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Hypersurfaces with L_r-Pointwise 1-Type Gauss Map

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In this paper, we study hypersurfaces in $\mathbb{E}^{n+1}$ whose Gauss map G satisfies the equation $L_rG = f(G + C)$ for a smooth function f and a constant vector C, where L_r is the linearized operator of the $(r+1)$-st mean curvature of the hypersurface, i.e., $L_r(f) = \mathrm{Tr}(P_r \circ \nabla^2 f)$ for $f \in \mathcal{C}^\infty(M)$, where P_r is the r-th Newton transformation, $\nabla^2 f$ is the Hessian of f, $L_rG = (L_rG_1, \dots, L_rG_{n+1})$ and $G = (G_1, \dots, G_{n+1})$. We focus on hypersurfaces with constant $(r + 1)$-st mean curvature and constant mean curvature. We obtain some classification and characterization theorems for these classes of hypersurfaces.

Key words: linearized operators L_r, L_r-pointwise 1-type Gauss map, r-minimal hypersurface.

Mathematical Subject Classification 2010: 53D02, 53C40, 53C42

1. Introduction

The study of submanifolds of finite type began in the late seventies with B.Y. Chen's attempts to find the best possible estimate of the total mean curvature of compact submanifolds of a Euclidean space and to find a notion of "degree" for submanifolds of a Euclidean space (see [8] for details). Since then the subject has had a rapid development and many mathematicians contributed to it (see the excellent survey of B.Y. Chen [6]). By definition, an isometrically immersed submanifold $x : M^n \to \mathbb{E}^{n+k}$ is said to be of finite type if x has a finite decomposition as $x - x_0 = \sum_{i=1}^{p} x_i$, for some positive integer p, such that $\Delta x_i = \lambda_i x_i, \lambda_i \in \mathbb{R}, 1 \le i \le p$, x_0 is constant, $x_i, 1 \le i \le p$, are non-constant smooth maps $x_i : M^n \to \mathbb{E}^{n+k}$ and Δ is the Laplace operator of M. In [10], this definition was similarly extended to differentiable maps, in particular, to the Gauss map of hypersurfaces. The notion of finite type Gauss map is an especially useful tool in the study of hypersurfaces (cf. [2–5, 9, 12, 16, 19]). If an oriented hypersurface M of a Euclidean space has a 1-type Gauss map G, then G satisfies $\Delta G = \lambda(G + C)$ for a constant $\lambda \in \mathbb{R}$ and a constant vector C. In [10], Chen and Piccinni made a general study on compact hypersurfaces of Euclidean spaces with finite type Gauss map; they proved that a compact hypersurface M of $\mathbb{E}^{n+1}$ has a 1-type Gauss map G if and only if M is a hypersphere in $\mathbb{E}^{n+1}$.

As is well known, the Laplace operator of a hypersurface M immersed into $\mathbb{E}^{n+1}$ is an (intrinsic) second-order linear differential operator which arises naturally as the linearized operator of the first variation of the mean curvature for normal variations of the hypersurface. From this point of view, the Laplace operator Δ can be seen as the first one of a sequence of n operators $L_0 = \Delta, L_1, \dots, L_{n-1}$, where L_r stands for the linearized operator of the first variation of the $(r+1)$-st mean curvature arising from normal variations of the hypersurface (see [22]). These operators are given by $L_r(f) = \mathrm{Tr}(P_r \circ \nabla^2 f)$ for any $f \in C^\infty(M)$, where P_r denotes the r-th Newton transformation associated to the second fundamental form of the hypersurface, and $\nabla^2 f$ is the Hessian of f (see the next section for details).

From this point of view, S.M.B. Kashani introduced the notion of L_r-finite type hypersurface in the Euclidean space [15], as an extension of the finite type theory. One can find our results in the last section of the last chapter of B.Y. Chen's book [8].

