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Estimates for the Gaussian Curvature of a Strictly Convex Surface and its Integral Parameters

V.I. Babenko

Closed and non-closed (with planar edges) strictly convex surfaces with continuous curvatures are considered. Upper and lower bounds are obtained for the Gaussian curvature under various restrictions imposed on integral parameters of a surface: the diameter and width of the surface, the volume of the enclosed body, the maximum area of planar cross-sections of the enclosed body, the radius of a circumscribed or inscribed ball, the height of non-closed surface and the area enclosed by the planar boundary of the surface.

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In the geometric theory of stability of shells, the question of determining the critical external pressure for a strictly convex closed (or rigidly fixed along the edges) shell is reduced to finding the minimum for the Gaussian curvature of its median surface [1, 5]. While modeling thin-walled structures, where only a few restrictions on the dimensions of the shell (non-canonical form) are given, a priori estimates for the critical pressure might be useful. Thus, it is important to obtain estimates for the Gaussian curvature of strictly convex surface depending on the restrictions imposed on its integral parameters. In this paper, in Theorems 1–3, several possible restrictions are considered and upper and lower bounds of the Gaussian curvature are obtained both for closed surfaces and for non-closed surfaces with flat edges. The estimates are obtained for the surfaces with continuous principal curvatures, i.e., for the surfaces of class $C^k (k \geq 2)$.

Notice that in a certain sense the considered problem is inverse to the problems where the impact of Gaussian curvature on the local shape of a surface [3] and on its integral parameters [2] is studied.

Theorem 1. *Let K be the Gaussian curvature of a closed strictly convex surface F with continuous principal curvatures, which bounds a body L whose diameter is not less than D_0. If one of the following conditions is fulfilled:*

1) *the volume of the body L is not less than V_0, where*

$$V_0 \leq \pi {D_0}^3/6; \tag{1}$$

2) *for the points P and Q on the surface F, the distance between which is equal to the diameter of the body L, the maximum area of the sections of L crossed by the planes orthogonal to PQ is not less than σ_0, where*

$$\sigma_0 \le \pi {D_0}^2/4; \tag{2}$$

3) *the surface F contains a ball whose radius is not less than R_0, where*

$$0 < R_0 \le D_0/2, \tag{3}$$

then the following estimate holds

$$\min_{(F)} K \le K_0, \tag{4}$$

where the minimum is taken over all points of the surface F; $K_0 \equiv const$ is the Gaussian curvature of a closed convex (spindle-shaped) surface of revolution F_0 such that its diameter is D_0 and either a volume of the body bounded by F_0 is equal to V_0 (in the first case) or the area of the equatorial circle of F_0 is equal to σ_0 (in the second case), or the radius of the equatorial circle of F_0 is equal to R_0 (in the third case). In (4) the equality holds true if F_0 is a sphere.

Remark 1. If the equality occurs in one of the inequalities (1)–(3), then, in the relevant case considered in the theorem, the surface F_0 is a sphere, and in this case the equality can be obtained in (4), taking a sphere F_0 as a surface F. If the inequalities (1)–(3) are strict, then (4) is strict as well. In this case, K_0 is the exact upper bound of the values $\min K$, which can be approached arbitrarily close if one takes the surface F sufficiently close to F_0 (for example, by rounding F_0 in neighborhoods of its two singular points).

Proof. Let Theorem 1 be false, i.e., suppose that there exists a surface $\widetilde{F}$ which satisfies the conditions of the theorem and whose Gaussian curvature $\widetilde{K}$ satisfies

$$\widetilde{K} \ge K_0, \tag{5}$$

where the inequality is strict if F_0 is a sphere.

Let $\widetilde{P}$ and $\widetilde{Q}$ be points on the surface $\widetilde{F}$, the distance between which is equal to the diameter $\widetilde{D}$. By using the Schwarz symmetrization, transform the body $\widetilde{L}$ bounded by the surface $\widetilde{F}$ into the rotation body L' with the axis of rotation passing through the points $\widetilde{P}$ and $\widetilde{Q}$. Then, by using the Steiner symmetrization, transform the body L' into the rotation body $\overline{L}$ symmetric relatively to a plane α which is orthogonal to the segment $\widetilde{P}\widetilde{Q}$. The diameters $\widetilde{D}$, D' and $\overline{D}$ and the volumes $\widetilde{V}$, V' and $\overline{V}$ of the bodies $\widetilde{L}$, L' and $\overline{L}$, respectively, will coincide [2]:

$$\overline{D} = D' = \widetilde{D}, \quad \overline{V} = V' = \widetilde{V}. \tag{6}$$

The surface of rotation $\overline{F}$ bounding the body $\overline{L}$, as well as the surface $\widetilde{F}$, satisfies the conditions of the theorem, and its Gaussian curvature $\overline{K}$ is continuous and bounded below [2]:

$$\min_{(\overline{F})} \overline{K} \ge \min_{(\widetilde{F})} \widetilde{K} \ge K_0. \tag{7}$$

