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Reachability and Controllability Problems for the Heat Equation on a Half-Axis

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In the paper, problems of controllability, approximate controllability, reachability and approximate reachability are studied for the control system $w_t = w_{xx}$, $w(0, \cdot) = u$, $x > 0$, $t \in (0, T)$, where $u \in L^\infty(0, T)$ is a control. It is proved that each end state of this system is approximately reachable in a given time T, and each its initial state is approximately controllable in a given time T. A necessary and sufficient condition for reachability in a given time T is obtained in terms of solvability a Markov power moment problem. It is also shown that there is no initial state that is null-controllable in a given time T. The results are illustrated by examples.

Key words: heat equation, controllability, approximate controllability, reachability, approximate reachability, Markov power moment problem.

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1. Introduction

Consider the heat equation on a half-axis

$$w_t = w_{xx}, \qquad x \in (0, +\infty),\ t \in (0, T), \tag{1.1}$$

controlled by the boundary condition

$$w(0, \cdot) = u, \qquad t \in (0, T), \tag{1.2}$$

under the initial condition

$$w(\cdot, 0) = w^0, \qquad x \in (0, +\infty), \tag{1.3}$$

and the steering condition

$$w(\cdot, T) = w^T, \qquad x \in (0, +\infty), \tag{1.4}$$

where $T > 0$, $u \in L^\infty(0, T)$ is a control, $\left(\frac{d}{dt}\right)^m w : [0, T] \to H_{⓪}^{-2m}$, $m = 0, 1$, $w^0, w^T \in H_{⓪}^0 = L^2(0, +\infty)$. Here, for $m = 0, 1, 2$,

$$H_{⓪}^m = \left\{\varphi \in L^2(0, +\infty) \mid \left(\forall k = \overline{0, m}\ \varphi^{(k)} \in L^2(0, +\infty)\right)\right.$$

$$\wedge \left(\forall k=\overline{0,m-1}\ \varphi^{(k)}(0^+)=0\right)\Big\}$$

with the norm

$$\|\varphi\|_{⓪}^{m}=\sqrt{\sum_{k=0}^{m}\binom{m}{k}\left(\|\varphi^{(k)}\|_{L^2(0,+\infty)}\right)^2},$$

and $H_{⓪}^{-m}=\left(H_{⓪}^{m}\right)^*$ with the strong norm $\|\cdot\|_{⓪}^{-m}$ of the adjoint space. We have $L^2(0,+\infty)=\left(H_{⓪}^{0}\right)^*=H_{⓪}^{-0}$.

In the paper, we study reachability and controllability problems for the heat equation on a half-axis. Note that these problems for the heat equation on domains bounded with respect to spatial variables were investigated rather completely in a number of papers (see, e.g., [3, 10, 12] and references therein). However, controlability problems for the heat equation on domains unbounded with respect to spatial variables have not been fully studied. The problems for this equation were studied in [1,2,8,9,11]. In particular, in [9], the null-controllability problem for control system (1.1)–(1.3) with L^2-control ($u\in L^2(0,T)$) was studied in a weighted Sobolev space of negative order. Using similarity variables and developing the solutions in the Fourier series with respect to the orthonormal basis $\{\phi_m\}_{m=1}^{\infty}$, the authors reduced the control problem to a moment problem

$$\int_0^S e^{ms}\widetilde{u}(s)\,ds=\alpha_m,\quad m=\overline{1,\infty},$$

where $\phi_m(y)=C_m\mathcal{H}_{2m-1}(y/2)e^{-y^2/4}$, $\mathcal{H}_{2m-1}$ is the Hermit polynomial, α_m is determined by the Fourier coefficient of the initial state of reduced control problem, $m=\overline{1,\infty}$. The solution to the moment problem determines a solution to the control problem and vice versa. The authors proved that the moment problem admits an L^2-solution iff α_m grows exponentially as $m\to\infty$. In particular, they proved that if $\alpha_m=O(e^{m\delta})$ as $m\to\infty$ for all $\delta>0$, then the initial state associated with $\{\alpha_m\}_{m=1}^{\infty}$ cannot be steered to the origin by L^2-control. In [9], it was also asserted that each initial state is approximately null-controllable in a given time $T>0$ by L^2-controls.

