$, the rank of the operator $\frac{\mathrm{B-B^*}}{i}$ equals $2(l+1)$.

II. Let us calculate the characteristic function [6],

$$S_\Delta(\lambda)=\mathrm{I}-i\mathrm{J_p}\left[\langle \mathrm{R_B}(\lambda)\mathrm{g_k},\mathrm{g_s}\rangle\right], \tag{15}$$

of the complex Δ (13). Let $f^k=(\mathrm{B}-\lambda\mathrm{I})^{-1}g_k\,(k=1,2)$, where $\{g_k(x)\}_1^2$ are given by (14). Then $\mathrm{B}f^k-\lambda f^k=g_k$ $(k=1,2)$.

For $k=1$, we have

$$i\int_0^x f_-(t)dt+\frac{i}{\varphi(x)}\int_0^x f_+(t)\varphi(t)dt-\lambda\left(f_+(x)+f_-(x)\right)=\sqrt[4]{\frac{\widetilde{b}}{4b}}, \tag{16}$$

(index $k=1$ by $f^1(x)=f(x)$ is omitted). Setting equal to zero the odd and even parts in this equality, we obtain

$$\begin{cases}\dfrac{i}{\varphi(x)}\displaystyle\int_0^x f_+(t)\varphi(t)dt-\lambda f_-(x)=0,\\ i\displaystyle\int_0^x f_-(t)dt-\lambda f_+(x)=\sqrt[4]{\dfrac{\widetilde{b}}{4b}}.\end{cases} \tag{17}$$

This implies that $f_+(0)=-\frac{1}{\lambda}\sqrt[4]{\frac{\widetilde{b}}{4b}}$ and

$$-\frac{1}{\lambda}\int_0^x\frac{dt}{\varphi(t)}\int_0^t f_+(\xi)\varphi(\xi)d\xi-\lambda f_+(x)=\sqrt[4]{\frac{\widetilde{b}}{4b}},$$

which, after differentiation, gives us

$$\frac{1}{\varphi(x)}\int_0^x f_+(\xi)\varphi(\xi)d\xi + \lambda^2 f'_+(x) = 0, \tag{18}$$

and thus $\left(f'_+(x)\varphi(x)\right)\big|_{x=0} = 0$. Differentiating (18) again, we obtain

$$\frac{-\varphi'(x)}{\varphi^2(x)}\int_0^x f_+(\xi)\varphi(\xi)d\xi + f_+(x) + \lambda^2 f''_+(x) = 0,$$

which, after using (18), gives us the equation

$$\lambda^2\frac{\varphi'(x)}{\varphi(x)}f'_+(x) + f_+(x) + \lambda^2 f''_+(x) = 0.$$

As a result, we obtain the Cauchy problem

$$\begin{cases} y''(x) + \frac{\varphi'(x)}{\varphi(x)}y'(x) + \frac{1}{\lambda^2}y(x) = 0; \\ y(0) = -\frac{1}{\lambda}\sqrt[4]{\frac{\widetilde{b}}{4b}};\ \left(\varphi(x)y'(x)\right)\big|_{x=0} = 0. \end{cases} \tag{19}$$

Knowing $f_+(x)$ as the solution of this problem, we find the function $f_-(x)$ from the first equation in (17), and thus we define $f(x) = f_+(x) + f_-(x)$ satisfying equation (16).

To find $S_\Delta(\lambda)$ (15), we calculate $\left\langle f^1, g_k\right\rangle$ $(k = 1, 2)$ where $f^1(x)$ is the solution of (16). Taking into account (17), we have

$$\left\langle f^1, g_1\right\rangle = \sqrt[4]{\frac{\widetilde{b}}{4b}}\int_{-a}^{a} f^1_+(x)|\varphi(x)|dx = \frac{\lambda}{i}\sqrt[4]{\frac{4\widetilde{b}}{b}}f^1_-(a)\varphi(a),$$

$$\left\langle f^1, g_2\right\rangle = 2\sqrt[4]{\frac{b}{4\widetilde{b}}}\int_0^a f^1_-(x)dx = \frac{1}{i}\sqrt[4]{\frac{4b}{\widetilde{b}}}\left(\lambda f^1_+(a) + \sqrt[4]{\frac{\widetilde{b}}{4b}}\right).$$

Therefore,

$$\left\langle f^1, g_1\right\rangle = -i\lambda\sqrt[4]{\frac{4\widetilde{b}}{b}}f^1_-(a)\varphi(a);\ \left\langle f^1, g_2\right\rangle = -i\left(\sqrt[-4]{\frac{4b}{\widetilde{b}}}\lambda f^1_+(a) + 1\right). \tag{20}$$

For $k = 2$, an analogue of equation (16) is given by

$$i\int_0^x f_-(t)dt + \frac{i}{\varphi(x)}\int_0^x f_+(t)\varphi(t)dt - \lambda\left(f_+(x) + f_-(x)\right) = \sqrt[4]{\frac{b}{4\widetilde{b}}}\left\{\frac{\chi_+(x)}{\varphi(x)} - \frac{\chi_-(x)}{\varphi(-x)}\right\}, \tag{21}$$

(index $k = 2$ by $f^2(x) = f(x)$ is omitted), which gives us the system of equations

$$\begin{cases} i\int\limits_0^x f_-(t)dt - \lambda f_+(x) = 0; \\ \frac{i}{\varphi(x)}\int\limits_0^x f_+(t)\varphi(t)dt - \lambda f_-(x) = \sqrt[4]{\frac{b}{4\widetilde{b}}}\frac{1}{\varphi(x)} \quad (x \in [0,a]). \end{cases} \tag{22}$$

Hence it follows that $\left.(\varphi(x)f_-(x))\right|_{x=0} = -\frac{1}{\lambda}\sqrt[4]{\frac{b}{4\widetilde{b}}}$, and

$$-\frac{1}{\lambda\varphi(x)}\int\limits_0^x\int\limits_0^t f_-(\xi)d\xi\varphi(t)dt - \lambda f_-(x) = \sqrt[4]{\frac{b}{4\widetilde{b}}}\frac{1}{\varphi(x)}. \tag{23}$$

As a result of differentiation, we obtain

$$\frac{1}{\lambda}\frac{\varphi'(x)}{\varphi^2(x)}\int\limits_0^x\int\limits_0^t f_-(\xi)d\xi\varphi(t)dt - \frac{1}{\lambda}\int\limits_0^x f_-(\xi)d\xi - \lambda f_-'(x) = -\sqrt[4]{\frac{b}{4\widetilde{b}}}\frac{\varphi'(x)}{\varphi(x)},$$

which, in virtue of (23), gives us

$$-\lambda\frac{\varphi'(x)}{\varphi(x)}f_-(x) - \frac{1}{\lambda}\int\limits_0^x f_-(\xi)d\xi - \lambda f_-'(x).$$

Taking into account the first equality in (22), we obtain that $f_+(x)$ is the solution of the Cauchy problem

$$\begin{cases} y''(x) + \frac{\varphi'(x)}{\varphi(x)}y'(x) + \frac{1}{\lambda^2}y(x) = 0; \\ y(0) = 0 \quad \left.(\varphi(x)y'(x))\right|_{x=0} = -\frac{i}{\lambda^2}\sqrt[4]{\frac{b}{4\widetilde{b}}}. \end{cases} \tag{24}$$

After finding $f_+(x)$ from (24) and then $f_-(x) = -i\lambda\, f_+'(x)$ from the first equality in (22), we obtain $f(x) = f_+(x) + f_-(x)$ as the solution of equation (21).