Similarly to any strictly convex surface of rotation, which is symmetric relatively to the equatorial plane and whose Gaussian curvature is continuous, the radius $\overline{R}$ of the equatorial circle of the surface $\overline{F}$ and its Gaussian curvature $\overline{K}$ have to satisfy the following inequalities:

$$1/\sqrt{\max_{(\overline{F})} \overline{K}} \le \overline{R} \le 1/\sqrt{\min_{(\overline{F})} \overline{K}},$$

therefore, by (7), we get

$$\overline{R} \le 1/\sqrt{K_0}.$$

The last inequality allows us to introduce a spindle-shaped surface of rotation $\overline{F_0}$ of constant Gaussian curvature K_0 with the same radius of the equatorial circle $\overline{R}$. Specify a Cartesian coordinate system (x, y, z) taking the plane α as the coordinate (x, y)-plane and the rotational axis of the surface $\overline{F}$ as the coordinate z-axis. The surface $\overline{F_0}$ is obtained by rotating its meridian $y = 0$ around the z-axis. Write the equation of the meridian in the parametric form as follows [2, 6]:

$$x = \overline{R}\cos u, \quad z = \int_0^u \sqrt{\frac{1}{K_0} - \overline{R}^2 \sin^2 v dv}, \quad y = 0, \quad | u | \le \frac{\pi}{2}, \tag{8}$$

where $u = l\sqrt{K_0}$, l is the arc length of the meridian.

The spindle-shaped surface $\overline{F_0}$ is convex closed. If $\overline{R} < \sqrt{K_0}$, then it has constant Gaussian curvature K_0 everywhere except two singular (conical) points $u = \pm\pi/2$ located on the axis of rotation. If $\overline{R} = 1/\sqrt{K_0}$, then the surface of $\overline{F}_0$ is a sphere.

The surface $\overline{F}$ has the same axis of rotation and equatorial circle as the surface $\overline{F}_0$, and its Gaussian curvature $\overline{K}$ is not less than the Gaussian curvature K_0 of $\overline{F}_0$ by (7). Therefore, see [2], the surface $\overline{F}$ is contained in the surface $\overline{F}_0$ and its diameter $\overline{D}$ is less than the diameter $\overline{D}_0$ of the surface $\overline{F}_0$:

$$\begin{aligned} &\overline{D} < \overline{D}_0 && \text{if} && \overline{R} < 1/\sqrt{K_0}, \\ &\overline{D} \le \overline{D}_0 && \text{if} && \overline{R} = 1/\sqrt{K_0}. \end{aligned} \tag{9}$$

Let us show that the radius $\overline{R}$ of the equatorial circles of the surfaces $\overline{F}_0$ and $\overline{F}$ is less than the radius R_0 of the equatorial circle of the spindle-shaped surface F_0:

$$\overline{R} < R_0. \tag{10}$$

Let $\overline{D}_0$ and D_0 be the diameters of the spindle-shaped surfaces $\overline{F}_0$ and F_0, respectively. Then, by (8), one gets

$$\overline{D}_0 = 2\int_0^{\pi/2} \sqrt{\frac{1}{K_0} - \overline{R}^2 \sin^2 v \, dv}, \quad D_0 = 2\int_0^{\pi/2} \sqrt{\frac{1}{K_0} - R_0^2 \sin^2 v \, dv}.$$

If $\overline{R} > R_0$, we have $\overline{D_0} < D_0$. Then (9) can be written in the form

$$\overline{D} \le \overline{D}_0 < D_0.$$

If $\overline{R} = R_0 < 1/\sqrt{K_0}$, we have $\overline{D}_0 = D_0$. Then (9) takes the form

$$\overline{D} < \overline{D}_0 = D_0.$$

Since $\widetilde{D} = \overline{D}$ in view of (6), in both cases the surface $\widetilde{F}$ has a diameter $\widetilde{D} < D_0$, which does not satisfy the conditions of the theorem.

If $\overline{R} = R_0 = 1/\sqrt{K_0}$, then we have $\overline{D_0} = D_0$, hence the surfaces $\overline{F}_0$ and F_0 coincide and they are spheres. By (6) and (9), we have

$$\widetilde{D} = \overline{D} \leq \overline{D}_0 = D_0.$$

But, by the conditions of the theorem, the diameter $\widetilde{D}$ of the surface $\widetilde{F}$ should be not less than D_0, therefore

$$\widetilde{D} = \overline{D} = \overline{D}_0 = D_0.$$

The surfaces $\overline{F}$ and $\overline{F}_0$ have the same diameter only if their Gaussian curvatures are equal [2], i.e., if $\overline{K} = K_0$. But then (7) implies the equality $\min \widetilde{K} = K_0$ which contradicts (5) (in our case F_0 is a sphere). Thus, the inequality (10) is proved.

