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Foliations of Codimension One and the Milnor Conjecture

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We prove that a fundamental group of leaves of a codimension one C^2-foliation with nonnegative Ricci curvature on a closed Riemannian manifold is finitely generated and almost Abelian, i.e., it contains finitely generated Abelian subgroup of finite index. In particular, we confirm the Milnor conjecture for manifolds which are leaves of a codimension one foliation with nonnegative Ricci curvature on a closed Riemannian manifold.

Key words: codimension one foliation, fundamental group, holonomy, Ricci curvature.

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1. Introduction

In 1963, Bishop proved the following theorem [2].

Bishop's Theorem. *A complete manifold with nonnegative Ricci curvature has a polynomial volume growth of balls.*

In 1968, Milnor showed that a fundamental group of a complete manifold with nonnegative Ricci curvature also has a polynomial growth (in the words metric) and stated the following conjecture [8].

Milnor's Conjecture. *The fundamental group of any complete Riemannian manifold with nonnegative Ricci curvature is finitely generated.*

In this paper, we confirm the Milnor conjecture for leaves of codimension one foliations with nonnegative Ricci curvature and prove the following theorem.

Main Theorem. *Let L be a leaf of a codimension one foliation $\mathcal{F}$ with nonnegative Ricci curvature on a closed manifold M. Then $\pi_1(L)$ is a finitely generated almost abelian group. In particular, it satisfies the Milnor conjecture.*

Remark 1.1. The Main theorem obviously holds for compact leaves (see Theorem 1.5 bellow). For noncompact leaves, the structure of foliations with nonnegative Ricci curvature is successfully used.

Remark 1.2. The Main theorem can not be strengthened up to Abelian groups. Indeed, consider the standard Reeb foliation $\mathcal{F}_{\mathcal{R}} = \{\mathcal{L}_\alpha\}$ on the round sphere S^3 (see [14]). It is well known that the leaves L_α in the induced metric have a nonnegative curvature. The Riemannian product S^3 with a closed nonnegative Ricci manifold M^n having non-Abelian fundamental group gives us a foliation $\mathcal{G} = \{L_\alpha \times M^n\}$ of codimension one with nonnegative Ricci curvature on $S^3 \times M^n$ with leaves having non-Abelian fundamental group.

In the proof, we essentially use the properties of almost without holonomy foliations and geometrical properties of the complete Riemannian manifolds with nonnegative Ricci curvature well studied by many famous mathematicians. Thus, in 1972, J. Cheeger and D. Gromoll generalized Toponogov's splitting theorem to the case of complete Riemannian manifolds with nonnegative Ricci curvature and obtained the following results.

Theorem 1.3 (Splitting Theorem [5]). *Assume that M is a complete Riemannian manifold with Ricci curvature $\mathrm{Ric}(M) \geq 0$ which has a stright line, i.e., a geodesic γ such that $d(\gamma(u), \gamma(v)) = |u - v|$ for all $u, v \in \mathbb{R}$. Then M is isometric to a Riemannian product space $N \times \mathbb{R}$, where N is a complete Riemannian manifold with $\mathrm{Ric}(N) \geq 0$.*

Theorem 1.4 ([5]). *Let M^n be a complete manifold with nonnegative Ricci curvature. Then:*

1. *M^n has at most two ends;*
2. *M^n is isometric to the Riemannian product $N \times E^k$ of the manifold N and Euclidian factor E^k, where N does not contain straight lines.*
3. *If M^n is closed, then its universal covering $\widetilde{M}^n$ is isometric to the Riemannian product $P \times E^k$, where P is compact and simply connected. Furthermore, the following extension holds:*

$$1 \to E \to \pi_1(M^n) \to \Gamma \to 1, \tag{1.1}$$

where E is a finite group and Γ is a crystallographic group.

Part 3 of Theorem 1.4 was generalized in 2000 by B. Wilking in the following theorem.

Theorem 1.5 ([15]). *Let M^n be a complete manifold with nonnegative Ricci curvature and $q : N \times E^l \to M^n$ be a regular isometric covering, where N has a compact isometry group (it holds in particular when N is closed). Then there exists a finitely sheeted covering $N \times T^p \times E^{l-p} \to M^n$. Moreover, this covering can be isometric for some deformed Riemanniam metric on M^n. If M^n is closed, then (1.1) is equivalent to the existence of the extension*

$$0 \to \mathbb{Z}^p \to \pi_1(M^n) \to F \to 1, \tag{1.2}$$

where F is a finite group.