Notice that sometimes the symbol $\Box$ is used to denote the operator L_1 which is the Cheng–Yau operator introduced in [11]. Later, in [17], the notion of pointwise 1-type Gauss map for the surfaces of the Euclidean 3-space $\mathbb{E}^3$ was extended in a natural way in terms of the Chen–Yau operator $\Box$ as follows:

Definition 1.1. A surface M of the Euclidean space $\mathbb{E}^3$ is said to have an L_1-pointwise 1-type Gauss map if its Gauss map satisfies

$$\Box G = f(G + C) \tag{1.1}$$

for a smooth function $f \in C^\infty(M)$ and a constant vector $C \in \mathbb{E}^3$. More precisely, an L_1-pointwise 1-type Gauss map is said to be of the first kind if (1.1) is satisfied for $C = 0$; otherwise, it is said to be of the second kind. Moreover, if (1.1) is satisfied for a constant function f, then we say *that M has an L_1-(global) 1-type Gauss map.*

Rotational, helicoidal and canal surfaces in $\mathbb{E}^3$ with L_1-pointwise 1-type Gauss map were studied in [18,21]. Motivated by this study, we define the hypersurfaces with L_r-pointwise 1-type Gauss map in this paper. In Section 2, we give the definition of a hypersurface with L_r-pointwise 1-type Gauss map and the basic definitions of the theory of hypersurfaces in $\mathbb{E}^{n+1}$. In Section 3, we focus on the hypersurfaces with constant $(r+1)$-st mean curvature and constant mean curvature. We obtain some classification and characterization theorems for the hypersurfaces with L_r-pointwise 1-type Gauss map.

2. Preliminaries

In this section, we recall the basic concepts of the theory of hypersurfaces [1]. Let $x : M^n \to \mathbb{E}^{n+1}$ be an isometrically immersed hypersurface in the Euclidean space with Gauss map G. We denote by ∇^0 and ∇ the Levi-Civita connections on $\mathbb{E}^{n+1}$ and M^n, respectively. The Gauss and Weingarten formulae are given by $\nabla^0_X Y = \nabla_X Y + \langle SX, Y\rangle G$ and $SX = -\nabla^0_X G$ for all tangent vector fields

$X, Y \in \mathcal{X}(M^n)$, where $S : \mathcal{X}(M^n) \to \mathcal{X}(M^n)$ is the shape operator (Weingarten endomorphism) of M^n with respect to the Gauss map G.

As is well known, for every point $p \in M^n$, S defines a linear self-adjoint endomorphism on the tangent space T_pM^n, and its eigenvalues $\lambda_1(p)$, $\lambda_2(p)$, $\ldots$, $\lambda_{n-1}(p)$, $\lambda_n(p)$ are the principal curvatures of the hypersurface. The characteristic polynomial $Q_S(t)$ of S is defined by

$$Q_S(t) = \det(tI - S) = (t - \lambda_1)(t - \lambda_2)\ldots(t - \lambda_{n-1})(t - \lambda_n) = \sum_{k=0}^{n}(-1)^k a_k t^{n-k},$$

where a_k is given by

$$a_k = \sum_{1 \le i_1 < \cdots < i_k \le n} \lambda_{i_1} \ldots \lambda_{i_k}, \quad \text{with } a_0 = 1.$$

The r-th mean curvature H_r of M^n in $\mathbb{E}^{n+1}$ is defined by $\binom{n}{r} H_r = a_r$, with $H_0 = 1$.

If $H_{r+1} = 0$, then we say *that M^n is an r-minimal hypersurface. The r-th Newton transformation of M^n* is the operator $P_r : \mathcal{X}(M^n) \to \mathcal{X}(M^n)$ defined by

$$P_r = \sum_{j=0}^{r}(-1)^j \binom{n}{r-j} H_{r-j} S^j = \sum_{j=0}^{r}(-1)^j a_{r-j} S^j.$$

Equivalently,

$$P_0 = I, \quad P_r = \binom{n}{r} H_r I - S \circ P_{r-1}.$$

Along with each Newton transformation P_r, we consider the second-order linear differential operator $L_r : C^\infty(M^n) \to C^\infty(M^n)$ given by $L_r(f) = \mathrm{Tr}(P_r \circ \nabla^2 f)$. Here, $\nabla^2 f : \mathcal{X}(M^n) \to \mathcal{X}(M^n)$ denotes the self-adjoint linear operator metrically equivalent to the Hessian of f and it is given by $\langle \nabla^2 f(X), Y\rangle = \langle \nabla_X(\nabla f), Y\rangle$, $X, Y \in \mathcal{X}(M^n)$.