In the present paper, we study control system (1.1)–(1.3) in $H_{⓪}^{0}=L^2(0,+\infty)$ with L^∞-control ($u\in L^\infty(0,T)$). Note that L^∞-controls allow us to consider initial states and solutions of the control system in the Sobolev space of order zero in contrast to [9], where the system was studied in a weighted Sobolev space of negative order as a result of using L^2-controls. In Section 3, considering the odd extension with respect to x of the initial state and the solution to (1.1)–(1.3), we reduce this system to control system (3.1), (3.2) in spaces $\widetilde{H}^m$ of all odd functions of H^m. Further, control system (3.1), (3.2) is considered instead of control system (1.1)–(1.3). In Section 4, we obtain the necessary and sufficient condition for an end state W^T to be reachable from the origin by using controls $u\in L^\infty(0,T)$ bounded by a given constant $L>0$. Next, the reachability problem is reduced to an infinite Markov power moment problem (Theorem 4.4). Moreover, it is proved that the solutions to the finite Markov power moment problem give us controls bounded by L and solving the approximate reachability problem (Theorem 4.5). The result of this theorem is illustrated by Examples 8.1 and 8.2 in Section 8. In Section 5, we prove that each end state $W^T \in \widetilde{H}^0$ is approximately reachable from the origin, using controls $u \in L^\infty(0,T)$, in a given time $T > 0$ (Theorem 5.2). To prove this theorem, we develop W^T in Fourier series with respect to $\{\psi_n^T\}_{n=0}^\infty$, $\psi_n^T(x) = \mathcal{H}_{2n+1}(x/\sqrt{2T})e^{-x^2/(4T)}$, $n = \overline{0,\infty}$. First, for each $n = \overline{0,\infty}$, we find a sequence of controls $\{u_l^n\}_{l=0}^\infty$ that solves the approximate reachability problem for the end state ψ_n^T. We use the Fourier transform with respect to x and find these controls from the relation

$$\left(\mathcal{F}\psi_n^T\right)(\sigma) = (-1)^{n+1} i\sqrt{2T}\mathcal{H}_{2n+1}(\sqrt{2T}\sigma)e^{-T\sigma^2} = -\sqrt{\frac{2}{\pi}} i\sigma \int_0^T e^{-\xi\sigma^2} u(T-\xi)\, d\xi.$$

Note that $u_l^n \to \delta^{(n)}$ as $l \to \infty$ in $\mathcal{D}'$ for each $n = \overline{0,\infty}$ (δ is the Dirac distribution). Then we find the controls u_N, $N \in \mathbb{N}$ solving the approximate reachability problem

$$u_N = \sum_p^N U_p^N u_{l_p^N}^p,$$

where $U_p^N \geq 0$ is a constant, $p = \overline{0,N}$. The results of this section are illustrated by Example 8.3 in Section 8. In Section 6, using Theorem 3.1 from [9], we prove that there is no initial state $W^0 \in \widetilde{H}^0$ that is null-controllable in a given time $T > 0$ by using controls $u \in L^\infty(0,T)$. In Section 7, from Theorem 5.2 of Section 5 it immediately follows that each initial state $W^0 \in \widetilde{H}^0$ is approximately controllable to any end state $W^T \in \widetilde{H}^0$, using controls $u \in L^\infty(0,T)$, in a given time $T > 0$.

2. Notation

Introduce the spaces used in the paper. For $m = 0, 1, 2$, denote

$$H^m = \left\{\varphi \in L^2(\mathbb{R}) \mid \forall k = \overline{0,m}\ \varphi^{(k)} \in L^2(\mathbb{R})\right\}$$

with the norm

$$\|\varphi\|^m = \sqrt{\sum_{k=0}^m \binom{m}{k} \left(\left\|\varphi^{(k)}\right\|_{L^2(\mathbb{R})}\right)^2},$$

and $H^{-m} = (H^m)^*$ with the strong norm $\|\cdot\|^{-m}$ of the adjoint space. We have $H^0 = L^2(\mathbb{R}) = \left(H^0\right)^* = H^{-0}$.