2. Foliations

Let us recall the notion of a foliation defined on an n-dimensional manifold M. We say that a family $\mathcal{F} = \{F_\alpha\}$ of path-wises connected subsets (leaves) of M defines a foliation of dimension p (or codimension q, where $p+q=n$) on M if

- $\mathcal{F}$ is a partition of M, i.e., $M = \coprod\limits_{\alpha} F_\alpha$.
- There is a *foliated atlas* $\mathcal{U} = \{(U_\lambda, \varphi_\lambda)\}_{\lambda\in\Lambda}$ on M. This means that each connective component of a leaf in the foliated chart with the coordinates $(x_1, \ldots, x_p, y_1, \ldots, y_q)$ has the form of a plane

$$y_1 = \text{const}, \ldots, y_q = \text{const},$$

and the transition maps

$$g_{ij} = \varphi_i \circ \varphi_j^{-1} : \varphi_j(U_i \cap U_j) \to \varphi_i(U_i \cap U_j)$$

have the form

$$g_{ij}(x, y) = (\hat{g}_{ij}(x, y), \overline{g}_{ij}(y)), \tag{2.1}$$

where $x \in \mathbb{R}^p, y \in \mathbb{R}^q$.

The atlas $\mathcal{U} = \{(U_\lambda, \varphi_\lambda)\}_{\lambda\in\Lambda}$ is supposed to be at least C^2-smooth and good. The latter means that:

1) $\mathcal{U}$ is locally finite;
2) U_λ is relatively compact in M, and $\varphi_\lambda(U_\lambda) = (-1, 1)^n \subset \mathbb{R}^n$;
3) $\overline{U_i \cup U_j} \subset W_{ij}$, where (W_{ij}, ψ_{ij}) is a foliated chart not necessarily belonging to $\mathcal{U}$.

Let $\pi : (-1, 1)^n \to (-1, 1)^q$ be a natural projection to the last q coordinates.

The preimage $P_\lambda := \varphi_\lambda^{-1}(\pi^{-1}(x))$ is called a local leaf. Denote the space of local leaves by Q_λ. Clearly, $Q_\lambda \simeq (-1, 1)^q$, and

$$U_\lambda = \bigcup_{x\in(-1,1)^q} \varphi_\lambda^{-1}(\pi^{-1}(x)).$$

A foliation $\mathcal{F}$ is said to be oriented if the tangent to $\mathcal{F}$ p-dimensional distribution $T^{\mathcal{F}}M \subset TM$ is oriented, and $\mathcal{F}$ is said to be transversely oriented if some transversal to $\mathcal{F}$ distribution of dimension $q = n - p$ is oriented. If the manifold M is supposed to be Riemannian, then the transverse orientability of $\mathcal{F}$ is equivalent to the transverse orientability of orthogonal distribution $T^{\mathcal{F}^\perp}M$.

3. Holonomy

We recall the notion of *holonomy*.

Let $l : [0, 1] \to L$ be a closed path in the leaf $L \in \mathcal{F}$. It was shown in [14] that there exists a chain of foliated charts $\mathcal{C} = \{U_0, \ldots, U_{n-1}, U_n = U_0\}$, which cover $l([0, 1])$ such that:

a) there exists a division of the segment $[0,1] : 0 = t_0 < t_1 < \cdots < t_n = 1$ such that $l([t_i, t_{i+1}]) \subset U_i,\ i = 0, \ldots, n-1$;

b) if the intersection $P_i \cap U_{i+1} \neq \emptyset$, then it is connective, which means that the local leaf P_{i+1} is correctly defined.

It can be shown that a set of points $z \in U_0$, which correctly define the chain $\mathcal{C}$ from the initial condition $z \in P_0(z)$ is open in U_0. Thus, there exists some neighborhood O of the P_0 consisting of local leaves for which the local diffeomorphism $\Gamma_l : V_0 \to (-1,1)^q$ of some neighborhood of zero $V_0 \subset (-1,1)^q$ to $(-1,1)^q$ is well defined as follows:

$$\Gamma_l(\pi \circ \varphi_0(P_0(z)) = \pi \circ \varphi_0(P_n(z)).$$

In [14], it was shown that the local diffeomorphisms $\{\Gamma_l : V_0 \to (-1,1)^q\}$ define the homomorphism

$$\Psi : \pi_1(L) \to G_q$$

of the fundamental group $\pi_1(L)$ to the group of diffeomorphism germs G_q in the origin $0 \in \mathbb{R}^q$. The homomorphism Ψ is defined up to inner automorphisms and it is called a *holonomy* of L. Its image is called a *group of holonomy* of L and denoted by $H(L)$.

Notice that if a foliation of codimension one is transversely oriented, then the one-sided holonomy

$$\Psi^+ : \pi_1(L) \to G_1^+$$

of the leaf L is also well defined, where G_1^+ denotes the group of germs of one-sided diffeomorphisms at 0 with a domain on the half-intervals $[0, \varepsilon)$.

Recall the following important results obtained on the holonomy of leaves.

Theorem 3.1 ([12]). *Let L be a leaf of a C^2-foliation of codimension one. If $H(L)$ has a polynomial growth, then $H(L)$ is a torsion free Abelian group.*

Nishimori proved the next theorem which describes the behavior of a codimension one foliation in the neighborhood of a compact leaf with Abelian holonomy.