Similarly to (20), we obtain

$$\left\langle f^2, g_1\right\rangle = -i\left(\sqrt[4]{\frac{4\widetilde{b}}{b}}\lambda f_-^2(x)\varphi(a) + 1\right); \quad \left\langle f^2, g_2\right\rangle = -i\sqrt[4]{\frac{4b}{\widetilde{b}}}\lambda f_+^2(a). \tag{25}$$

(20) and (25) imply

$$S_\Delta(\lambda) = \mathrm{I} - i\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} -i\lambda\sqrt[4]{\frac{4\widetilde{b}}{b}}f_-^1(a)\varphi(a) & -i\left(\sqrt[4]{\frac{4b}{\widetilde{b}}}\lambda f_+^1(a) + 1\right) \\ -i\left(\sqrt[4]{\frac{4\widetilde{b}}{b}}\lambda f_-^2(a)\varphi(a) + 1\right) & -i\sqrt[4]{\frac{4b}{\widetilde{b}}}\lambda f_+^2(a) \end{bmatrix}$$

$$= -\lambda \begin{bmatrix} \sqrt[4]{\frac{4\tilde{b}}{b}} f_-^2(a)\varphi(a) & \sqrt[4]{\frac{4b}{\tilde{b}}} f_+^2(a) \\ \sqrt[4]{\frac{4\tilde{b}}{b}} f_-^1(a)\varphi(a) & \sqrt[4]{\frac{4b}{\tilde{b}}} f_+^1(a) \end{bmatrix}.$$

Theorem 1. *The characteristic function $S_\Delta(\lambda)$ (15) of the complex Δ (13) equals*

$$S_\Delta(\lambda) = -\sqrt{2}\lambda \begin{bmatrix} cf_-^2(a)\varphi(a) & \frac{1}{c} f_+^2(a) \\ cf_-^1(a)\varphi(a) & \frac{1}{c} f_+^1(a) \end{bmatrix}, \tag{26}$$

where $c = \left(\tilde{b}/b\right)^{\frac{1}{4}}$; $f_+^1(x)$ and $f_+^2(x)$ are the solutions of the Cauchy problems (19) *and* (24)*, respectively, besides,*

$$f_-^k(a) = -i\lambda \left(f_+^k(x) \right)' \Big|_{x=a} \quad (k = 1, 2). \tag{27}$$

II. If we choose $\varphi(x) = x^\nu$ $(\nu > -1)$, then the conditions (1) are met and the main equation of the Cauchy problems (19), (24) is given by

$$y''(x) + \frac{\nu}{x} y'(x) + \frac{1}{\lambda^2} y(x) = 0, \tag{28}$$

the solution of which is expressed in terms of Bessel functions.

Let the equation from (19), (24) be written in the form

$$\left(\varphi(x)y'(x)\right)' + \frac{1}{\lambda^2}\varphi(x)y(x) = 0, \tag{29}$$

and let

$$y(x) = z(\xi), \quad \xi(x) \stackrel{\text{def}}{=} \int_0^x \frac{dt}{\varphi(t)} + C \quad (C \in \mathbb{R}). \tag{30}$$

Then equation (29) becomes

$$z''(\xi) + \frac{1}{\lambda^2} q(\xi) Z(\xi) = 0, \tag{31}$$

where

$$q(\xi) = \varphi^2\left(x(\xi)\right), \tag{32}$$

and $x(\xi)$ is the inverse function of $\xi(x)$ (30).

E x a m p l e 1. Let

$$\varphi(x) = \sqrt{a^2 - x^2} \quad \left(x \in [0, a]\right). \tag{33}$$

Therefore,

$$q(\xi) = a^2 - a^2 \sin^2 \xi = \frac{a^2}{2}(1 + \cos 2\xi).$$

And in this case we arrive (see (31)) at the well-known Mathieu equation [8],

$$z''(\xi) + \frac{a^2}{2\lambda^2}(1 + \cos 2\xi) z(\xi) = 0.$$

E x a m p l e 2. Let

$$\varphi(x) = \sqrt{4x^3 - g_2 x - g_3} \quad (g_2, g_3 \in \mathbb{R}), \tag{34}$$

where $g_3 > 0$, the roots of the polynomial $4x^3 - g_2 x - g_3$ are different and simple, and the number a is chosen so that this polynomial is positive on $(0, a)$. Then

$$\xi(x) = \int^{x} \frac{dt}{\sqrt{4t^2 - g_2 t - g_3}}, \quad x = \wp(\xi),$$

where $\wp(\xi)$ is the elliptic Weierstrass function [9]. Using the equation for $\wp(\xi)$, $\left(\wp'(\xi)\right)^2 = 4\wp^3(\xi) - g_2\wp(\xi) - g_3$, we obtain that equation (31) becomes

$$Z''(\xi) + \frac{1}{\lambda^2}\left(\wp'(\xi)\right)^2 = 0. \tag{35}$$

This coincides with the well-known Lam*é* equation [8].