Now, let the surface F bounds a body whose volume is not less than the volume V_0 of the body bounded by the spindle-shaped surface F_0 (this is the first version of the restrictions imposed on the integral parameters of the surface F in question). The surface $\widetilde{F}$ bounds a body $\widetilde{L}$. In view of (6), its volume $\widetilde{V}$ is equal to the volume of the body $\overline{L}$, which is not greater than the volume $\overline{V}_0(\overline{R})$ of the body $\overline{L}_0$ bounded by the surface $\overline{F}_0$, because the body $\overline{L}$ is contained in the body $\overline{L}_0$. Namely, we have

$$\widetilde{V} = \overline{V} \leq \overline{V}_0(\overline{R}) = 2\pi\overline{R}^2 \int_0^{\pi/2} \cos^2 u \sqrt{\frac{1}{K_0} - \overline{R}^2 \sin^2 u}\, du.$$

This implies the following:

$$\widetilde{V} < \overline{V}_0(R_0) = V_0$$

because $\overline{V}_0(\overline{R})$ grows monotonically with increasing $\overline{R}$ and, moreover, $\overline{R} < R_0$ by (10). Therefore the surface $\widetilde{F}$, whose Gaussian curvature satisfies $\widetilde{K} \geq K_0$ by (10), bounds a body with the volume $\widetilde{V}$ which is less than the volume V_0 of the body L_0. Hence, $\widetilde{F}$ does not satisfy the assumptions of the theorem. Thus, Theorem 1 is proved for the first version of restrictions imposed on the surface F.

Next, let $\widetilde{\sigma}$ be the maximum area of the sections of the body $\widetilde{L}$ crossed by the planes orthogonal to the segment $\widetilde{P}\widetilde{Q}$. Let the plane, where the maximum area is achieved, be chosen as the plane α used above for the Steiner symmetrization of the body L'. Under the Schwarz and the Steiner symmetrizations, the area of the sections of corresponding bodies crossed by the plane α remains the same [2]. Hence the symmetrization of the body $\widetilde{L}$ results in the body of rotation $\overline{L}$ with the area of the equatorial circle $\pi\overline{R}^2$ equal to $\widetilde{\sigma}$. On the other hand,

$$\widetilde{\sigma} = \pi\overline{R}^2 < \pi R_0^2 = \sigma_0,$$

because $\overline{R} < R_0$ by (10), and hence the maximum area $\widetilde{\sigma}$ of cross-sections of the body $\widetilde{L}$ is less than the area σ_0 of the equatorial circle of the surface F_0. Therefore, the surface $\widetilde{F}$ with the Gaussian curvature verifying $\widetilde{K} \geq K_0$ by (5) does not satisfy the assumptions of the theorem. Thus, Theorem 1 is proved for the second version of the restrictions imposed on the integral parameters of the surface F.

Finally, suppose the surface $\widetilde{F}$ contains a ball with the radius $\widetilde{R}$. Applying the symmetrization as above, we obtain a surface of rotation $\overline{F}$ which also contains a ball with the radius $\widetilde{R}$. In this case, $\widetilde{R}$ is not greater than the radius $\overline{R}$ of the equatorial circle of the surface $\overline{F}$ [2]. Therefore, we have

$$\widetilde{R} \leq \overline{R} < R_0.$$

Hence, by (5), the surface $\widetilde{F}$ with the Gaussian curvature verifying $\widetilde{K} \geq K_0$ cannot contain a ball with the radius $\widetilde{R} \geq R_0$, i.e., it does not satisfy the assumptions of the theorem. Thus, Theorem 1 is proved for the third version of the restrictions imposed on the integral parameters of the surface F, and this completes the proof. □

Remark 2. Hereinafter *the height* of a convex surface F with a flat edge ∂F is defined as the distance between the plane containing the edge ∂F and the support plane to F that is parallel to the plane containing ∂F. Besides, *a segment* of a closed strictly convex surface F is defined as a part of F that is cut off by a plane. If a segment of a strictly convex surface with planar edge can be uniquely projected on the plane containing the edge, then this segment is called *a cap*, otherwise it is called *a dome*.

Let F' be a surface which has the largest Gaussian curvature K^0 among all the spindle-shaped rotation surfaces of constant Gaussian curvature containing an axially symmetric segment of height h and base radius r. Denote the above segment of the surface F' by F^0. If $H \leq r$, then the segment F^0 is a spherical cap F_s with the radius $R = (r^2 + H^2)/(2H)$ and the Gaussian curvature $K^0 = 1/R^2$. If $H > r$, then the segment F^0 is a dome of the surface F' and $K^0 > 1/R^2$. We claim that the following statement holds true.

Theorem 2. *Let K be the Gaussian curvature of a simply connected strictly convex surface F with continuous principal curvatures, whose height is H and whose planar edge bounds a planar region of area S. Then the following estimates hold:*

$$\begin{aligned} &\min_{(F)} K \leq K^0 \qquad &\text{if} \qquad &H \leq \sqrt{S/\pi}, \\ &\min_{(F)} K < K^0 \qquad &\text{if} \qquad &H > \sqrt{S/\pi}, \end{aligned} \tag{11}$$

where the minimum is taken over all points of the surface F, and $K^0 \equiv const$ is the Gaussian curvature of the segment F^0 described above, whose height is $h = H$ and base radius is $r = \sqrt{S/\pi}$. For $H \leq r$, there is the equality in (11) if F

coincides with F_s. For $H > r$, there is a strict inequality in (11) which can be arbitrarily close to an equality if we chose F to be sufficiently close to F^0.