Theorem 3.2 ([9]). *Let $\mathcal{F}$ be a transversely oriented C^r-foliation of codimension one on the oriented n -dimensional manifold M and F_0 be a compact leaf of $\mathcal{F}$. Suppose that $2 \leq r \leq \infty$. Let T be a tubular neighborhood of F_0 and U_+ be a union of F_0 and a connected component $T \backslash F_0$. Suppose that $H(F_0)$ is an Abelian group. Then only one of the three cases holds.*

1. *For any neighborhood V of F_0, the restricted foliation $\mathcal{F}|_{V \bigcap U_+}$ has a compact leaf which is not F_0.*

2. *There is a neighborhood V of F_0 such that all leaves $\mathcal{F}|_{V \bigcap U_+}$ except F_0 are dense in $V \bigcap U_+$. In this case, $H(F_0)$ is a free Abelian group of rank ≥ 2.*

3. *There is a neighborhood V of F_0 and a connected oriented submanifold N of codimension one in F_0 with the following properties. Denote by F_* a compact*

manifold with boundary obtained by attaching two copies N_1 and N_2 of N to $F_0\backslash N$ satisfying $\partial F_* = N_1 \bigcup N_2$. Let $f : [0,\varepsilon) \to [0,\delta)$ be a contracting C^r-diffeomorphism such that $f(0) = 0$. Denote by X_f a manifold obtained from $F_* \times [0,\varepsilon)$ by identifying $(x,t) \in N_1 \times [0,\varepsilon)$ and $(x, f(t)) \in N_2 \times [0,\delta)$. After factorization, we obtain the foliation F_f on X_f. It is claimed that for some f as above, there is a C^r-diffeomorphism $h : V \bigcap U_+ \to X_f$ which maps each leaf of $\mathcal{F}|_{V \bigcap U_+}$ onto some leaf of F_f. The foliation $\mathcal{F}|_{V\bigcap U_+}$ uniquely defines the homology class $[N] \in H_{n-2}(F_0, \mathbb{Z})$, and the germ at zero of the map f is unique up to conjugation. In this case, $H(F_0)$ is an infinite cyclic group.

A foliation is said to be *a foliation without holonomy* if the holonomy of each leaf is trivial, and it is said to be a *foliation almost without holonomy* if the holonomy of noncompact leaves is trivial. For example, the Reeb foliation $\mathcal{F}_R$ is a foliation almost without holonomy on S^3 since all leaves of $\mathcal{F}_R$, except a single compact leaf homeomorphic to torus, are homeomorphic to $\mathbb{R}^2$ and thus have a trivial fundamental group.

Let us call by *block* a compact saturated subset B of codimension one foliated n-dimensional manifold which is an n-dimensional submanifold with a boundary. Recall that *a saturated set* of the foliation $\mathcal{F}$ on a manifold M is called a subset of M which is a union of leaves of $\mathcal{F}$. Clearly that ∂B is a finite union of compact leaves.

The following theorem is a reformulation of the results of Novikov [10] and Imanishi [7] obtained for foliations without holonomy and for foliations almost without holonomy, respectively.

Theorem 3.3. *Let L be a noncompact leaf of a codimension one foliation $\mathcal{F}$ almost without holonomy on a closed n-dimensional manifold M. Then one of the following holds:*

a) *$\mathcal{F}$ is a foliation without holonomy whose all leaves are diffeomorphic to the typical leaf L and dense in M. We have the group extension*

$$1 \to \pi_1(L) \to \pi_1(M) \to \mathbb{Z}^k \to 0, \tag{3.1}$$

where $k > 0$ and $k = 1$ iff the foliation $\mathcal{F}$ is a locally trivial fibration over the circle.

The universal covering $\widetilde{M}$ has the form

$$\widetilde{M} \cong \widetilde{L} \times \mathbb{R}.$$

b) *L belongs to some block B whose all leaves in the interior $\mathring{B}$ are diffeomorphic to the typical leaf L and are either dense in $\mathring{B}$ (B is called a dense block in this case) or proper in $\mathring{B}$ (B is called a proper block in this case). We have the group extension*

$$1 \to \pi_1(L) \to \pi_1(B) \to \mathbb{Z}^k \to 0, \tag{3.2}$$

where $k > 0$ and $k = 1$ iff B is proper and the foliation in $\mathring{B}$ is a locally trivial fibration over the circle.

The universal covering of $\mathring{B}$ has the form

$$\widetilde{\mathring{B}} \cong \widetilde{L} \times \mathbb{R}. \tag{3.3}$$

Let us call the block from b) an *elementary block*.

4. Growth of leaves

A *minimal set* of the foliation $\mathcal{F}$ is a closed saturated set which has no other closed saturated sets.

The following Plant's theorem describes minimal sets of foliations of codimension one with leaves of subexponential growth. The growth means the volume growth of balls $B_x(R) \subset L_x \in \mathcal{F}$ as a function of the radius R.