III. Consider the matrix-function

$$S(x, \lambda) \stackrel{\text{def}}{=} -\sqrt{2}\lambda \begin{bmatrix} c f_-^2(x)\varphi(x) & \frac{1}{c} f_+^2(x) \\ c f_-^1(x)\varphi(x) & \frac{1}{c} f_+^1(x) \end{bmatrix} \quad (x \in [0, a]) . \tag{36}$$

Since

$$\left(f_-^k(x)\varphi(x)\right)' = \frac{i}{\lambda} f_+^k(x)\varphi(x), \quad \left(f_+^k(x)\right)' = \frac{i}{\lambda} f_-^k(x) \quad (k = 1, 2),$$

in virtue of (17) and (22), then

$$\frac{d}{dx} S(x, \lambda) = -\sqrt{2}\lambda \frac{i}{\lambda} \begin{bmatrix} c f_+^2(x)\varphi(x) & f_-^2(x)\frac{1}{c} \\ c f_+^1(x)\varphi(x) & f_-^1(x)\frac{1}{c} \end{bmatrix}$$

$$= \frac{i}{\lambda} S(x, \lambda) \begin{bmatrix} 0 & \frac{1}{c^2\varphi(x)} \\ c^2\varphi(x) & 0 \end{bmatrix} = \frac{i}{\lambda} S(x, \lambda) b(x) \mathrm{J}_p,$$

where

$$b(x) \stackrel{\text{def}}{=} \begin{bmatrix} \frac{1}{c^2\varphi(x)} & 0 \\ 0 & c^2\varphi(x) \end{bmatrix} . \tag{37}$$

Theorem 2. *The function* $S_\Delta(\lambda)$ (26) *is the monodromy function* [5,6], $S_\Delta(\lambda) = S(a,\lambda)$, *of the Cauchy problem*

$$\begin{cases} \frac{d}{dx}S(x,\lambda) = \frac{i}{\lambda}S(x,\lambda)b(x)\mathrm{J_p} \\ S(0,\lambda) = \mathrm{I}; \quad x \in [0,a], \end{cases} \tag{38}$$

where $b(x)$ *is given by* (37).

Using $b(x)$ (37), we specify [6] the Hilbert space

$$L^2_{2,a}\left(b(x)\right) \stackrel{\mathrm{def}}{=} \left\{ f(x) = [f_1(x), f_2(x)] : \int\limits_0^a f(x)b(x)f^*(x)dx < \infty \right\} \tag{39}$$

assuming that the factorization by the kernel of metric is done. Define in this space the linear operator

$$\left(\mathrm{B}_m f\right)(x) \stackrel{\mathrm{def}}{=} i \int\limits_0^x f(t)b(t)dt\mathrm{J_p}. \tag{40}$$

Since

$$\left(\frac{\mathrm{B}_m - \mathrm{B}_m^*}{i} f\right)(x) = \int\limits_0^a f(t)b(t)dt\mathrm{J_p} = \sum_{\alpha,\beta=1}^{2} \langle \mathrm{f}, \mathrm{e}_\alpha \rangle (\mathrm{J_p})_{\alpha,\beta} \mathrm{e}_\beta,$$

where $e_1 = [1,0],\ e_2 = [0,1]$, then the totality

$$\Delta_m = \left(\mathrm{B}_m, L^2_{2,a}\left(b(x)\right), \{e_\alpha\}_1^2, \mathrm{J_p}\right) \tag{41}$$

is an operator complex.

It is easy to show that the characteristic function

$$S_{\Delta_m}(\lambda) = \mathrm{I} - i\mathrm{J_p}\left[\langle \mathrm{R_{B_m}}(\lambda)\mathrm{e}_\alpha, \mathrm{e}_\beta \rangle\right]$$

of the complex Δ_m (41) coincides with $S(a,\lambda)$ (38),

$$S_{\Delta_m}(\lambda) = S(a,\lambda),$$

and thus, in view of the Theorem on unitary equivalence [6], the simple parts of the complexes Δ (13) and Δ_m (41) are unitarily equivalent.

Let us show the explicit form of the operator that realizes this equivalence. For every function $f(x) = f_-(x) \oplus f_+(x)$ from $L^2_\varphi(-a,a)$ (in view of decomposition (3)),

$$\|f(x)\|^2 = 2\int\limits_0^a |f_-(x)|^2 \varphi(x)dx + 2\int\limits_0^a |f_+(x)|^2 \varphi(x)dx$$

takes place. Therefore we can associate the function $f(x) \in L^2_\varphi(-a,a)$ with the vector-function $\sqrt{2}\,[f_-(x), f_+(x)]$ given on $[0,a]$, whose components $f_-(x)$ and $f_+(x)$ are continued on $[-a,0]$ in the odd and even ways, respectively. Define the operator $\mathrm{U} : L^2_\varphi(-a,a) \to L^2_{2,a}\,(b(x))$ by the formula

$$(\mathrm{U}f)\,(x) = \sqrt{2}\left[cf_-(x)\varphi(x),\ \frac{1}{c}f_+(x)\right], \tag{42}$$

besides, $\|\mathrm{U}f\|^2_{L^2_{2,a}(b(x))} = \|f\|^2_{L^2_\varphi(-a,a)}$.
It is easy to see that

$$(\mathrm{U}^*f)\,(x) = \frac{1}{\sqrt{2}}\left(\frac{1}{c\varphi(x)}f_1(x) + cf_2(x)\right), \tag{43}$$

assuming that the components $f_1(x)$ and $f_2(x)$ of the vector-function $f(x) = [f_1(x), f_2(x)] \in L^2_{2,a}\,(b(x))$ are continued on $[-a,0]$ in the odd and even ways, respectively.

The operator U (42) realizes the unitary equivalence between the simple parts of the operators B (5) and B_m (40),

$$\mathrm{UB} = \mathrm{B}_m\mathrm{U}.$$

IV. Consider the integration operator

$$(\mathbb{J}f)(x) = i\int_0^x f(t)dt \tag{44}$$

in the space without weight $L^2(-a,a)$ (which coincides with $L^2_\varphi(-a,a)$ (2) for $\varphi(x) \equiv 1, x \in [-a,a]$)

Theorem 3. *Let for $\varphi(x)$ satisfying the relations* (1), (8) *the conditions*

$$\varphi(x) \neq 0, \quad \frac{1}{\varphi(x)} \neq 0 \quad (\forall x \in (0,a)) \tag{45}$$

be met instead of (9), *then the operator* B (5) *is similar to the operator* $\mathbb{J}$ (44).

P r o o f. Taking into account decomposition (2) ($\varphi(x) \equiv 1$) and Lemma 1, we can write the integration operator $\mathbb{J}$ (44) in the unitarily equivalent form

$$(\mathbb{J}_m f)\,(x) = i\int_0^x f(t)dt\mathrm{J}_\mathrm{p}, \tag{46}$$

where $f = [f_1, f_2] \in L^2_{2,a}(\mathrm{I})$ (39), $(b(x) \equiv \mathrm{I})$; besides, $f_1(x) = \sqrt{2} f_-(x)$, $f_2(x) = \sqrt{2} f_+(x)$. This operator differs from B_m (40) by the weight $b(x)$.