Proof. Assume that the stated theorem is false, i.e., there exists a surface $\widetilde{F}$ which satisfies the assumptions of the theorem, but whose Gaussian curvature $\widetilde{K}$ satisfies the converse of (11):

$$\begin{aligned} \widetilde{K} > K^0 = 1/R^2 \quad &\text{if} \quad H \le r, \\ \widetilde{K} \ge K^0 > 1/R^2 \quad &\text{if} \quad H > r, \end{aligned} \tag{12}$$

i.e.,

$$\widetilde{K} > 1/R^2$$

for any value of H/r.

Let the plane α contain the edge of $\widetilde{F}$. Introduce a Cartesian coordinate system (x, y, z) by choosing the plane α as the coordinate (x, y)-plane and placing the origin of coordinates inside the planar domain bounded by the edge $\widetilde{F}$. Let the coordinate z-axis be oriented towards the surface $\widetilde{F}$.

Consider the convex body $\widetilde{L}$ bounded by the surface $\widetilde{F}$ and the plane α. The Schwarz symmetrization applied to the body $\widetilde{L}$ results in a body of rotation $\overline{L}$ which has the coordinate z-axis as the axis of rotation. It is bounded by the plane α and a strictly convex surface of rotation, which will be denoted by $\overline{F}$. Clearly, the height of the surface $\overline{F}$ is equal to H. The edge $\partial\overline{F}$ is a circle of radius $r = \sqrt{S/\pi}$ which is located in the plane α. The Gaussian curvature $\overline{K}$ of $\overline{F}$ is continuous and satisfies the following inequalities, (see [2], (12)):

$$\min_{(\overline{F})} \overline{K} \ge \min_{(\widetilde{F})} \widetilde{K} > 1/R^2. \tag{13}$$

Similarly to the surface $\overline{F}$, consider a spherical segment F_s whose height is equal to H and base radius is equal to r. Place $\overline{F}$ and F_s so that they have the same axis of rotation, the coordinate z-axis, and the same edge $\partial\overline{F} = \partial F_s$. Then the surfaces in question will have the same vertex O_s, where they are tangent to each other. Let C_S and $\overline{C}$ be the meridians of F_s and $\overline{F}$ determined by $y = 0$. Represent the curves C_S and $\overline{C}$ in parametric forms, $x = x_s(\tau), z = z_s(\tau)$ and $x = \bar{x}(\tau), z = \bar{z}(\tau)$, respectively. Moreover, specify τ to be the angle between the coordinate x-axis and the straight lines tangent to the curves such that $\tau = 0$ at the vertex O_s. For $x \ge 0$, one has $\tau \in [0, \pi]$.

Let us show that the following inequality holds true for any $0 < \tau \le \pi/2$:

$$\bar{x}(\tau) < x_s(\tau). \tag{14}$$

For this purpose, consider the segments of the surfaces F_s and $\overline{F}$, obtained by rotating the arcs of the curves C_s and $\overline{C}$, corresponding to $\tau \in (0, \pi/2]$, around the z-axis. To these segments, apply orthogonal projection onto the coordinate plane α and evaluate the areas of the obtained planar domains:

$$\pi[x_s(\tau)]^2 = \iint R^2 \cos\tau \, d\omega,$$

$$\pi[\bar{x}(\tau)]^2 = \iint \frac{\cos\tau}{\overline{K}}\, d\omega, \tag{15}$$

where $d\omega$ is the area element of the spherical image of the surface, and the integration is over the corresponding segments of spherical images. Comparing the right-hand sides in (15) and taking into account (13), one verifies (14) for any $0 < \tau \leq \pi/2$.

For the curvature radii, $\bar{\rho}(\tau)$ and $\rho_s(\tau) \equiv R$, of the meridional curves $\overline{C}$ and C_s of $\overline{F}$ and F_s, respectively, the following inequality holds true at $\tau = 0$ due to (13):

$$\bar{\rho}(\tau) < \rho_s(\tau).$$

Due to continuity, it remains valid in a sufficiently small neighborhood of $\tau = 0$, i.e., the curve $\overline{C}$ is tangent to the curve C_s at O_s from within such that the neighborhood of the point $\tau = 0$ in the curve $\overline{C}$ is within a region M_s bounded by the curve C_s and the x-axis. We claim that the whole curve $\overline{C}$ is not located inside M_s and not tangent to the curve C_s at the points $\tau > 0$ different from the vertex O_s. Indeed, if $\overline{C}$ is tangent to C_s at some point $\tau = \tau' > 0$, then, in view of (13), the segment $\tau < \tau'$ of the surface $\overline{F}$ is contained in the segment $\tau < \tau'$ of the surface F_s [2], and hence the surfaces $\overline{F}$ and F_s are not tangent to each other at $\tau = 0$. Therefore the curve $\overline{C}$ can intersect the curve C_s at some point $\tau = \bar{\tau}^*$ so that an arc $\tau < \bar{\tau}^*$ of the curve $\overline{C}$ is located inside M_s. Let $P^* \in \overline{C}$ be the point of this intersection and τ_s^* be the value of the parameter τ on C_s corresponding to the point P^*. Then $\bar{\tau}^* < \tau_s^*$ does not hold for $\tau_s^* \leq \pi/2$, and thus $x_s(\bar{\tau}^*) < x_s(\tau_s^*) = \bar{x}(\bar{\tau}^*)$ which contradicts to (14).