Theorem 4.1 ([11]). *Assume that a C^2-foliation of codimension one on a compact manifold has the leaves of subexponential growth. Then each minimal set of the foliation is either the whole manifold or a compact leaf.*

We say that $\mathcal{F}$ is a *foliation with nonnegative Ricci curvature* on a Riemannian manifold if each leaf of $\mathcal{F}$ has a nonnegative Ricci curvature in the induced metric.

We obtain the following corollary.

Corollary 4.2. *One of the following holds:*

1. *All leaves of a C^2-foliation of codimension one with nonnegative Ricci curvature are dense.*

2. *The closure of each leaf contains a compact leaf.*

A leaf L of a transversely oriented foliation $\mathcal{F}$ of codimension one is called *resilient* if there is a transversal arc $[x, y)$, $x \in L$, and a loop σ such that $\Gamma_\sigma : [x, y) \to [x, y)$ is the contraction to x, and $L \cap (x, y) \neq \emptyset$.

It turns out that the theorem below holds.

Theorem 4.3 ([6]). *Let M be a compact manifold and $\mathcal{F}$ be a C^2-foliation of codimension one on M. Then a resilient leaf of $\mathcal{F}$ must have exponential growth.*

Corollary 4.4. *A C^2-foliation of codimension one with nonnegative Ricci curvature on a compact manifold has no resilient leaves.*

5. Mappings into foliations

Let us recall the definition of an exponential map along the leaf of a foliation $\mathcal{F}$ on the Riemannian manifold M.

Each vector a tangent to a leaf L_x at the point $x \in M$ is mapped to the end of the geodesic of L_x which has the length $|a|$ and is directed to a at the initial point x. Since the foliation is supposed to be smooth, the constructed exponential map $\exp^{\mathcal{F}} : T^{\mathcal{F}}M \to M$ must also be smooth. By analogy, $\exp^{\perp}$ denotes the orthogonal exponential map where each orthogonal vector p at x is mapped to the end of the orthogonal to $\mathcal{F}$ trajectory of the length $|p|$ which is directed to p at the initial point x. Let us consider the composition of continuous maps

$$F : I \times D^{n-1}(R) \xrightarrow{i} V(J,R) \xrightarrow{\exp^{\mathcal{F}}} M, \tag{5.1}$$

where $I = [0,1]$, $D^{n-1}(R)$ is a Euclidean ball of radius R and $i : I \times D^{n-1}(R) \to V(J,R) \subset T^{\mathcal{F}}M$ is a homeomorphism on the set $V(J,R) = \{(x,v_x) : x \in J = i(I \times 0), v_x \in T_x^{\mathcal{F}}M, |v_x| \le R\}$. The interval $J = i(I \times 0)$ is embedded and orthogonal to $\mathcal{F}$. For each $x \in \mathring{J}$, call the set $F(I, D^{n-1}(R))$ a $V(J,R)$-neighborhood of x.

Remark 5.1. Since the length of J and the real number R can be taken arbitrarily small, for any point x there exists J containing x and a $V(J,\varepsilon)$-neighborhood W of x which is inside of a foliated chart (U_α, ϕ_α) of the foliation. The restriction of ϕ_α on $U'_\alpha \subset W$ gives us a foliated chart $(U'_\alpha, \phi'_\alpha)$ with diameters of local leaves less than ε, where U'_α is a neighborhood of x homeomorphic to $(-1,1)^n$.

Proposition 5.2. *Let $F : I \times D^{n-1}(R) \to M$ as in (5.1). Then there exists $\delta > 0$ such that for each $0 \le t < \delta$ the balls $B_t(R) := \exp^{\mathcal{F}} \circ i(t \times D^{n-1}(R))$ belong to an arbitrary ε-collar of the ball $B_0(R+\varepsilon) \supset B_0(R)$. By the ε-collar of $B_0(R+\varepsilon)$, we understand the set $\Delta = \exp^{\perp}(a)$, $|a| < \varepsilon$, where a are normal to $B_0(R+\varepsilon)$ vectors directed toward a half-space defined by $J = F(I \times 0)$, and ε is small enough for the deformation retraction $\mathrm{pr} : \Delta \to B_0(R+\varepsilon)$ to be well defined. Here pr maps the point $x \in \Delta$ to the initial point of the orthogonal trajectory.*

Proof. The proof follows from the fact that $I \times D^{n-1}(R)$ is compact for each $R \in \mathbb{R}_+$ and F is uniformly continuous. □

Recall also the definition by S. Adams and G. Stuck.

Definition 5.3 ([1])**.** Let $\mathcal{F}$ be a foliation of a Riemannian manifold M. Let X be a connected locally compact Hausdorff space. Define $C_{\mathcal{F}}(X,M)$ to be the space of continuous maps of X into the leaves of $\mathcal{F}$. Let us consider $C_{\mathcal{F}}$ (X,M) as a subspace of the space $C(X,M)$ of continuous maps of X into M, where $C(X,M)$ is endowed with the topology of uniform convergence on compact sets.

The following important result was obtained.