For $n = \overline{-2,2}$, denote

$$H_n = \left\{\psi \in L^2_{\mathrm{loc}}(\mathbb{R}) \mid \left(1+\sigma^2\right)^{n/2} \psi \in L^2(\mathbb{R})\right\}$$

with the norm

$$\|\psi\|_n = \left\|\left(1+\sigma^2\right)^{n/2} \psi\right\|_{L^2(\mathbb{R})}.$$

Evidently, $H_{-n} = (H_n)^*$.

By $\mathcal{F} : H^{-2} \to H_{-2}$, denote the Fourier transform operator with the domain H^{-2}. This operator is an extension of the classical Fourier transform operator which is an isometric isomorphism of $L^2(\mathbb{R})$. The extension is given by the formula

$$\langle \mathcal{F}f, \varphi\rangle = \langle f, \mathcal{F}^{-1}\varphi\rangle, \quad f \in H^{-2},\ \varphi \in H_2.$$

This operator is an isometric isomorphism of H^m and H_m, $m = \overline{-2, 2}$ [5, Chap. 1].

A distribution $f \in H^{-2}$ (or H_{-2}) is said to be *odd* if $\langle f, \varphi(\cdot)\rangle = -\langle f, \varphi(-(\cdot))\rangle$, $\varphi \in H^2$ (or H_2 respectively).

By $\widetilde{H}^n$, denote the subspace of all odd distributions in H^n, $n = \overline{-2, 2}$. Evidently, $\widetilde{H}^n$ is a closed subspace of H^n, $n = \overline{-2, 2}$.

Remark 2.1. Note that, for $\varphi \in H^m_{⓪}$, its odd extension $\varphi(\cdot) - \varphi(-(\cdot))$ belongs to $\widetilde{H}^m$, $m = 0, 1, 2$. The converse assertion is true only for $m = 0, 1$, and it is not true for $m = 2$. That is why the odd extension of a distribution $f \in H^{-m}_{⓪}$ may not belong to $\widetilde{H}^{-m}$, $m = 1, 2$. However, the following theorem holds.

Theorem 2.2 ([4])**.** *Let $f \in H^0_{⓪}$ and there exist $f(0^+) \in \mathbb{R}$. Then $f'' \in H^{-2}_{⓪}$ can be extended to the odd distribution F, and $F \in \widetilde{H}^{-2}$. This distribution is given by the formula*

$$F = \big(f(\cdot) - f(-(\cdot))\big)'' - 2f(0^+)\delta', \tag{2.1}$$

where δ is the Dirac distribution.

3. Preliminary

Consider control problem (1.1)–(1.3). Let W^0 and $W(\cdot, t)$ be the odd extensions of w^0 and $w(\cdot, t)$ with respect to x, $t \in [0, T]$. If w is a solution to problem (1.1)–(1.3), then W is a solution to the following problem:

$$W_t = W_{xx} - 2u\delta', \qquad x \in \mathbb{R},\ t \in (0, T), \tag{3.1}$$

$$W(\cdot, 0) = W^0, \qquad x \in \mathbb{R}, \tag{3.2}$$

according to Theorem 2.2. Here $W^0 \in \widetilde{H}^0$, $\left(\frac{d}{dt}\right)^m W : [0, T] \to \widetilde{H}^{-2m}$, $m = 0, 1$, δ is the Dirac distribution with respect to x. The converse assertion is also true: if W is a solution to (3.1), (3.2), then its restriction $w = W|_{(0,+\infty)}$ is a solution to (1.1)–(1.3), and

$$W(0^+, t) = u(t) \quad \text{a.e. on } [0, T] \tag{3.3}$$

(see below (3.10)). Evidently, (1.4) holds iff

$$W(\cdot, T) = W^T \tag{3.4}$$

holds, where W^T is an odd extension of w^T.