Now we state the following lemma from [1], which we will need later.

Lemma 2.1. *Let $x : M^n \to \mathbb{E}^{n+1}$ be a connected orientable hypersurface immersed into the Euclidean space with Gauss map G. Then the Gauss map G of M satisfies*

$$L_r G = -\binom{n}{r+1} \nabla H_{r+1} - \binom{n}{r+1}(nH_1 H_{r+1} - (n-r-1)H_{r+2})G. \tag{2.1}$$

Next we will give the definition for a hypersurface with L_r-pointwise 1-type Gauss map.

Definition 2.2. An oriented hypersurface M of a Euclidean space $\mathbb{E}^{n+1}$ is said *to have an L_r-pointwise 1-type Gauss map* if its Gauss map satisfies

$$L_r G = f(G + C) \tag{2.2}$$

for a smooth function $f \in C^\infty(M)$ and a constant vector $C \in \mathbb{E}^{n+1}$. More precisely, an L_r-pointwise 1-type Gauss map is said to be of the first kind if (2.2) is satisfied for $C = 0$; otherwise, it is said to be of the second kind. Moreover, if (2.2) is satisfied for a constant function f, then we say *M has a (global) 1-type Gauss map.*

A function (or mapping) ϕ defined on M is said to be harmonic if its Laplacian vanishes identically, i.e., if $\Delta\phi = 0$. After changing the Laplace operator Δ by the operator L_r, we give the following definition.

Definition 2.3. An oriented hypersurface M of a Euclidean space $\mathbb{E}^{n+1}$ is said *to have an L_r-harmonic Gauss map* if its Gauss map satisfies $L_r G = 0$.

We also need the following remark, theorem and lemma for later use.

Remark 2.4 ([7]). A hypersurface of a Euclidean space $\mathbb{E}^{n+1}$ is called *isoparametric* if its principal curvatures are constant counting multiplicities. An isoparametric hypersurface of $\mathbb{E}^{n+1}$ has q distinct principal curvatures with $q \leq 2$. If $q = 2$, one of principal curvatures must be 0. Isoparametric hypersurfaces of $\mathbb{E}^{n+1}$ are locally hyperspheres, hyperplanes or a standard product embedding of $S^k \times \mathbb{E}^{n-k}$. This result was proved in [20] for $n = 2$, and in [23], for arbitrary n.

Theorem 2.5 ([14])**.** *Let M^3 be an oriented 3-dimensional complete Riemannian manifold, and $x : M^3 \to \mathbb{E}^4$ be a minimal isometric immersion with constant Gauss–Kronecker curvature. Then the Gauss–Kronecker curvature is identically zero.*

Lemma 2.6. *Let M be an oriented hypersurface in $\mathbb{E}^{n+1}$ with at most 2 distinct principal curvatures of multiplicities q and $n-q$ $(1 \leqslant q \leqslant n)$. Suppose that $\{e_1, \ldots, e_n\}$ is an orthonormal frame corresponding to the principal directions and the principal curvatures κ_1, κ_2 such that $Se_i = \kappa_1 e_i$, $1 \leqslant i \leqslant q$ and $Se_j = \kappa_2 e_j$, $q+1 \leqslant j \leqslant n$. If a vector field $C \in C^\infty(M, \mathbb{E}^{n+1})$ is constant, then*

$$e_i(C_{n+1}) = -\kappa_1 C_i, \qquad 1 \leqslant i \leqslant q, \tag{2.3}$$

$$e_i(C_{n+1}) = -\kappa_2 C_i, \qquad q+1 \leqslant i \leqslant n, \tag{2.4}$$

where $C_i = \langle C, e_i\rangle$ and $C_{n+1} = \langle C, G\rangle$.