Thus, the curve $\overline{C}$ can not intersect the curve C_s for $\tau_s^* \leq \pi/2$. Consequently, if $H \leq r$, then the edge of the spherical segment F_s can not be the same as the edge of the surface $\overline{F}$ whose Gaussian curvature satisfies restriction (13), and this completes the proof for the case $H \leq r$.

Now consider the case $H > r$. Let $\overline{C}$ meet the edge $\partial\overline{F}$ at a point P^{**} with $\tau = \bar{\tau}^{**}$ such that $\bar{x}(\bar{\tau}^{**}) = r$. We claim that $\bar{\tau}^{**} > \pi/2$ and hence the surface $\overline{F}$ is a dome.

Indeed, let $\bar{\tau}^{**} \leq \pi/2$. Consider a spindle-shaped surface of revolution $\widetilde{F}_0$ with the following properties: the coordinate z-axis is the axis of rotation, the intersection of $\widetilde{F}_0$ with the coordinate plane $z = 0$ is the equatorial circle of radius r, the Gaussian curvature is equal to $1/R^2$. Let $x = \tilde{x}_0(\tau)$, $z = \tilde{z}_0(\tau)$ be the parametric representation of the meridional curve $\widetilde{C}_0$ obtained by intersecting $\widetilde{F}_0$ with the coordinate plane $y = 0$. Then, in view of (8), the diameter $\widetilde{D}_0$ of $\widetilde{F}_0$ is expressed and estimated as follows:

$$\widetilde{D}_0 = 2\int_0^{\pi/2} \sqrt{R^2 - r^2\sin^2\sigma}\, d\sigma < 2(R + \sqrt{R^2 - r^2}) = 2H. \tag{16}$$

Indeed, the equality holds in (16) for $r = R$. Then, by using asymptotic expansions for elliptic integrals (see [4]), it is easy to see that (16) holds true for $0 < 1 - r/R \ll 1$. For $r < R$, we have

$$\frac{\partial(2H)}{\partial x} = -\frac{1}{\sqrt{R^2 - x}}, \qquad (x \equiv r^2),$$

$$\frac{\partial \widetilde{D}_0}{\partial x} = -\int_0^{\pi/2} \frac{\sin^2\sigma}{\sqrt{R^2 - x\sin^2\sigma}}\,d\sigma > -\int_0^{\pi/2} \frac{\sin^2\sigma}{\sqrt{R^2 - x}}\,d\sigma = -\frac{\pi}{4\sqrt{R^2 - x}},$$

and hence

$$\frac{\partial(2H)}{\partial x} < \frac{\partial \widetilde{D}_0}{\partial x} < 0.$$

Therefore, if r decreases, then the diameter $\widetilde{D}_0$ grows more slowly than $2H$, and this implies inequality (16), q.e.d.

Inequality (16) can be verified directly by calculating the involved integral.

Notice that the body $\overline{L}$ is contained inside the body $\widetilde{L}_0$ bounded by the surface $\widetilde{F}_0$ and the plane containing the edge ∂F. If $\overline{\tau}^{**} = \pi/2$, then this statement follows from Blaschke's theorem [2, IV, §25]. The statement is obvious if $\overline{\tau}^{**}$ is less than the angle $\widetilde{\tau}_0$ corresponding to the point where the meridian $\widetilde{C}_0$ meets the coordinate z-axis (at the conical singular point of the surface $\widetilde{F}_0$, we have $x = \widetilde{x}_0(\widetilde{\tau}_0) = 0$). For the case $\widetilde{\tau}_0 < \overline{\tau}^{**} < \pi/2$, we consider the strips of the surfaces $\overline{F}$ and $\widetilde{F}_0$ obtained by rotating the arcs of $\overline{C}$ and $\widetilde{C}_0$ corresponding to $\tau \in [0, \bar{\tau}^{**}]$. Apply the orthogonal projection onto the plane $z = 0$ so that the strips are mapped onto two planar ring-like domains. Comparing the areas of these domains, we get

$$\begin{aligned} \pi(r^2 - \bar{x}^2(\tau)) &= \int_\tau^{\bar{\tau}^{**}} \frac{\cos\tau}{\overline{K}}\,d\tilde{\omega} < \int_\tau^{\bar{\tau}^{**}} R^2\cos\tau\,d\tilde{\omega} \\ &< \int_\tau^{\pi/2} R^2\cos\tau\,d\tilde{\omega} = \pi(r^2 - \tilde{x}_0(\tau)), \end{aligned} \tag{17}$$

where $d\tilde{\omega} = 2\pi\sin\tau d\tau$ is the area element of the spherical image. Consequently,

$$\widetilde{x}_0(\tau) < \overline{x}(\tau), \quad \tau < \overline{\tau}^{**}.$$