Let $a(x) \geqslant 0$ be the matrix-function such that $a(x) = \{b(x)\}^{\frac{1}{2}}$,

$$a(x) = \begin{bmatrix} \frac{1}{C\sqrt{\varphi(x)}} & 0 \\ 0 & N\sqrt{\varphi(x)} \end{bmatrix} \tag{47}$$

Specify the isometric operator $V : L^2_{2,a}(b(x)) \to L^2_{2,a}(\mathrm{I})$ by the formula

$$(Vf)(x) = f(x)a(x). \tag{48}$$

Then $(V^* f)(x) = f(x)a^{-1}(x)$. It is easy to see that the operator $\widehat{B}_m = V\mathrm{B}_m V^*$ acting now in the space $L^2_{2,a}(\mathrm{I})$ equals

$$\left(\widehat{B}_m f\right)(x) = i\int_0^x f(t)a(t)\mathrm{J_p a(x)} = \mathrm{i}\int_0^{\mathrm{x}} \mathrm{f(t)} \begin{bmatrix} \frac{\sqrt{\varphi(x)}}{\sqrt{\varphi(t)}} & 0 \\ 0 & \frac{\sqrt{\varphi(t)}}{\sqrt{\varphi(x)}} \end{bmatrix} \mathrm{dtJ_p}. \tag{49}$$

The boundedness of $\widehat{B}_m$ (49) in $L^2_{2,a}(\mathrm{I})$ is guaranteed by the quadratic summability of the kernels $K(x,f) = \frac{\sqrt{\varphi(x)}}{\sqrt{\varphi(t)}}$ $\left(K(x,f) = \frac{\sqrt{\varphi(t)}}{\sqrt{\varphi(x)}}\right)$ in view of (9). Taking into account the form of $\mathbb{J}_m$ (46) and $\widehat{B}_m$ (49), to conclude the proof we need to ascertain the similarity of the operators

$$i\int_0^x g(t)\frac{dt}{\sqrt{\varphi(t)}}\sqrt{\varphi(x)} \left(i\int_0^x g(t)\sqrt{\varphi(t)}dt\frac{1}{\sqrt{\varphi(x)}}\right), \quad i\int_0^x g(t)ft,$$

where $g(t) \in L^2_{(0,a)}$ (which corresponds to the component-wise similarity of (46) and (49)). But this is obvious since in the first case the similarity in $L^2_{(0,a)}$ is given by the operator $(T_1 g)(x) = \frac{g(x)}{\sqrt{\varphi(x)}}$, and in the second, by $(T_2 g)(x) = g(x)\sqrt{\varphi(x)}$. Moreover, it is obvious that T_1 and T_2 are bounded and boundedly invertible.

2. L. de Branges Transform

I. Let $\Theta(x,\lambda) = S\left(x, \frac{1}{\lambda}\right)$. Then (38) implies

$$\begin{cases} \frac{d}{dx}\Theta(x,\lambda) = i\lambda\Theta(x,\lambda)b(x)\mathrm{J}_p, \\ \Theta(0,\lambda) = \mathrm{I}, \quad x \in [0,a], \end{cases} \tag{50}$$

and thus $\Theta(x,\lambda)$ is the solution of the integral equation

$$\Theta(x,\lambda) - i\lambda \int_0^x \Theta(t,\lambda) b(t) dt \mathrm{J}_p = \mathrm{I}. \tag{51}$$

(50) implies

$$\Theta(x,\lambda)\mathrm{J}_p\Theta^*(x,\omega) - \mathrm{J}_p = i(\lambda - \overline{\omega}) \int_0^x \Theta(t,\lambda) b(t) \Theta^*(t,\omega) dt; \tag{52}$$

$\forall x \in [0,a]$. Consider the row-vector

$$L(x,\lambda) \stackrel{\text{def}}{=} [0,1]\Theta(x,\lambda) = \left[B(x,\lambda), A(x,\lambda)\right]. \tag{53}$$

Then (51) implies

$$L(x,\lambda) - i\lambda \int_0^x L(t,\lambda) b(t) dt \mathrm{J}_\mathrm{p} = [0,1]. \tag{54}$$

Therefore the kernel

$$K_x(\lambda,\omega) \stackrel{\text{def}}{=} \frac{L(x,\lambda)\mathrm{J}_\mathrm{p}L^*(x,\omega)}{i(\lambda - \overline{\omega})} = \int_0^x L(t,\lambda) b(t) L^*(t,\omega) dt \tag{55}$$

is positive definite [6,7], and thus

$$L(x,\lambda)\mathrm{J}_\mathrm{p}L^*(x,\lambda) = \begin{cases} \geqslant 0, \ \lambda \in \mathbb{C}_-; \\ = 0, \ \lambda \in \mathbb{R}; \\ \leqslant 0, \ \lambda \in \mathbb{C}_+. \end{cases} \tag{56}$$

Write the involution J_p as $\mathrm{J}_\mathrm{p} = P_+ - P_-$, where $P_\pm$ are orthoprojectors,

$$P_\pm = \frac{1}{2}\left(\mathrm{I} \pm \mathrm{J}_\mathrm{p}\right) = \frac{1}{2}\begin{bmatrix} 1 & \pm 1 \\ \pm 1 & 1 \end{bmatrix}.$$

Then

$$L(x,\lambda)P_+ = E(x,\lambda)L_\circ^+; \qquad L(x,\lambda)P_- = \widetilde{E}(x,\lambda)L_\circ^-,$$

where

$$E(x,\lambda) \stackrel{\text{def}}{=} A(x,\lambda) + B(x,\lambda); \qquad \widetilde{E}(x,\lambda) \stackrel{\text{def}}{=} A(x,\lambda) - B(x,\lambda), \tag{57}$$

besides,

$$L_0^{\pm} = L_0 P_{\pm} = \frac{1}{2}[\pm 1, 1].$$

Since

$$L(x,\lambda)\mathrm{J_p}L^*(x,\omega) = \frac{1}{2}\left\{E(x,\lambda)\overline{E(x,\omega)} - \widetilde{E}(x,\lambda)\overline{\widetilde{E}(x,\omega)}\right\},$$

then the kernel $K_x(\lambda,\omega)$ (55) is given by

$$K_x(x,\lambda) = \frac{E(x,\lambda)\overline{E(x,\omega)} - \widetilde{E}(x,\lambda)\overline{\widetilde{E}(x,\omega)}}{2i(\lambda-\overline{\omega})}, \tag{58}$$

and

$$|E(x,\lambda)| - \left|\widetilde{E}(x,\lambda)\right| = \begin{cases} \geqslant 0, & \lambda \in \mathbb{C}_-; \\ = 0, & \lambda \in \mathbb{R}; \\ \leqslant 0, & \lambda \in \mathbb{C}_+. \end{cases}$$

Theorem 4. *Let* $L(x,\lambda) = [B(x,\lambda), A(x,\lambda)]$ *be the nontrivial* ($L(x,\lambda) \not\equiv [1,0]$) *solution of integral equation* (54)*, then*

1) $L(x,\lambda) \in L^2_{2,l}(b(x))$ $\forall L \in [0,a]$ *and* $\forall \lambda \in \mathbb{C}$*;*

2) *the function* $E(x,\lambda)$ $\left(\widetilde{E}(x,\lambda)\right)$ (57) *has no zeroes in* $\mathbb{C}_+$ *(respectively in* $\mathbb{C}_-$*), besides,*

$$|E(x,\lambda)| - \left|\widetilde{E}(x,\lambda)\right| = \begin{cases} > 0, & \lambda \in \mathbb{C}_-; \\ = 0, & \lambda \in \mathbb{R}; \\ < 0, & \lambda \in \mathbb{C}_+; \end{cases} \tag{59}$$

and $E(x,0) = \widetilde{E}(x,0) = 1$ $\forall x \in [0,a]$.