Proof. By the definition above, we have $C = \sum_{i=1}^n C_i e_i + C_{n+1} G$. Suppose that $\omega_{ij}^k = \langle \nabla_{e_i} e_j, e_k\rangle$, $1 \leqslant i, j, k \leqslant n$; by a direct calculation, we have

$$\begin{aligned}
\nabla^0_{e_i} C &= \sum_{j=1}^n e_i(C_j) e_j + e_i(C_{n+1}) G + \sum_{j=1}^n C_j \nabla^0_{e_i} e_j + C_{n+1} \nabla^0_{e_i} G \\
&= \sum_{j=1}^n e_i(C_j) e_j + e_i(C_{n+1}) G + \sum_{j,k=1}^n C_j \omega_{ij}^k e_k + C_i \kappa_l G - C_{n+1} \kappa_l e_i,
\end{aligned}$$

where $l = 1$ if $1 \leqslant i \leqslant q$ and $l = 2$ if $q+1 \leqslant i \leqslant n$. Since $\nabla^0_{e_i} C = 0$, we get the result. □

3. Characterization theorems on hypersurfaces with L_r-pointwise 1-type Gauss map

In this section, we will give some characterization theorems on the hypersurfaces of $\mathbb{E}^{n+1}$ in terms of their Gauss map. We focus on the hypersurfaces with constant $(r+1)$-st mean curvature and on hypersurfaces with constant mean curvature.

3.1. Hypersurfaces with constant $(r+1)$-st mean curvature.

Theorem 3.1. *If an oriented hypersurface M of a Euclidean space $\mathbb{E}^{n+1}$ has L_r-harmonic Gauss map, then the $(r+1)$-st mean curvature of M is constant, in particular, if $n = r+1$, then M is minimal or $(n-1)$-minimal, i.e., $H_n = 0$.*

Proof. By Lemma 2.1, M has the L_r-harmonic Gauss map if the $(r+1)$-st mean curvature of M is constant. If $n = r+1$, then Lemma 2.1 implies that $HH_n = 0$, hence M is minimal or $(n-1)$-minimal. □

In particular, when $n = 2$, we deduce from Theorem 3.1 and Remark 2.4 the following corollary proved by Kim and Turgay in [17].

Corollary 3.2. *An oriented surface M in $\mathbb{E}^3$ has an L_1-harmonic Gauss map if and only if it is flat, i.e., its Gaussian curvature vanishes identically.*

From Theorem 3.1 and Theorem 2.5 we can easily deduce the following corollary.

Corollary 3.3. *If an oriented complete hypersurface M in $\mathbb{E}^4$ has an L_2-harmonic Gauss map, then M is 2-minimal.*

In [10], Chen and Piccinni proved that there is no compact hypersurface in $\mathbb{E}^{n+1}$ with harmonic Gauss map. To extend this result to the case of L_r-harmonic Gauss map, we state and prove the following theorem.

Theorem 3.4. *There is no compact hypersurface in $\mathbb{E}^{n+1}$ with L_r-harmonic Gauss map.*

Proof. Let M be a compact hypersurface in $\mathbb{E}^{n+1}$ with L_r-harmonic Gauss map. By Lemma 2.1, the $(r+1)$-st mean curvature of M is constant. It is well known that every compact hypersurface in a Euclidean space has elliptic points, that is, the points where all the principal curvatures are positive (or negative). In particular, this implies that there exists no compact hypersurface in $\mathbb{E}^{n+1}$ with vanishing $(r+1)$-st mean curvature for every $r = 0, \dots, n-1$. Since M^n has elliptic points, after an appropriate choice of the Gauss map G of M^n, if r is odd, we can suppose that $H_{r+1} > 0$. Also, if $H_{r+1} > 0$, then $H_j > 0$ for all $j = 1, \dots, r$. Moreover,

$$H_{i-1}H_{i+1} \leq H_i^2 \quad \text{and} \quad H_1 \geq H_2^{1/2} \geq H_3^{1/3} \geq \cdots \geq H_i^{1/i}, \qquad i = 1, \dots, r$$

(see page 52 of [13]). Thus, the above inequalities yield

$$H_1 H_{r+1} \geq H_{r+2}. \tag{3.1}$$

On the other hand, by using formula (2.1), when $r = n-1$, we get $H_1 = 0$, which is impossible. When $r < n-1$, we have $H_1 H_{r+1} < H_{r+2}$, which contradicts (3.1). □

Using Definition 2.2 and equation (2.1), we now state the following theorems which characterize the hypersurfaces of Euclidean spaces with L_r-1-type Gauss map of the first kind.