Consider control problem (3.1), (3.2). Denote $V^0 = \mathcal{F}W^0$ and $V(\cdot,t) = \mathcal{F}_{x\to\sigma}W(\cdot,t)$, $t \in [0,T]$. We have

$$V_t = -\sigma^2 V - \sqrt{\frac{2}{\pi}} i\sigma\, u, \qquad \sigma \in \mathbb{R},\ t \in (0,T), \tag{3.5}$$

$$V(\cdot,0) = V^0, \qquad \sigma \in \mathbb{R}. \tag{3.6}$$

Therefore,

$$V(\sigma,t) = e^{-t\sigma^2}V^0(\sigma) - \sqrt{\frac{2}{\pi}} i\sigma \int_0^t e^{-(t-\xi)\sigma^2} u(\xi)\, d\xi, \quad \sigma \in \mathbb{R},\ t \in [0,T], \tag{3.7}$$

is the unique solution to (3.5), (3.6). Since $u \in L^\infty(0,T)$, we have

$$|V(\sigma,t)| \le |V^0(\sigma)| + \sqrt{\frac{2}{\pi}}\, \|u\|_{L^\infty(0,T)} \frac{1 - e^{-t\sigma^2}}{|\sigma|}, \quad \sigma \in \mathbb{R},\ t \in [0,T]. \tag{3.8}$$

Hence $V(\cdot,t) \in \widetilde{H}_0$, $t \in [0,T]$. From (3.7), we obtain

$$W(x,t) = \frac{e^{-\frac{x^2}{4t}}}{\sqrt{4\pi t}} * W^0(x) + \sqrt{\frac{2}{\pi}} x \int_0^t e^{-\frac{x^2}{4\xi}} \frac{u(t-\xi)}{(2\xi)^{3/2}} d\xi. \tag{3.9}$$

Since for any $t \in (0,T]$ the function $\frac{e^{-\frac{x^2}{4t}}}{\sqrt{2t}} * W^0(x)$ is odd and continuous, we obtain

$$\frac{e^{-\frac{x^2}{4t}}}{\sqrt{2t}} * W^0(x) \to 0 \quad \text{as } x \to 0^+.$$

Setting $\mu = \frac{|x|}{2\sqrt{\xi}}$, we get

$$x \int_0^t e^{-\frac{x^2}{4\xi}} \frac{u(t-\xi)}{(2\xi)^{3/2}} d\xi = \sqrt{2}\operatorname{sgn} x \int_{|x|/(2\sqrt{t})}^\infty e^{-\mu^2} u\left(t - \frac{x^2}{4\mu^2}\right) d\mu.$$

According to Lebesgue's dominated convergence theorem, we get

$$W(0^+,t) = \frac{2}{\sqrt{\pi}} u(t) \int_0^\infty e^{-\mu^2} = u(t) \quad \text{a.e. on } [0,T], \tag{3.10}$$

i.e., (3.3) holds.

Thus control systems (1.1)–(1.3) and (3.1), (3.2) are equivalent. Therefore, basing on this reason, we will further consider control system (3.1), (3.2) instead of original system (1.1)–(1.3).

4. Reachability

Definition 4.1. For control system (3.1), (3.2), a state $W^T \in \widetilde{H}^0$ is said to be reachable from a state $W^0 \in \widetilde{H}^0$ in a given time $T > 0$ if there exists a control $u \in L^\infty(0,T)$ such that there exists a unique solution to (3.1), (3.2), (3.4).

By $\mathcal{R}_T(W^0)$, denote the set of all states $W^T \in \widetilde{H}^0$ reachable from W^0 in the time T.