Therefore, see [2, IV, §25], the curvature radii, $\widetilde{\rho}_0(\tau)$ and $\overline{\rho}(\tau)$, of the meridional curves $\widetilde{C}_0 \subset \widetilde{F}_0$ and $\overline{C} \subset \overline{F}$, respectively, satisfy the inequality

$$\widetilde{\rho}_0(\tau) > \overline{\rho}(\tau), \quad \tau < \overline{\tau}^{**}.$$

Let $\widetilde{h}_0(\tau)$ and $\overline{h}(\tau)$ be the support functions of the curves $\widetilde{C}_0$ and $\overline{C}$ with respect to the point P^{**}. Then, for $\tau < \overline{\tau}^{**}$, we have, see [2, IV, §25]:

$$\widetilde{h}_0(\tau) = \int_\tau^{\pi/2} \widetilde{\rho}_0(\sigma)\sin(\sigma - \tau)\,d\sigma,$$

$$\overline{h}(\tau) = \int_\tau^{\overline{\tau}^{**}} \overline{\rho}(\sigma)\sin(\sigma - \tau)\,d\sigma,$$

$$\widetilde{h}_0(\tau) - \overline{h}(\tau) = \int_\tau^{\overline{\tau}^{**}} (\widetilde{\rho}_0(\sigma) - \overline{\rho}(\sigma))\sin(\sigma - \tau)\,d\sigma + \int_{\overline{\tau}^{**}}^{\pi/2} \widetilde{\rho}_0(\sigma)\sin(\sigma - \tau)\,d\sigma > 0.$$

Therefore, if $\widetilde{\tau}_0 < \overline{\tau}^{**} < \pi/2$, then the curves $\widetilde{C}_0$ and $\overline{C}$ intersect at the point P^{**} only, and hence the body $\overline{L}$ is contained inside the body $\widetilde{L}_0$. In view of (16), this implies that the height of the surface $\overline{F}$ is less than H, but this contradicts the assumption of the theorem. Consequently, $\bar{\tau}^{**} > \pi/2$ holds along the edge $\partial\overline{F}$, and thus the surface $\overline{F}$ is a convex dome.

Denote by $\overline{R}$ the radius of the equatorial circle of the surface $\overline{F}$ and consider a spindle-shaped surface of revolution $\overline{F}^0$, which has the same equatorial radius $\overline{R}$ and contains an axisymmetric segment $\bar{\bar{F}}^0$ with the base radius r and the height H. Let $\overline{K}^0$ be the Gaussian curvature of $\overline{F}^0$, which is determined according to (8) from the equations

$$r = \overline{R}\cos v, \quad H = \int_v^{\pi/2} \sqrt{1/\overline{K^0} - \overline{R}^2 \sin^2 u}\, du, \quad -\pi/2 < v < 0.$$

In view of the assumptions of the theorem and the restrictions (12) and (13), we have

$$\overline{K}^0 \leq K^0 \leq \overline{K},$$

where $\overline{K}^0 \equiv \text{const}$ and $K^0 \equiv \text{const}$. Consider two possible cases.

1. Let $\overline{K} \not\equiv \text{const}$. Then $\overline{F}$ lies inside the surface $\bar{\bar{F}}^0$, see [2, IV, §25], therefore the height of $\overline{F}$ (as well as the height of $\widetilde{F}$) is less than H.

2. Let $\overline{K} \equiv \text{const}$. Since the surface $\overline{F}$ is regular, it is a spherical segment of radius $\bar{R} < R$ due to (13). Therefore $\overline{F}$ is inside the spherical segment F_s, and hence the height of $\overline{F}$ is less than H.

Thus, if the assumption (12) is true, then the surface $\overline{F}$ (and the surface $\widetilde{F}$) do not satisfy the assumptions of the theorem. Hence (12) is false, and this completes the proof of Theorem 2. □

Finally, consider the width of the convex surface F which is defined as the minimum of the distances between parallel supporting planes of F [2].

Theorem 3. *Let F be a strictly convex closed surface F with continuous principal curvatures. Assume that the width of F is not greater than Δ and, moreover, F is located in the ball whose radius is not greater than R, where $R \geq \Delta/2$. Let F_{00} be a convex surface of rotation with constant Gaussian curvature $K_{00} \equiv const$, whose width is equal to Δ and equatorial radius is equal to R. Then the following estimate for the Gaussian curvature K of F holds true:*

$$\max_{(F)} K \geq K_{00}, \tag{18}$$

where the maximum is taken over all points of F. Besides, if $R = \Delta/2$, then F_{00} is a sphere and the equality is achieved in (18) provided that F coincides with F_{00}. If $R > \Delta/2$, then inequality (18) is strict and K_{00} is the infimum of values of $\max K$, which can be approached arbitrarily closely by choosing the surface F to be sufficiently close to F_{00}.