Theorem 3.5. *An oriented hypersurface M in $\mathbb{E}^{n+1}$ has an L_r-pointwise 1-type Gauss map of the first kind if and only if it has a constant $(r+1)$-st mean curvature.*

Theorem 3.6. *An oriented hypersurface M in $\mathbb{E}^{n+1}$ has an L_r-(global) 1-type Gauss map of the first kind if and only if both H_{r+1} and $nH_1H_{r+1} - (n-r-1)H_{r+2}$ are constant.*

We can deduce the following corollary on hypersurfaces with L_r-1-type Gauss map.

Corollary 3.7. *All oriented isoparametric hypersurfaces of a Euclidean space $\mathbb{E}^{n+1}$ have an L_r-(global) 1-type Gauss map.*

So, by Remark 2.4, hyperplanes, hyperspheres and the generalized cylinder $S^{n-k} \times \mathbb{E}^k$ of $\mathbb{E}^{n+1}$ have the L_r-1-type Gauss map. We can also state some characterization corollaries about hypersurfaces with at most 2 distinct principal curvatures.

Corollary 3.8. *An oriented hypersurface M in $\mathbb{E}^{n+1}$ with at most 2 distinct principal curvatures has an L_r-(global) 1-type Gauss map of the first kind, where $n \neq r+1$, if and only if it is an open domain of a hypersphere, a hyperplane or a generalized cylinder.*

Proof. By Theorem 3.6, we conclude that M has the L_r-(global) 1-type Gauss map of the first kind if and only if it is isoparametric, and thus Remark 2.4 gives the result. □

Corollary 3.9. *An oriented hypersurface M in $\mathbb{E}^{n+1}$ with at most 2 distinct principal curvatures has an L_{n-1}-(global) 1-type Gauss map of the first kind if and only if it is either an $(n-1)$-minimal hypersurface or an open domain of a hypersphere, a hyperplane or a generalized cylinder.*

Proof. By Theorem 3.6, M has the L_{n-1}-(global) 1-type Gauss map of the first kind if and only if H_n and HH_n are constant. If $H_n \neq 0$, then H is constant. Therefore, M is isoparametric, so, Remark 2.4 gives the result. □

In particular, when $n = 2$, we have the following result that was also proved by Kim and Turgay in [17].

Corollary 3.10. *An oriented surface M in $\mathbb{E}^3$ has an L_1-(global) 1-type Gauss map of the first kind if and only if it is either a flat surface or an open domain of a sphere.*

3.2. Hypersurfaces with constant mean curvature. Now, we study the hypersurfaces in $\mathbb{E}^{n+1}$ with at most 2 distinct principal curvatures and a constant mean curvature which have an L_r-pointwise 1-type Gauss map. First, we focus on minimal hypersurfaces.

Corollary 3.11. *An oriented minimal hypersurface M in $\mathbb{E}^{n+1}$ with at most 2 distinct principal curvatures has an L_r-pointwise 1-type Gauss map of the first kind if and only if it is an open domain of a hyperplane.*

Proof. The proof follows directly from Theorem 3.5 and Remark 2.4. □

In particular, the following result follows directly from Corollary 3.11 that was proved for minimal surfaces by Kim and Turgay [17].

Corollary 3.12. *An oriented minimal surface M in $\mathbb{E}^3$ has an L_1-pointwise 1-type Gauss map of the first kind if and only if it is an open domain of a plane.*

Next, we prove the following proposition.