Theorem 5.4 ([1])**.** *Let M, $\mathcal{F}$ and X be as in Definition (5.3). Then $C_{\mathcal{F}}(X,M)$ is a closed subspace of $C(X,M)$.*

6. Foliations with nonnegative Ricci curvature

Proposition 6.1. *Let $\mathcal{F}$ be a transversely oriented C^2-foliation of codimension one with nonnegative Ricci curvature on a closed oriented manifold M. Then $\mathcal{F}$ is a foliation almost without holonomy.*

Proof. As noted at the beginning of the article, Milnor proved that a fundamental group of a compact manifold with nonnegative Ricci curvature has a polynomial growth in the words metric. Therefore, by Theorem 3.1, each compact leaf has an Abelian holonomy and thus Theorem 3.2 can be applied. Suppose that there exists a noncompact leaf L with nontrivial holonomy. Since $\mathcal{F}$ is supposed to be transversely oriented, the holonomy mast be infinite. Let γ be a closed path in L which represents a nontrivial holonomy element of $\pi_1(L)$. Consider two cases of holonomy mappings.

Case 1. The one-sided holonomy $\Psi^+([\gamma])$ is nontrivial and the holonomy map $\Gamma_\gamma : [0,\varepsilon) \to [0,\varepsilon')$ does not have any fixed points in $[0,\varepsilon)$ for some ε.

In this case, $\Psi^+(\pm[\gamma])$ is represented by a contracting map. If L is locally dense, then L must be a resilient leaf, which is impossible by Corollary 4.4. Otherwise, another noncompact leaf P winds around L and, by Corollary 4.2, there exists a compact leaf $K \subset \overline{L} \bigcap \overline{P}$. From part 3 of Theorem 3.2, it simply follows that the leaf P must have infinitely many ends, which contradicts to Theorem 1.4.

Case 2. The half-interval $[0,\varepsilon)$ has a sequence of fixed points $\{F_i\}$ of the holonomy map $\Gamma_\gamma : [0,\varepsilon) \to [0,\varepsilon')$ which converges to zero.

Since $\{F_i\} \bigcap [0,F_k]$ is closed in $[0,\varepsilon)$, where $F_k \in \{F_i\} \bigcap [0,\varepsilon)$, then either the holonomy of L must be trivial or we have to find a half-interval $[a,\delta) \subset [0,\varepsilon)$ on which $\Gamma_{\gamma'}$ is a contracting map. The leaf L' corresponding to the point $a \in [0,\varepsilon)$ has a contracting holonomy on $\gamma' \subset L'$, where γ' is a closed path corresponding to the fixed point a of the map Γ_γ. Since the set of compact leaves is closed (see [14]) and M is the normal topological space, we can choose ε small enough for the leaf L' to be noncompact. Thus, we have arrived to Case 1 and the proposition is proven. □

Corollary 6.2. *The structure of transversely oriented foliations of codimension one with nonnegative Ricci curvature on compact oriented manifolds is described by Theorem 3.3.*

Proposition 6.3. *Let $\mathcal{F}$ be a transversely oriented C^2-foliation of codimension one which is almost without holonomy (in particular, a foliation with nonnegative Ricci curvature) on the oriented Riemannian manifold M. Let $\{[x_i,y_i]\} \subset L_i \in \mathcal{F}$ be a sequence of the shortest (in L_i) geodesic segments of the length $l_i \to \infty$ and $\{z_i \in [x_i,y_i]\}$ be a sequence of points converging to some point $z \in L$, where L is a noncompact leaf of $\mathcal{F}$. Suppose that the lengths of the segments $\{[x_i,z_i]\}$ and $\{[z_i,y_i]\}$ approach to ∞. Then there is a subsequence of $\{[x_i,y_i]\}$ converging to a straight line $l \in L$, which passes through z.*