Proof. Assume that the stated theorem is false, i.e., there exists a surface $\widetilde{F}$ which satisfies the assumptions of the theorem, but whose Gaussian curvature $\widetilde{K}$ satisfies the converse of (18):

$$\begin{aligned} &\widetilde{K} < K_{00} && \text{if} && R = \Delta/2, \\ &\widetilde{K} \le K_{00} && \text{if} && R > \Delta/2. \end{aligned} \tag{19}$$

Transform the surface $\widetilde{F}$ into a center symmetrical surface of rotation $\overline{F}$ which still satisfies the assumptions of the theorem. For this purpose, take the support function $\widetilde{H}(\varphi, \psi)$ of the surface $\widetilde{F}$, find the function

$$\widetilde{\widetilde{H}}(\psi) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \widetilde{H}(\varphi, \psi)\, d\varphi,$$

by averaging $\widetilde{H}(\varphi, \psi)$ with respect to φ, and then construct a surface $\widetilde{\widetilde{F}}$ with the support function $\widetilde{\widetilde{H}}(\psi)$, see [2, IV, §26]. It is assumed here that φ and ψ are the longitude and the latitude of the geographical coordinate system on the unit sphere centered at a point inside $\widetilde{F}$ so that the polar axis α is chosen to be orthogonal to a pair of parallel planes tangent to $\widetilde{F}$, the distance between which is equal to the width $\widetilde{\Delta} \le \Delta$ of the surface $\widetilde{F}$. After that consider the function

$$\overline{H}(\psi) = \frac{1}{2}\widetilde{\widetilde{H}}(\psi) + \frac{1}{2}\widetilde{\widetilde{H}}(-\psi)$$

and construct a surface with the support function $\overline{H}(\psi)$; this is just the desired strictly convex closed surface of the rotation surface $\overline{F}$ which is center symmetrical, see [2, IV, §26]. Moreover, $\overline{F}$ is located inside a ball of radius R and has a symmetry plane which is orthogonal to the axis α. Besides, the Gaussian curvature $\overline{K}$, the width $\overline{\Delta}$ and the equatorial radius $\overline{R}$ of $\overline{F}$ satisfy the following estimates (see [2, §26] and take into account (19)):

$$\begin{aligned} &\max_{(\overline{F})} \overline{K} \le \max_{(\widetilde{F})} \widetilde{K} < K_{00} && \text{if} && R = \Delta/2 \\ &\max_{(\overline{F})} \overline{K} \le \max_{(\widetilde{F})} \widetilde{K} \le K_{00} && \text{if} && R > \Delta/2, \end{aligned} \tag{20}$$

$$\overline{\Delta} = \widetilde{\Delta}, \qquad \overline{R} \le R. \tag{21}$$

We claim that in view of (19) the widths of the surfaces $\overline{F}$ and $\widetilde{F}$ are greater than Δ.

Indeed, since the surface $\overline{F}$ is strictly convex and center symmetrical, then its continuous Gaussian curvature $\overline{K}$ and the radius of the equatorial circle $\overline{R}$ have to satisfy the inequality

$$\overline{R} \ge 1/\sqrt{\max_{(\overline{F})} \overline{K}}.$$

Therefore, by virtue of (20), we get

$$\overline{R} \ge 1/\sqrt{K_{00}}. \tag{22}$$

Similarly to the proof of Theorem 1, consider a surface of rotation $\overline{F}_{00}$ with constant Gaussian curvature, which has the following properties: the axis of rotation of $\overline{F}_{00}$ is the straight line α, the radius of the equatorial circle of $\overline{F}_{00}$ is equal to $\overline{R}$. Denote by K_{00} the Gaussian curvature of $\overline{F}_{00}$. Introduce a Cartesian coordinate system (x, y, z) by choosing the plane containing the equatorial circle of $\overline{F}_{00}$ for the coordinate (x, y)-plane and the axis of rotation α for the coordinate z-axis. The surface $\overline{F}_{00}$ is obtained by rotating its meridional curve $y = 0$ around the z-axis. This curve can be represented parametrically in the following form, see [2, 6]:

$$x = \gamma(\psi), \quad z = \delta(\psi), \quad |\psi| \le \pi/2,$$

where

$$\begin{aligned} \gamma(\psi) &= \sqrt{\overline{R}^2 - 1/K_{00}\sin^2\psi}, \\ \delta(\psi) &= 1/\sqrt{K_{00}}\left(\int_0^{\psi}\sqrt{K_{00}\overline{R}^2 - \sin^2 u}\,du - \right. \\ &\qquad\qquad \left. -(K_{00}\overline{R}^2 - 1)\int_0^{\psi}\frac{du}{\sqrt{K_{00}\overline{R}^2 - \sin^2 u}}\right), \end{aligned}$$