Proposition 3.13. *Let M be a connected orientable hypersurface in $\mathbb{E}^{n+1}$ with at most 2 distinct principal curvatures. Suppose that $nH_1H_{r+1} = (n-r-1)H_{r+2}$. Then M has an L_r-pointwise 1-type Gauss map of the second kind if and only if it is an open domain of a hyperplane.*

Proof. Let M be a connected orientable hypersurface in $\mathbb{E}^{n+1}$ with at most 2 distinct principal curvatures of multiplicities q and $n-q$, $1 \leqslant q \leqslant n$. If M has an L_r-pointwise 1-type Gauss map of the second kind, then (2.2) is satisfied for a constant vector C and a smooth function f. Let $\mathcal{O} = \{p \in M \mid f(p) \neq 0\}$. We now suppose $\mathcal{O} \neq \varnothing$. Since $nH_1H_{r+1} = (n-r-1)H_{r+2}$, (2.1) and (2.2) imply $f(G+C) = -\binom{n}{r+1}\nabla H_{r+1}$. Therefore, we have $C_{n+1} = \langle C, G\rangle = -1$ on $\mathcal{O}$. Thus, from (2.3) and (2.4), we obtain

$$\kappa_1 C_1 = \cdots = \kappa_1 C_q = \kappa_2 C_{q+1} = \cdots = \kappa_2 C_n = 0 \quad \text{on } \mathcal{O}.$$

Let $\mathcal{O}_1 = \{p \in \mathcal{O} \mid \kappa_1\kappa_2(p) \neq 0\}$. Then, $C_1 = \cdots = C_n = 0$ on $\mathcal{O}_1$. Thus, the constant vector $C = -G$ on $\mathcal{O}_1$ and thus $\mathcal{O}_1$ is a part of a hyperplane, which is a contradiction. Therefore, we have $\mathcal{O}_1 = \varnothing$, which implies $\kappa_1\kappa_2 = 0$. Since $nH_1H_{r+1} = (n-r-1)H_{r+2}$, $\mathcal{O}$ is an open domain of a hyperplane. Moreover, by the continuity, we have $M = \mathcal{O}$.

Conversely, suppose M is an open domain of a hyperplane. Then its Gauss map G is a non-zero constant vector, which implies $L_rG = 0$. Therefore, (2.2) is satisfied for $C = -G \neq 0$ and an arbitrary smooth function f. Hence, M has the L_r-pointwise 1-type Gauss map of the second kind. □

By combining Corollary 3.11 and Proposition 3.13, we obtain.

Corollary 3.14. *An oriented connected minimal hypersurface M in $\mathbb{E}^{n+1}$ with at most 2 distinct principal curvatures has an L_{n-1}-pointwise 1-type Gauss map if and only if it is an open domain of a hyperplane.*

In particular, when $n = 2$, by combining Corollary 3.12 and Proposition 3.13, we obtain the following corollary proved by Kim and Turgay in [17].

Corollary 3.15. *An oriented connected minimal surface M in $\mathbb{E}^3$ has an L_1-pointwise 1-type Gauss map if and only if it is an open domain of a plane.*

Next, we give a complete classification of hypersurfaces with constant mean curvature and at most 2 distinct principal curvatures whose Gauss map satisfies $L_r G = \lambda(G + C)$ for a constant λ and a constant vector C.

Theorem 3.16. *Let M be a hypersurface with constant mean curvature and at most 2 distinct principal curvatures in $\mathbb{E}^{n+1}$. Then M has an L_r-(global) 1-type Gauss map if and only if it is an open domain of a hypersphere, a hyperplane or a generalized cylinder.*

Proof. Let M be an oriented hypersurface in $\mathbb{E}^{n+1}$ with at most 2 distinct principal curvatures of multiplicities q and $n - q$, $1 \leqslant q \leqslant n$. Suppose that $\{e_1, \dots, e_n\}$ is an orthonormal frame of its principal directions to the principal curvatures κ_1, κ_2 such that $Se_i = \kappa_1 e_i$, $1 \leqslant i \leqslant q$, $Se_j = \kappa_2 e_j$, $q + 1 \leqslant j \leqslant n$. Let us consider an open set $U = \{p \in M : \nabla H_{r+1}(p) \neq 0\}$. Our objective is to show that U is empty. Since M has a constant mean curvature, we have $q\kappa_1 + (n - q)\kappa_2 = h_0$ for a constant h_0, which implies

$$e_i(\kappa_1) = \frac{q - n}{q} e_i(\kappa_2), \quad i = 1, \dots, n. \tag{3.2}$$