The proof of the theorem is similar to that of the corresponding statement in [6].

Let us recall [11, 12] that $F(x)$ is said to be a function of bounded type in $\mathbb{C}_+(\mathbb{C}_-)$ if it is the quotient of two holomorphic functions in $\mathbb{C}_+(\mathbb{C}_-)$.

The mean type [10, 11] for these functions is calculated by the formula

$$h \stackrel{\text{def}}{=} \lim_{y\uparrow\infty} \frac{\ln|f(iy)|}{y}.$$

Let a pair of entire functions $A(\lambda)$ and $B(\lambda)$ be given such that $E(\lambda) = A(\lambda)+B(\lambda)$ and $\widetilde{E}(\lambda) = A(\lambda)-B(\lambda)$ have no zeroes in $\mathbb{C}_-$ and $\mathbb{C}_+$, respectively, besides,

$$|E(\lambda)| - \left|\widetilde{E}(\lambda)\right| = \begin{cases} > 0, \ \lambda \in \mathbb{C}_-; \\ = 0, \ \lambda \in \mathbb{R}; \\ < 0, \ \lambda \in \mathbb{C}_+. \end{cases} \tag{60}$$

With this pair of functions, $A(\lambda)$ and $B(\lambda)$, it is customary [6, 11] to associate the Hilbert space.

Definition. *A linear variety of entire functions $F(\lambda)$ such that*

1) $F(\lambda)/E(\lambda)$ $\left(F(\lambda)/\widetilde{E}(\lambda)\right)$ *is an entire function of bounded type in $\mathbb{C}_-$ (in $\mathbb{C}_+$);*

2)
$$\int\limits_{\mathbb{R}} \frac{|F(t)|^2}{|E(t)|^2} dt = \int\limits_{\mathbb{R}} \frac{|F(t)|^2}{\left|\widetilde{E}(t)\right|^2} dt < \infty$$

is said to be the Louis de Branges space $\mathfrak{B}(A, B)$.

The Hilbert properties of this function space follow from the Phragmén–Lindelöf principle, and the inner product in $\mathfrak{B}(A, B)$ is given by

$$\langle F(\lambda), G(\lambda)\rangle = \int\limits_{\mathbb{R}} F(t)\overline{G(t)} \frac{dt}{|E(t)|^2}.$$

It is easy to show that the kernel

$$K(\lambda, \omega) = \frac{E(\lambda)\overline{E}(\omega) - \widetilde{E}(\lambda)\overline{\widetilde{E}(\omega)}}{2\pi i(\lambda - \overline{\omega})} \tag{61}$$

is reproducing [6, 11], namely,

$$\langle F(t), K(t, \omega)\rangle = F(\omega)$$

$\forall F(\lambda) \in \mathfrak{B}(A, B)$ and $\forall \omega \in \mathbb{C}$.

The L. de Branges Theorem [11, 6]. *Let $L(x, \lambda) = [B(x, \lambda), A(x, \lambda)]$ be the solution of integral equation (54), where $x \in [0, a]$ $(0 < a < \infty)$ and $b(x) \geqslant 0$. With every row-vector $f(x) \in L^2_{2,a}(b(x))$ (39), we associate the function*

$$F(\lambda) \stackrel{\text{def}}{=} \frac{1}{\pi} \int\limits_0^l f(t)b(t)L^*(t, \overline{\lambda})dt, \tag{62}$$

where $0 < l \leqslant a$. *Then* $F(\lambda) \in \mathfrak{B}\left(A(l,\lambda), B(l,\lambda)\right)$, *and the Parseval equality*

$$\pi \left\| F(\lambda) \right\|^2_{\mathfrak{B}(A(l,\lambda),B(l,\lambda))} = \| f(x) \|^2_{L^2_{2,l}(b(x))} \tag{63}$$

is true.

For every function $F(\lambda) \in \mathfrak{B}\left(A(l,\lambda), B(l,\lambda)\right)$, *there is a function* $f(x) \in L^2_{2,a}\left(b(x)\right)$ *with support on* $[0,l]$ *such that for* $F(\lambda)$ *representation* (62) *takes place.*

It is easy to show that the shift operator [6],

$$(\widetilde{B}F)(\lambda) = \frac{F(\lambda) - F(0)}{\lambda} \qquad \left(F(\lambda) \in \mathfrak{B}\left(A(a,\lambda), B(a,\lambda)\right)\right),$$

is the L. de Branges transform of the operator B_m (40), and the model complex

$$\widetilde{\Delta}_m \left(\widetilde{B}, \mathfrak{B}\left(A(a,\lambda), B(a,\lambda)\right), \left\{\widetilde{e}_\alpha(\lambda)\right\}_1^2, \mathrm{J_p} \right) \tag{64}$$

is the L. de Branges transform of Δ_m, where

$$\widetilde{e}_1(\lambda) = \frac{B^*(a,\lambda)}{\lambda}, \qquad \widetilde{e}_2(\lambda) = \frac{1 - A^*(a,\lambda)}{\lambda},$$

which is customary [6] regarded as a functional model of complex (13).