Proof. Starting from some $i > i_0$, we can replace the sequence of the segments $[x_i,y_i]$ by the sequence of the segments $[x_i^l,y_i^l] \subset [x_i,y_i]$ of the length l such that z_i is the midpoint of $[x_i^l,y_i^l]$. We have $\rho_{L_i}(x_i,y_i) \geq \rho_M(x_i,y_i)$, where ρ_{L_i} denotes a Riemannian metric on the leaf L_i induced by the Riemannian metric ρ_M on M. By Arzela–Ascoli theorem (see [3, Theorem 2.5.14]) and Theorem 5.4, we can see that the sequence $\{[x_i^l, y_i^l]\}$ converges uniformly to the rectifiable curve $r_l \subset L$ passing though z with the endpoints x^l and y^l in L. Show that r_l is the shortest geodesic segment $[x^l, y^l]$. Indeed, if it is not, then consider the loop $r_l \cup [x^l, y^l]$ and a smooth transversal to $\mathcal{F}$ embedding of the square $[x^l, y^l] \times I \subset M$ such that $[x_t^l, y_t^l] := [x^l, y^l] \times t \subset L_t \in \mathcal{F}$ and $[x_0^l, y_0^l] = [x^l, y^l]$. The length l_t of $[x_t^l, y_t^l]$ continuously depends on t (recall that $\mathcal{F}$ is C^2-smooth and so is the induced foliation on $[x^l, y^l] \times I$) and there exists ε and δ such that for $0 \le t < \delta$ the inequality $l_t < l - 2\varepsilon$ holds. Since $x_i^l \to x^l$, we can assume that for $i > i_0$, the points $x_{t_i}^l$ and x_i^l belong to the same local leaf of a foliated chart (U_α, ϕ_α) with diameters of local leaves less than ε (see Remark 5.1). Therefore, $\rho_{L_i}(x_i^l, x_{t_i}^l) < \varepsilon$. We also assume, without loss of generality, that if $i > i_0$, then $t_i < \delta$ and $\lim_{i\to\infty} t_i = 0$. But $y_i^l \to y^l$, and starting from some $i \ge i_1 > i_0$ the sequence $\{y_i^l\}$ belongs to a foliated chart (U_β, ϕ_β) which contains y^l and the diameters of local leaves of (U_β, ϕ_β) are less than ε. If $y_{t_i}^l$ belong to the same local leaf as y_i^l, then this contradicts our assumption that the segments $[x_i^l, y_i^l]$ are the shortest segments:

$$l = \rho_{L_i}(x_i^l, y_i^l) \le \rho_{L_i}(x_i^l, x_{t_i}^l) + \rho_{L_i}(x_{t_i}^l, y_{t_i}^l) + \rho_{L_i}(y_i^l, y_{t_i}^l) = 2\varepsilon + l_{t_i} < l.$$

It means that for $i > i_1$, the points y_i^l and $y_{t_i}^l$ belong to different local leaves of (U_β, ϕ_β), and the loop $r_l \cup [x^l, y^l]$ represents a nontrivial holonomy of L. This is impossible since $\mathcal{F}$ is supposed to be a foliation almost without holonomy. This implies that $r_l = [x^l, y^l]$. So we have a sequence of the shortest geodesic segments $\{[x^{l_j}, y^{l_j}]\}$, which pass through the midpoint $z \in L$, that are limits of sequences of the shortest segments $\{[x_i^{l_j}, y_i^{l_j}]\} \subset L_i$, where $l_j \to \infty$. The sequence $\{[x^{l_j}, y^{l_j}]\}$ contains a subsequence converging to a straight line in L (see, for example, [3]). □

7. The main result

Definition 7.1. Let L be a noncompact leaf of the elementary block B and $K \in \partial B$ be a compact leaf. We call the curve $\gamma(t) \subset L$, $t \in [0, \infty)$ *outgoing* (to K) if for each $\varepsilon > 0$ there is t_0 such that $\gamma(t), t \in [t_0, \infty)$ is inside of the ε-collar of K. It is clear that $\overline{\gamma} \cap K \ne \emptyset$.

Definition 7.2. Let L be a noncompact leaf of the elementary block B and $K \in \partial B$ be a compact leaf. We say that an element $[\alpha] \in \pi_1(L, x_0)$ is not peripheral along the outgoing curve $\gamma \subset L$ (to K) if there is some ε-collar $U_\varepsilon K$ of K and there is no loop α_t with base point $x_t = \gamma(t)$ such that α_t is free homotopic to α and α_t belongs to $U_\varepsilon K$. Otherwise, we say that α is peripheral along γ.

Remark 7.3. The term "peripheral" was used in [4]. But our definition of peripheral elements of $\pi_1(L)$ along the outgoing curve γ is in some sense a foliated analogue of elements having geodesic loops to infinity property along a ray γ used in [13].

Let us call a dimension k of E^k in the splitting theorem, a *splitting dimension* of the complete manifold M with nonnegative Ricci curvature.

Now we turn to the proof of Main theorem.

Proof. Main theorem holds for compact leaves (see Remark 1.1). Suppose L is noncompact. Suppose also that $\mathcal{F}$ is transversely oriented and M is oriented. We have two cases.

Case 1. *$\mathcal{F}$ does not contain compact leaves.*

In this case, by Theorem 3.3 and Corollary 6.2, all leaves are dense and diffeomorphic. The result follows from the main theorem of [1], which claims that almost all leaves have a structure of a Riemannian product $S \times E^n$, where S is compact.