Now we suppose that M has an L_r-(global) 1-type Gauss map. Therefore, from (2.1) and (2.2), we obtain

$$-\nabla H_{r+1} - (h_0 H_{r+1} - (n - r - 1)H_{r+2})G = \lambda(G + C). \tag{3.3}$$

From (3.2), we conclude that there exist polynomials f and g with constant coefficients such that

$$e_i(H_{r+1}) = f(\kappa_1)e_i(\kappa_1), \; e_i(H_{r+2}) = g(\kappa_1)e_i(\kappa_1), \quad i = 1, \dots, n. \tag{3.4}$$

From (3.2)–(3.4), we get

$$\lambda C_i = -e_i(H_{r+1}) = e_i(\kappa_1)f(\kappa_1), \quad i = 1, \dots, n, \tag{3.5}$$

$$\lambda(C_{n+1} + 1) = -h_0 H_{r+1} + (n - r - 1)H_{r+2}. \tag{3.6}$$

By using (3.4) and (3.6), we obtain

$$\lambda e_i(C_{n+1} + 1) = (-h_0 f(\kappa_1) + (n - r - 1)g(\kappa_1))e_i(\kappa_1).$$

Therefore, from (2.3) and (3.5), we get

$$\lambda C_i\left[-\kappa_1 + h_0 - (n-r-1)\frac{g(\kappa_1)}{f(\kappa_1)}\right] = 0, \quad i = 1, \ldots, q, \quad \text{on } U, \tag{3.7}$$

$$\lambda C_i\left[-\kappa_2 + h_0 - (n-r-1)\frac{g(\kappa_1)}{f(\kappa_1)}\right] = 0, \quad i = q+1, \ldots, n, \quad \text{on } U. \tag{3.8}$$

Note that if $\lambda = 0$, then we have $L_r G = 0$, and it implies that the $(r+1)$-st mean curvature is constant on U, which is a contradiction. If $\kappa_1 = h_0 - (n-r-1)\frac{g(\kappa_1)}{f(\kappa_1)}$ or $\kappa_2 = h_0 - (n-r-1)\frac{g(\kappa_1)}{f(\kappa_1)}$, we conclude that H_{r+1} is constant on U, which is a contradiction. Therefore, $C_1 = \cdots = C_n = 0$ on U. Thus, (3.5) implies that H_{r+1} is constant on U, which is a contradiction. Hence, U is empty and H_{r+1} is constant on M. Since M has a constant mean curvature, we get that M is isoparametric, and therefore Remark 2.4 gives the result. □

In particular, when $n = 2$, we obtain the following corollary that was proved by Kim and Turgay in [17].

Corollary 3.17. *Let M be a surface with constant mean curvature in $\mathbb{E}^3$. Then M has an L_1-(global) 1-type Gauss map if and only if it is an open domain of a sphere, a plane or a cylinder.*

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Гіперповерхні з L_r-точковим типу 1 гауссовим відображенням

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У статті вивчаються гіперповерхні в $\mathbb{E}^{n+1}$, гауссове відображення G яких задовольняє рівняння $L_rG = f(G + C)$ для гладкої функції f і постійного вектора C, де L_r є лінеаризованим оператором $(r + 1)$-ої середньої кривизни гіперповерхні, тобто $L_r(f) = \mathrm{Tr}(P_r \circ \nabla^2 f)$ для $f \in \mathcal{C}^\infty(M)$, а P_r є r-им перетворенням Ньютона, $\nabla^2 f$ є гессіаном f, $L_rG = (L_rG_1, \ldots, L_rG_{n+1})$ і $G = (G_1, \ldots, G_{n+1})$. Наша увага зосереджена на гіперповерхнях з постійною $(r+1)$-ою середньою кривизною і постійною середньою кривизною. Для цих класів гіперповерхонь отримано теореми класифікації і характеризації.

Ключові слова: лінеаризовані оператори L_r, L_r-точкове типу 1 гауссове відображення, r-мінімальна гіперповерхня.