II. Following the considerations of item II from Section 1, as an example, we chose $\varphi(x)$ of the form

$$\varphi(x) = x^\nu \qquad (-1 < \nu < 1). \tag{65}$$

Then the numbers $b, \widetilde{b}$ (9) are

$$b = \frac{a^{\nu+1}}{\nu+1}, \qquad \widetilde{b} = \frac{a^{1-\nu}}{1-\nu} = \frac{1+\nu}{1-\nu} a^{-2\nu} b.$$

The differential equation of the Cauchy problems (19), (24) coincides with the Bessel equation

$$y''(x) + \frac{\nu}{x} y'(x) + \frac{1}{\lambda^2} y(x) = 0,$$

the solution of which is expressed in terms of the Bessel functions

$$y(x) = \left(\frac{x}{\xi}\right)^{\frac{1-\nu}{2}} \left\{ C_1 \mathrm{J}_{\frac{1-\nu}{2}}\left(\frac{x}{\lambda}\right) + C_2 \mathrm{J}_{\frac{\nu-1}{2}}\left(\frac{x}{\lambda}\right) \right\},$$

where C_1 and C_2 are constants, besides,

$$y(0) = C_2 \frac{2^{\frac{\nu-1}{2}}}{\Gamma\left(\frac{\nu+1}{2}\right)}; \qquad \left(x^\nu y'(x)\right)\big|_{x=0} = C_1 \frac{\lambda^{\frac{\nu-1}{2}} 2^{\frac{\nu+1}{2}}}{\Gamma\left(\frac{1-\nu}{2}\right)}.$$

Hence we find

$$f_+^1(x) = -\frac{1}{\lambda\sqrt{2}}\sqrt[4]{\frac{1+\nu}{1-\nu}}\Gamma\left(\frac{\nu+1}{2}\right)a^{\frac{-\nu}{2}}\left(\frac{x}{2\lambda}\right)^{\frac{1-\nu}{2}}\mathrm{J}_{\frac{\nu-1}{2}}\left(\frac{x}{\lambda}\right), \tag{66}$$

and thus

$$f_-^1(x) = \frac{i}{\lambda\sqrt{2}}\sqrt[4]{\frac{1+\nu}{1-\nu}}\Gamma\left(\frac{\nu+1}{2}\right)a^{-\frac{\nu}{2}}\left(\frac{x}{2\lambda}\right)^{\frac{1-\nu}{2}}\mathrm{J}_{\frac{\nu+1}{2}}\left(\frac{x}{\lambda}\right) \tag{67}$$

in virtue of $f_-^1(x) = -i\lambda\left(f_+^1(x)\right)'$. Analogously,

$$f_+^2(x) = -\frac{i}{\lambda}\frac{\Gamma\left(\frac{1-\nu}{2}\right)}{(2\lambda)^\nu}\sqrt[4]{\frac{1-\nu}{1+\nu}}a^{\frac{\nu}{2}}\left(\frac{x}{2\lambda}\right)^{\frac{1-\nu}{2}}\mathrm{J}_{\frac{1-\nu}{2}}\left(\frac{x}{\lambda}\right). \tag{68}$$

Therefore $f_-^2(x)\ \left(= -\lambda i f_+^2(x)\right)'$ equals

$$f_-^2(x) = -\frac{1}{\lambda}\frac{\Gamma\left(\frac{1-\nu}{2}\right)}{(2\lambda)^\nu}\sqrt[4]{\frac{1-\nu}{1+\nu}}a^{\frac{\nu}{2}}\left(\frac{x}{2\lambda}\right)^{\frac{1-\nu}{2}}\mathrm{J}_{\frac{-1-\nu}{2}}\left(\frac{x}{\lambda}\right). \tag{69}$$

The characteristic function $S_\Delta(\lambda)$ (26) of the complex Δ (13) for the case $\varphi(x) = x^\nu$ equals

$$\begin{aligned}
&S_\Delta(\lambda)\\
&= \begin{bmatrix} \frac{\sqrt{2}\Gamma\left(\frac{1-\nu}{2}\right)a^\nu}{(2\lambda)^\nu}\left(\frac{a}{2\lambda}\right)^{\frac{1-\nu}{2}}\mathrm{J}_{\frac{-\nu-1}{2}}\left(\frac{a}{\lambda}\right) & i\sqrt{2}\sqrt{\frac{1-\nu}{1+\nu}}\frac{\Gamma\left(\frac{1-\nu}{2}\right)a^\nu}{(2\lambda)^\nu}\left(\frac{a}{2\lambda}\right)^{\frac{1-\nu}{2}}\mathrm{J}_{\frac{1-\nu}{2}}\left(\frac{a}{\lambda}\right) \\ i\sqrt{\frac{1+\nu}{1-\nu}}\Gamma\left(\frac{1+\nu}{2}\right)\left(\frac{a}{2\lambda}\right)^{\frac{1-\nu}{2}}\mathrm{J}_{\frac{\nu+1}{2}}\left(\frac{a}{\lambda}\right) & \Gamma\left(\frac{\nu+1}{2}\right)\left(\frac{a}{2\lambda}\right)^{\frac{1-\nu}{2}}\mathrm{J}_{\frac{\nu-1}{2}}\left(\frac{a}{\lambda}\right) \end{bmatrix}\\
&= \begin{bmatrix} \frac{\Gamma\left(\frac{1-\nu}{2}\right)}{2^{\frac{\nu}{2}}}\left(\frac{a}{\lambda}\right)^{\frac{1+\nu}{2}}\mathrm{J}_{\frac{-1-\nu}{2}}\left(\frac{a}{\lambda}\right) & i\sqrt{\frac{1-\nu}{1+\nu}}\frac{\Gamma\left(\frac{1-\nu}{2}\right)}{2^{\frac{\nu}{2}}}\left(\frac{a}{\lambda}\right)^{\frac{1+\nu}{2}}\mathrm{J}_{\frac{1-\nu}{2}}\left(\frac{a}{\lambda}\right) \\ i\sqrt{\frac{1+\nu}{1-\nu}}\frac{\Gamma\left(\frac{1+\nu}{2}\right)}{2^{\frac{1-\nu}{2}}}\left(\frac{a}{\lambda}\right)^{\frac{1-\nu}{2}}\mathrm{J}_{\frac{1+\nu}{2}}\left(\frac{a}{\lambda}\right) & \frac{\Gamma\left(\frac{1+\nu}{2}\right)}{2^{\frac{1-\nu}{2}}}\left(\frac{a}{\lambda}\right)^{\frac{1-\nu}{2}}\mathrm{J}_{\frac{\nu-1}{2}}\left(\frac{a}{\lambda}\right) \end{bmatrix}.
\end{aligned} \tag{70}$$

Therefore the functions $A(a,\lambda)$ and $B(a,\lambda)$ in (57) are given by

$$A(a,\lambda) = \frac{\Gamma\left(\frac{1+\nu}{2}\right)}{2^{\frac{1-\nu}{2}}}(a\lambda)^{\frac{1-\nu}{2}}\mathrm{J}_{\frac{\nu-1}{2}}(a\lambda);\ B(a,\lambda) = i\sqrt{\frac{1+\nu}{1-\nu}}\frac{\Gamma\left(\frac{1+\nu}{2}\right)}{2^{\frac{1-\nu}{2}}}(a\lambda)^{\frac{1-\nu}{2}}\mathrm{J}_{\frac{\nu+1}{2}}(a\lambda). \tag{71}$$

Hence, the function $E(x,\lambda)$ (57) equals

$$E(x,\lambda) = \frac{\Gamma\left(\frac{1+\nu}{2}\right)}{2^{\frac{1-\nu}{2}}}(x\lambda)^{\frac{1-\nu}{2}}\left[\mathrm{J}_{\frac{\nu-1}{2}}(\mathrm{x}\lambda) + \mathrm{i}\sqrt{\frac{1+\nu}{1-\nu}}\mathrm{J}_{\frac{\nu+1}{2}}(\mathrm{x}\lambda)\right]. \tag{72}$$

Expression (73) coincides per se with the well-known Dunkl kernel studied in [4], particularly for $\nu = 0$, $E(x,\lambda) = e^{i\lambda x}$.