According to (3.9), we have

$$\mathcal{R}_T(W^0) = \Big\{ W^T \in \widetilde{H}^0 \mid \exists v \in L^\infty(0,T) \quad W^T = \frac{1}{\sqrt{2\pi}} \frac{e^{-\frac{x^2}{4T}}}{\sqrt{2T}} * W^0(x) + \sqrt{\frac{2}{\pi}} x \int_0^T e^{-\frac{x^2}{4\xi}} \frac{v(\xi)}{(2\xi)^{3/2}} d\xi \Big\}, \tag{4.1}$$

in particular,

$$\mathcal{R}_T(0) = \left\{ W^T \in \widetilde{H}^0 \mid \exists v \in L^\infty(0,T) \;\; W^T = \sqrt{\frac{2}{\pi}} x \int_0^T e^{-\frac{x^2}{4\xi}} \frac{v(\xi)}{(2\xi)^{3/2}} d\xi \right\}. \tag{4.2}$$

First, we study $\mathcal{R}_T(0)$. Denote also

$$\mathcal{R}_T^L(0) = \Big\{ W^T \in \widetilde{H}^0 \mid \exists v \in L^\infty(0,T) \Big(\|v\|_{L^\infty(0,T)} \le L \wedge W^T = \sqrt{\frac{2}{\pi}} x \int_0^T e^{-\frac{x^2}{4\xi}} \frac{v(\xi)}{(2\xi)^{3/2}} d\xi \Big) \Big\}. \tag{4.3}$$

Evidently, the following theorem holds

Theorem 4.2. *We have*

(i) $\mathcal{R}_T(0) = \cup_{L>0} \mathcal{R}_T^L(0)$;

(ii) $\mathcal{R}_T^L(0) \subset \mathcal{R}_T^{L'}(0)$, $L \le L'$;

(iii) $f \in \mathcal{R}_T^1(0) \Leftrightarrow Lf \in \mathcal{R}_T^L(0)$.

We can obtain the following necessary condition for f to belong to $\mathcal{R}_T^L(0)$.

Theorem 4.3. *If $W^T \in \mathcal{R}_T^L(0)$, then for any $T^* > T$,*

$$\int_0^\infty e^{\frac{x^2}{4T^*}} \left| W^T(x) \right| dx \le L \sqrt{\frac{T^*}{\pi}} \ln \frac{\sqrt{T^*} + \sqrt{T}}{\sqrt{T^*} - \sqrt{T}}. \tag{4.4}$$

Proof. Using (4.3), we have

$$\begin{aligned}
\int_0^\infty e^{\frac{x^2}{4T^*}} \left| W^T(x) \right| dx &\le \sqrt{\frac{2}{\pi}} L \int_0^\infty e^{\frac{x^2}{4T^*}} x \int_0^T e^{-\frac{x^2}{4\xi}} \frac{d\xi}{(2\xi)^{3/2}} \\
&= \sqrt{\frac{2}{\pi}} L \int_0^T \frac{1}{(2\xi)^{3/2}} \int_0^\infty e^{-x^2\left(\frac{1}{4\xi} - \frac{1}{4T^*}\right)} x\, dx\, d\xi \\
&= \frac{L}{\sqrt{2\pi}} \int_0^T \frac{1}{(2\xi)^{3/2}} \frac{1}{\frac{1}{4\xi} - \frac{1}{4T^*}} d\xi = L \sqrt{\frac{T^*}{\pi}} \ln \frac{\sqrt{T^*} + \sqrt{T}}{\sqrt{T^*} - \sqrt{T}}.
\end{aligned}$$

□

Theorem 4.4. *Let $W^T \in \widetilde{H}^0$ and (4.4) hold. Let*

$$\omega_n = \frac{n!}{(2n+1)!} \int_0^\infty x^{2n+1} W^T(x)dx, \qquad n = \overline{0,\infty}. \tag{4.5}$$

Then $W^T \in \mathcal{R}_T^L(0)$ iff there exists $v \in L^\infty(0,T)$ such that $\|v\|_{L^\infty(0,T)} \le L$ and

$$\int_0^T \xi^n v(\xi)d\xi = \omega_n, \qquad n = \overline{0,\infty}. \tag{4.6}$$

Proof. According to (4.3), $W^T \in \mathcal{R}_T^L(0)$ iff there exists $v \in L^\infty(0,T)$ such that $\|v\|_{L^\infty(0,T)} \le L$ and

$$W^T = \sqrt{\frac{2}{\pi}} x \int_0^T e^{-\frac{x^2}{4\xi}} \frac{v(\xi)}{(2\xi)^{3/2}} d\xi.$$