Case 2. *$\mathcal{F}$ contains compact leaves and L is a typical leaf of the elementary block B.*

Recall that we have the monomorphism $\pi_1(L) \to \pi_1(B)$ (see (3.2)). Fix a typical leaf L and consider the connective component of its universal covering $\widetilde{L} \subset \widetilde{\mathring{B}}$. Recall that the universal covering of $\mathring{B}$ has the form (3.3). According to Theorem 1.4,

$$\widetilde{L} \cong N \times E^k, \; k \geq 0, \tag{7.1}$$

where N does not contain straight lines. Let us show that each leaf $\widetilde{L}_{\widetilde{x}}$ passing though $\widetilde{x}$ of the pull back foliation $\widetilde{\mathcal{F}}$ of $\widetilde{\mathring{B}}$, which is a preimage of $L_x \in \overline{L} \in \mathring{B}$, splits with the same splitting dimension k as $\widetilde{L}$. Indeed, let $\widetilde{x} \in \widetilde{L}_{\widetilde{x}}$ be an arbitrary point. We can find a sequence $\widetilde{x}_i \in \widetilde{L}_{\widetilde{x}_i}$ such that $\lim_{i\to\infty} \widetilde{x}_i = \widetilde{x}$, where $\widetilde{L}_{\widetilde{x}_i}$ are the preimages of L passing through $\widetilde{x}_i \in \widetilde{\mathring{B}}$ with respect to the covering $p : \widetilde{\mathring{B}} \to \mathring{B}$. Notice that actually nontrivial is only the case of dense B. By (7.1), we can find a sequence of k orthogonal straight lines passing through $\widetilde{x}_i$, which converges to k orthogonal straight lines passing through $\widetilde{x}$. This follows from some modification of Proposition 6.3. Thus, we obtain that $\widetilde{L}_{\widetilde{x}} \cong N_x \times E^l$, $l \geq k$. By exchanging L and L_x, we obtain $l = k$. It is clear that the splitting directions $\mathbb{R}^k$ form a subdistribution in $T^{\mathcal{F}}M$ called a splitting distribution. Respectively, a distribution orthogonal to a splitting distribution in $T^{\mathcal{F}}M$ is called a horizontal distribution.

Suppose that N is not compact in the splitting $\widetilde{L} \cong N \times E^k$. Then there is a horizontal ray $\widetilde{\gamma} \in N \times \vec{0}$. Show that $p(\widetilde{\gamma}) =: \gamma$ is an outgoing curve. Suppose also that γ is not an outgoing curve. Then we can find a sequence of points $\gamma(t_i)$, $t_i \to \infty$, which converges to the point $x \in \mathring{B}$. It means that we can find a sequence of points $\widetilde{x}_i := \widetilde{\gamma}_i(t_i)$ which converges to the point $\widetilde{x}$ such that $p(\widetilde{x}) = x$. Here $\widetilde{\gamma}_i$ are the corresponding preimages of γ with respect to the covering map $p : \widetilde{\mathring{B}} \to \mathring{B}$. Since $t_i \to \infty$, we can find a sequence of the shortest horizontal segments $[a_i, b_i] \in \widetilde{\gamma}_i$ passing through $\widetilde{x}_i$ such that the lengths $[a_i, \widetilde{x}_i]$ and $[b_i, \widetilde{x}_i]$ tend to ∞. Thus, the sequence of the shortest horizontal segments contains a subsequence converging to a horizontal straight line passing through $\widetilde{x}$ by Proposition 6.3, which is a contradiction as $N_{\widetilde{x}}$ in the splitting $\widetilde{L}_{\widetilde{x}} \cong N_{\widetilde{x}} \times E^k$ does not contain straight lines. We can conclude that γ is outgoing.

Consider two possibilities.

a) *Each element* $[\alpha] \in \pi_1(L)$ *is peripheral along an outgoing to* $K \in \partial B$ *curve* γ.

Suppose $\gamma(t) \subset U_\varepsilon K$ for $t \in [t_0, \infty)$. In this case, in L, we can change each loop α with the base point to the homotopic loop $\alpha' \subset U_\varepsilon K$ with the same base point. Since K is a deformation retract of $U_\varepsilon K$, we have that $i_*(\pi_1(L))$ is isomorphic to a subgroup of $i_*(\pi_1(K))$, where i_* denotes in both cases homomorphisms induced by the inclusions $i : L \to B$ and $i : K \to B$. But a subgroup and an image of a finitely generated almost Abelian group is finitely generated almost Abelian, thus the result follows from (1.2) and Theorem 1.4.

b) *There is a nonperipheral element* $[\alpha] \in \pi_1(L)$ *along* γ.