3. The Function $e(x,\lambda)$

I. In [4], the basis property of the system of functions generated by the Dunkl kernels $d_\alpha(i\lambda_k t)$ is studied. The functions are given by the formula $d(izt) = (1 - zB)^{-1}\mathbb{I}$, where B is given by (5) and $\varphi(x) = x^\nu$. Consider the general case (not supposing that $\varphi(x)$ has degree dependence).

Denote by $e(x,\lambda)$ ($e(x,\lambda) = d(i\lambda x)$ as $\varphi(x) = x^\nu$) the function

$$e(x,\lambda) = (I - \lambda B)^{-1}\mathbb{I}, \tag{73}$$

where B is given by (5), and $\mathbb{I}$ is identical to 1 on $[-a,a]$. Then

$$Be(x,\lambda) = \frac{e(x,\lambda) - \mathbb{I}}{\lambda}, \tag{74}$$

moreover, $e(x,\lambda)$ satisfies the equation

$$e(x,\lambda) - \lambda i \int_0^x e_-(t,\lambda)dt - \frac{\lambda i}{\varphi(x)} \int_0^x e_+(t,\lambda)\varphi(t)dt = 1. \tag{75}$$

Using Lemma 1, we obtain the system of integral equations

$$\begin{cases} e_+(x,\lambda) - \lambda i \int\limits_0^x e_-(t,\lambda)dt = 1; \\ e_-(x,\lambda) - \frac{\lambda i}{\varphi(x)} \int\limits_0^x e_+(t,\lambda)\varphi(x)dt = 0. \end{cases} \tag{76}$$

Hence we obtain the integral equation for $e_+(x,\lambda)$,

$$e_+(x,\lambda) - (i\lambda)^2 \int_0^x \frac{dt}{\varphi(t)} \int_0^t e_+(s,\lambda)\varphi(s)ds = 1. \tag{77}$$

By differentiating (77), we have

$$e'_+(x,\lambda) - \frac{(i\lambda)^2}{\varphi(x)} \int_0^x e_+(s,\lambda)\varphi(s)ds = 0. \tag{78}$$

Therefore,

$$e''_+(x,\lambda) + \frac{\varphi'(x)}{\varphi(x)}(i\lambda)^2 \frac{1}{\varphi(x)} \int_0^x e_+(s,\lambda)\varphi(s)ds - (i\lambda)^2 e_+(x,\lambda) = 0,$$

which results in the differential equation

$$\begin{cases} e''_+(x,\lambda) + \frac{\varphi'(x)}{\varphi(x)} e'_+(x,\lambda) - (i\lambda)^2 e_+(x,\lambda) = 0 \\ e_+(0,\lambda) = 1, \qquad \varphi(x) e'_+(x,\lambda)\big|_{x=0} = 0. \end{cases} \tag{79}$$

Realizing the transformation $x \to x\lambda$ in (79), we obtain

$$e_+(x,\lambda) = f(x\lambda), \tag{80}$$

where $f(\xi)$ is the solution of the Cauchy problem

$$\begin{cases} f''(\xi) + \frac{\varphi'(\xi)}{\varphi(\xi)} f'(\xi) + f(\xi) = 0 \\ f(0) = 1, \quad \varphi(\xi) f'(\xi)|_{\xi=0} = 0. \end{cases} \tag{81}$$

The first equation in (76) yields $e'_+(x,\lambda) = i\lambda e_-(x,\lambda)$, therefore,

$$e_-(x,\lambda) = -if'(x\lambda). \tag{82}$$

Lemma 2. *The function* $e(x,\lambda)$ (73) *is given by*

$$e(x,\lambda) = f(x\lambda) - if'(x\lambda), \tag{83}$$

where $f(\xi)$ *is the solution of the Cauchy problem* (81).

R e m a r k 2. *If* $\varphi(x) = C$, *then the solution of the Cauchy problem is given by* $f(\xi) = \cos\xi$, *consequently,*

$$e(x,\lambda) = e^{i\lambda x}.$$

II. Remark 2 implies that the function $e(x,\lambda)$ (83) is an entire function of exponential type for $\varphi(x) = \text{const}$. Let us show that this fact also takes place in the general case. The integral equation

$$f(x) = 1 - \int_0^x \frac{dt}{\varphi(t)} \int_0^t f(s)\varphi(s)ds \tag{84}$$

is equivalent to the Cauchy problem (81). We write (84) as

$$f(x) = 1 - \int_0^x K(x,s) f(s) ds, \tag{85}$$

where

$$K(x,s)=\varphi(s)\int\limits_s^x \frac{dt}{\varphi(t)}. \tag{86}$$

The standard method of solution of integral equation (85), which uses iteration, gives

$$f(x)=1-\int\limits_0^x K(x,s)ds+\int\limits_0^x K_1(x,s)ds-\dots, \tag{87}$$

where

$$K_n(x,s)=\int\limits_s^x K_{n-1}(x,\xi)K(\xi,s)ds \quad (n\in\mathbb{Z}_+), \tag{88}$$

besides, $K_0(x,s)=K(x,s)$ (86).