ψ is the angle between the normal vector to the surface $\overline{F}_{00}$ and the equatorial (x, y)-plane. The equality in (22) implies that $\overline{F}_{00}$ is a sphere. If (22) is a strict inequality, then

$$x = \gamma(\pm\pi/2) = \sqrt{\overline{R}^2 - 1/K_{00}} \ne 0,$$

and hence the surface $\overline{F}_{00}$ is not closed. Make it closed by adding two discs of radius $\sqrt{\overline{R}^2 - 1/K_{00}}$ placed at the planes $z = \delta(\pm\pi/2)$. The resulting closed surface of rotation resembles the head of cheese, therefore $\overline{F}_{00}$ will be referred to as "cheese-shaped" following the terminology of [2, IV, §26]. Notice that if $R > \Delta/2$, then F_{00} is also "cheese-shaped" [6], and in this case one gets

$$R > 1/\sqrt{K_{00}}. \tag{23}$$

Since $\overline{K} \le K_{00}$ due to (20), then $\overline{F}$ contains a "cheese-shaped" surface $\overline{F}_{00}$, see [2, IV, §26]. Therefore, the width $\overline{\Delta}$ of the surface $\overline{F}$ is not less than the width $\overline{\Delta}_{00}(K_{00}, \overline{R}) = 2\delta(\pi/2)$ of the surface $\overline{F}_{00}$, i.e.,

$$\begin{aligned} &\overline{\Delta} \ge \overline{\Delta}_{00}(K_{00}, \overline{R}) && \text{if} && \overline{R} = 1/\sqrt{K_{00}}, \\ &\overline{\Delta} > \overline{\Delta}_{00}(K_{00}, \overline{R}) && \text{if} && \overline{R} > 1/\sqrt{K_{00}}. \end{aligned} \tag{24}$$

Notice that if $\overline{R} = R$, then $\overline{F}_{00}$ and F_{00} coincide, hence

$$\overline{\Delta}_{00}(K_{00}, R) = \Delta. \tag{25}$$

The width of a "cheese-shaped" surface monotonically decreases when the radius of its equatorial circle increases [2, IV, §26]. Therefore,

$$\overline{\Delta}_{00}(K_{00}, \overline{R}) > \overline{\Delta}_{00}(K_{00}, R) = \Delta \quad \text{if} \quad \overline{R} < R. \tag{26}$$

Consider the cases depending on the structure of the surface $\overline{F}$. We have the estimates (21)–(26) and the inequalities $R \geq \overline{R} \geq 1/\sqrt{K_{00}}$.

1. Let
$$\overline{R} > 1/\sqrt{K_{00}}. \tag{27}$$
Then inequality (24) is strict, hence
$$\widetilde{\Delta} = \overline{\Delta} > \overline{\Delta}_{00}(K_{00}, \overline{R}) \geq \overline{\Delta}_{00}(K_{00}, R) = \Delta. \tag{28}$$

2. Let
$$\overline{R} = 1/\sqrt{K_{00}} < R. \tag{29}$$
Then inequality (26) is strict, hence
$$\widetilde{\Delta} = \overline{\Delta} \geq \overline{\Delta}_{00}(K_{00}, \overline{R}) > \overline{\Delta}_{00}(K_{00}, R) = \Delta. \tag{30}$$

3. Let
$$\overline{R} = 1/\sqrt{K_{00}} = R. \tag{31}$$
In this case, the surfaces $\overline{F}_{00}$ and F_{00} coincide and are spheres whose width (diameter) is equal to $\overline{\Delta}_{00} = \Delta = 2R$, so $R = \Delta/2$. Therefore, inequality (19) is strict, i.e., $\overline{K} < K_{00}$ by (20). The surface $\overline{F}$ and the sphere $\overline{F}_{00}$ contained inside $\overline{F}$ can not be tangent at a point located on the axis of rotation, see [2], therefore $\overline{\Delta} > \overline{\Delta}_{00}(K_{00}, \overline{R})$. Then we obtain
$$\widetilde{\Delta} = \overline{\Delta} > \overline{\Delta}_{00}(K_{00}, \overline{R}) = \overline{\Delta}_{00}(K_{00}, R) = \Delta.$$

Thus, due to (27), (29), and (31) the width of the surface is greater than the width Δ of the surface F_{00}. This means that the surface $\widetilde{F}$ does not satisfy the assumptions of the theorem, hence the assumption (19) leads to a contradiction. This completes the proof of Theorem 3. □

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V.I. Babenko,

B. Verkin Institute for Low Temperature Physics and Engineering of the National Academy of Sciences of Ukraine, 47 Nauky Ave., Kharkiv, 61103, Ukraine,
E-mail: babenko@ilt.kharkov.ua

Оцінки гаусової кривини строго опуклої поверхні та її інтегральні параметри

В.І. Бабенко

Розглядаються як замкнені, так і незамкнені з плоским краєм строго опуклі поверхні з неперервною кривиною. Одержано оцінки зверху та знизу для гаусової кривини в залежності від заданих обмежень на деякі інтегральні параметри поверхні, такі як: діаметр або ширина поверхні, об'єм тіла, яке обмежує поверхня, максимальна площа "поперечного" перерізу тіла, радіус описаного чи вписаного шару, висота незамкненої поверхні та площа області, яку обмежує плоский край поверхні.

Ключові слова: строго опуклі поверхні, гаусова кривина, інтегральні параметри.