From the definition, it follows that there is a countable family of minimal length geodesic loops $\{\alpha_i\}$ with the base points $x_i = \gamma(t_i)$, $t_i \to \infty$, which is free homotopic to the loop α, and the sequence $\{z_i \in \alpha_i\}$ such that $z_i \in B \setminus U_\varepsilon K$ for some ε-collar $U_\varepsilon K$ of $K \in \partial B$. Let us suppose that the sequence $\{z_i\}$ converges to $z \in B \setminus U_\varepsilon K$. Since γ is an outgoing curve, we can choose a subsequence $\{x_j\}$ in $\{x_i\}$ converging to $x \in K$ to get $\rho(x_j, z_j) \to \infty$. Indeed, by Proposition 5.2, for an arbitrary fixed $R > \operatorname{diam} K$, there exists $\delta > 0$ such that for each $0 \le t < \delta$, the balls $B_t(R) := \exp^{\mathcal{F}} \circ i(t \times D^{n-1}(R))$ belong to the ω-collar $U_\omega K$ $(\omega < \varepsilon)$, where $i(0,0) = x$. Since $x_j \to x$, there exists i_0 such that $x_j \in B_t(\sigma) \subset B_t(R)$ for $j > i_0$ and a small enough $\sigma > 0$ (see Remark 5.1) such that $\rho_{L_i}(x_j, z_j) \ge R - \sigma$. As R is taken arbitrarily, we get that $\rho(x_j, z_j) \to \infty$. It means that we have a sequence of the shortest geodesic segments $[\widetilde{x}_j, \widetilde{y}_j] \subset \widetilde{L} \subset \widetilde{\mathring{B}}$ of the length $l_j \to \infty$ with the ends $\widetilde{x}_j \in \widetilde{\gamma} \subset \widetilde{L}$ and $\widetilde{y}_j \in \widetilde{\gamma}_\alpha := \Theta([\alpha])(\widetilde{\gamma}) \subset \widetilde{L}$ such that $p(\widetilde{x}_j) = p(\widetilde{y}_j) = x_j$, where Θ defines an isometric action of $\pi_1(B)$ on $\widetilde{\mathring{B}}$.

Let $\widetilde{z}_j \in [\widetilde{x}_j, \widetilde{y}_j]$ be such that $p(\widetilde{z}_j) = z_j$. Since $\rho(x_j, z_j) \to \infty$, we have $\rho(\widetilde{x}_j, \widetilde{z}_j) \to \infty$ and $\rho(\widetilde{y}_j, \widetilde{z}_j) \to \infty$. Choose such $g_j \in \pi_1(B)$ that $\lim_{j\to\infty} \Theta(g_j)\widetilde{z}_j \to \widetilde{z} \in \widetilde{\mathring{B}}$ and $p(\widetilde{z}) = z$. Pay attention to that $[\widetilde{x}_j, \widetilde{y}_j]$ do not need to be horizontal since $\widetilde{\gamma}_\alpha$ does not need to belong to $N \times \vec{0}$. However, $[\widetilde{x}_j, \widetilde{y}_j]$ belong to $N \times \vec{a}$ for some $\vec{a} \in \mathbb{R}^k$. Substitute the ray $\widetilde{\gamma}$ with the ray $\widetilde{\gamma}' := \widetilde{\gamma} + \vec{a}$ and the points $\widetilde{x}_j$ with $\widetilde{x}'_j := \widetilde{x}_j + \vec{a}$. Clearly that the shortest geodesic segments $[\widetilde{x}'_j, \widetilde{y}_j]$ are the projections of $[\widetilde{x}_j, \widetilde{y}_j]$ along $\vec{a}$. Let $z'_j \in [\widetilde{x}'_j, \widetilde{y}_j]$ be the images of z_j with respect to the projections. We can see that $[x_j, x'_j, z_j]$ is a right triangle. Since $\rho(x_j, x'_j) = |a|$, we have that $\lim_{j\to\infty} \rho(x'_j, y_j) = \infty$ and $\rho(z_j, z'_j) < |a|$. Proposition 6.3 immediately implies that a sequence of the shortest horizontal geodesic segments $\Theta(g_j)[\widetilde{x}'_j, \widetilde{y}_j]$ has a subsequence converging to the horizontal straight line, which is impossible.

If N in the splitting $\widetilde{L} \cong N \times \mathbb{R}^k$ is compact, then the result follows from Theorem 1.5.

If M is not oriented and (or) $\mathcal{F}$ is not transversely oriented, we can pass to the finitely sheeted oriented isometric covering $p : \overline{M} \to M$ such that the pull-back foliation $\overline{\mathcal{F}}$ is transversely oriented for which the result has been proven.

Each leaf $L \in \mathcal{F}$ is finitely covered by some leaf $\overline{L} \in \overline{\mathcal{F}}$. And the expected result follows from the fact that $p_* : \pi_1(\overline{L}) \to \pi_1(L)$ is a monomorphism and $\pi_1(\overline{L})$ is isomorphic to some subgroup of finite index in $\pi_1(L)$. □

Corollary 7.4. *The leaves of a foliation of codimension one with nonnegative Ricci curvature on a closed Riemannian manifold satisfy the Milnor conjecture.*

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Шарування ковимірності один та гіпотеза Мілнора

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Ми доводимо, що фундаментальна група шарів C^2-шарування ковимірності один невід'ємної кривини Річчі замкнутого ріманова многовиду є скінченно породженою та майже абелевою, тобто містить скінченно породжену абелеву підгрупу скінченного індексу. Зокрема, ми підтверджуємо гіпотезу Мілнора щодо многовидів, які є шарами шарування ковимірності один невід'ємної кривини Річчі замкнутого ріманова многовиду.

Ключові слова: шарування ковимірності один, фундаментальна група, голономія, кривина Річчі.