Lemma 3. *For the kernels $K_n(x,s)$, the estimations*

$$K_n(x,s)\leqslant M^n\varphi(s)\int\limits_s^x \frac{(xt)^n}{n!}\frac{dt}{\varphi(t)} \quad (n\in\mathbb{Z}_+) \tag{89}$$

are true where

$$\frac{1}{\varphi(x)}\int\limits_0^x \varphi(t)dt<M \quad (\forall x\in[0,a]). \tag{90}$$

P r o o f. We prove inequalities (89) using induction. For $n=1$, (86), (88), after the integration by parts, imply

$$K_1(x,s)=\int\limits_s^x\int\limits_\xi^x \frac{dt}{\varphi(t)}\varphi(\xi)\int\limits_s^\xi \frac{d\mu}{\varphi(\mu)}\varphi(s)d\xi=\varphi(s)\int\limits_s^x d\xi\int\limits_\xi^x \frac{dt}{\varphi(t)}\int\limits_s^\xi \frac{d\mu}{\varphi(\mu)}d_\xi\int\limits_0^\xi \varphi(\nu)d\nu$$

$$=-\varphi(s)\int\limits_s^x\int\limits_0^\xi \varphi(\nu)d\nu\left\{-\frac{1}{\varphi(\xi)}\int\limits_s^\xi \frac{d\mu}{\varphi(\mu)}+\frac{1}{\varphi(\xi)}\int\limits_\xi^x \frac{dt}{\varphi(t)}\right\}d\xi.$$

Taking into account that $\varphi(x)>0$ $(x\in\mathbb{R}_+)$, we obtain

$$K_1(x,s)\leqslant\varphi(x)\int\limits_s^x \frac{1}{\varphi_0(\xi)}\int\limits_0^\xi \varphi(\nu)d\nu\int\limits_s^\xi \frac{d\mu}{\varphi(\mu)}d\xi$$

$$\leqslant M\varphi(s)\int\limits_s^x d\xi\int\limits_s^\xi \frac{d\mu}{\varphi(\mu)}=M\varphi(s)\int\limits_s^x (x-t)\frac{dt}{\varphi(t)},$$

which gives (89) for $n=1$.

Suppose that the estimation (89) for n is proved, show that it takes place for $n+1$. (89) and (90) imply

$$K_{n+1}(x,s) \leqslant \int_s^x M^n \varphi(\xi) \int_\xi^x \frac{(x-t)^n}{n!} \frac{dt}{\varphi(t)} \int_s^\xi \frac{d\mu}{\varphi(\mu)} \varphi(s) d\xi$$

$$= \frac{M^n}{n!} \varphi(s) \int_s^x \int_\xi^x (x-t)^n \frac{dt}{\varphi(t)} \int_s^\xi \frac{d\mu}{\varphi(\mu)} d\xi \int_0^\xi \varphi(\nu) d\nu$$

$$= -\frac{M^n \varphi(s)}{n!} \int_s^x \int_0^\xi \varphi(\nu) d\nu \left\{ -\frac{(x-\xi)^n}{\varphi(\xi)} \int_s^\xi \frac{d\mu}{\varphi(\mu)} + \frac{1}{\varphi(\xi)} \int_\xi^s (x-t)^n \frac{dt}{\varphi(t)} \right\} d\xi$$

$$\leqslant \frac{M^n \varphi(s)}{n!} \int_s^x \frac{1}{\varphi(\xi)} \int_0^\xi \varphi(\nu) d\nu (x-\xi)^n \int_s^\xi \frac{d\mu}{\varphi(\mu)} d\xi.$$

Using (90) and changing the order of integration, we obtain

$$K_{n+1}(x,s) \leqslant \frac{M^{n+1}\varphi(s)}{n!} \int_s^x \int_\mu^x (x-\xi)^n d\xi \frac{d\mu}{\varphi(\mu)} = \frac{M^{n+1}\varphi(s)}{(n+1)!} \int_s^x \frac{(x-\mu)^{n+1} d\mu}{\varphi(\mu)},$$

which concludes the proof.

Using Lemma 3, we obtain that series (87) for the solution $f(x)$ of equation (84) is dominated by the series

$$|f(x)| < 1 + \int_0^x \varphi(s) \left\{ M \int_s^x (x-t) \frac{dt}{\varphi(t)} + M^2 \int_s^x \frac{(x-t)^2}{2!} \frac{dt}{\varphi(t)} + \cdots \right\} ds$$

$$= 1 + \int_0^x \varphi(s) \int_s^x \left(\exp M(x-t) - 1 \right) \frac{dt}{\varphi(t)} ds.$$

Using the obvious inequality $e^x - 1 < xe^x$, which is valid $\forall x > 0$, we obtain

$$|f(x)| < 1 + \int_0^x \varphi(s) M \int_s^x (x-t) e^{M(x-t)} \frac{dt}{\varphi(t)} ds \leqslant 1 + M \int_0^x \varphi(s) \int_s^x \frac{dt}{\varphi(t)} ds \, x e^{Mx}$$
$$\leqslant 1 + M^2 x^2 e^{Mx}.$$

Thus the estimation

$$|f(x)| \leqslant 1 + M^2 x^2 e^{Mx} \tag{91}$$

is true for $f(x)$.

Theorem 5. *If a function* $\varphi(x)$ *satisfying the conditions* (1) *is such that the integrals* (9) *converge and* (90) *takes place, then estimation* (91) *is true for* $f(x)$*, the solution of the Cauchy problem* (81).

Hence it follows that $e(x,\lambda)$ (73) is an entire function of exponential type.

References

[1] *B.S. Pavlov,* Basicity of an Exponential System and Muckenhoupt's Condition. — *Doklady Akademii Nauk SSSR*, **247** (1979), No. 1, 37–40. (Russian)

[2] *S.V. Hruščev, N.K. Nikol'skii, and B. S. Pavlov,* Unconditional Bases of Exponentials and of Reproducing Kernels. — *Complex Analysis and Spectral Theory. Lectures Notes in Math.* **864** (1981), 214–335.

[3] *G.M. Gubreev,* Selected works. "Serednyak T. K.", Dnepropetrovsk, 2014. (Russian)

[4] *G.M. Gubreev and V.N. Levchuk,* Description of Unconditional Bases Formed by Values of the Dunkl Kernels. — *Functional Analysis and Its Application* **49** (2015), issue 1, 60–63.

[5] *R. Hatamleh and V.A. Zolotarev,* On Model Representations of Non-Selfadjoint Operator with Infinitely Dimensional Imaginary Component. — *J. Math. Phys., Anal., Geom.* **11** (2015), No. 2, 174–186.

[6] *V.A. Zolotarev,* Analytical Methods of Spectral Representations of Non-Selfadjoint and Non-Unitary Operators. Kharkiv National University Publishing House, Mag-Press, Kharkiv, 2003. (Russian)

[7] *I.C. Gohberg and M.G. Krein,* Theory and Applications of Volterra Operators in Hilbert Space. Translations of Mathematical Monographs **24**, 1970.

[8] *E. Coddington and N. Levinson,* Theory of Ordinary Differential Equations, McGraw-Hill Book Company, Inc., 1955.

[9] *N.I. Akhiezer,* Elements of the Theory of Elliptic Functions. American Mathematical Society, Providence, RI, 1990.

[10] *B.Ya. Levin,* Lectures on Entire Functions. Transl. Math. Soc., Providence, RI, 1997.

[11] *L. de Branges,* Hilbert Spaces of Entire Functions. Prentice Hall, London, 1968.