Denoting $V^T = \mathcal{F}W^T$, we have

$$V^T(\sigma) = -\sqrt{\frac{2}{\pi}} i\sigma \int_0^T e^{-\xi\sigma^2} v(\xi)d\xi.$$

We see that $V^T(\sigma)$ is an odd entire function. Therefore,

$$\sum_{n=0}^\infty \frac{\left(V^T\right)^{(2n+1)}(0)}{(2n+1)!} \sigma^{2n+1} = V^T(\sigma) = -\sqrt{\frac{2}{\pi}} i\sigma \sum_{n=0}^\infty \frac{(-1)^n}{n!} \sigma^{2n} \int_0^T \xi^n v(\xi)d\xi.$$

Since

$$\left(V^T\right)^{(2n+1)}(0) = \sqrt{\frac{2}{\pi}} \int_0^\infty (-ix)^{2n+1} W^T(x)dx = -i\sqrt{\frac{2}{\pi}} (-1)^n \frac{(2n+1)!}{n!} \omega_n, \tag{4.7}$$

we conclude the assertion of the theorem. □

Theorem 4.5. *Let $W^T \in \widetilde{H}^0$ and (4.4) hold. Let $\{\omega_n\}_{n=0}^\infty$ be defined by (4.5). If for each $N \in \mathbb{N}$ there exists $v_N \in L^\infty(0,T)$ such that $\|v_N\|_{L^\infty(0,T)} \le L$ and*

$$\int_0^T \xi^n v_N(\xi)d\xi = \omega_n, \qquad n = \overline{0,N}, \tag{4.8}$$

then $W^T \in \overline{\mathcal{R}_T^L(0)}$ (the closure is considered in $\widetilde{H}^0$).

Proof. By W_N, denote the solution to problem (3.1), (3.2) with $W^0 = 0$ and $u(t) = v_N(T-t)$. Denote also $V^T = \mathcal{F}W^T$, $V_N(\cdot,t) = \mathcal{F}_{x\to\sigma} W_N(\cdot,t)$, $t \in [0,T]$. Then V_N is the unique solution to (3.5), (3.6) with $V^0 = 0$ and the same u. Evidently,

$$\int_a^\infty \left|V^T(\sigma)\right|^2 d\sigma \to 0 \quad \text{as } a \to \infty. \tag{4.9}$$

Let $T > T^*$. Put

$$W_{T^*} = \int_0^\infty e^{\frac{x^2}{4T^*}} \left|W^T(x)\right| dx.$$

For $n=\overline{0,\infty}$, we have

$$\left(V^T\right)^{(2n)}(0)=0,\quad \left(V^T\right)^{(2n+1)}(0)=(-1)^n i\sqrt{\frac{2}{\pi}}\int_0^\infty x^{2n+1}W^T(x)dx. \tag{4.10}$$

Therefore, using the Stirling formula:

$$\sqrt{2\pi}n^{n+\frac{1}{2}}e^{-n}\le n!\le en^{n+\frac{1}{2}}e^{-n},\qquad n\in\mathbb{N}, \tag{4.11}$$

we get

$$\begin{aligned}\left|\left(V^T\right)^{(2n+1)}(0)\right| &\le \sqrt{\frac{2}{\pi}}\int_0^\infty \left(x^{2n+1}e^{-\frac{x^2}{4T^*}}\right)\left(e^{\frac{x^2}{4T^*}}\left|W^T(x)\right|\right)dx\\ &\le \sqrt{\frac{2}{\pi}}W_{T^*}\left(\frac{2n+1}{2e}\right)^{\frac{2n+1}{2}}(4T^*)^{\frac{2n+1}{2}}\\ &\le W_{T^*}\frac{(2n+1)!}{\pi\sqrt{2n+1}}\left(\frac{2T^*e}{2n+1}\right)^{\frac{2n+1}{2}}.\end{aligned} \tag{4.12}$$

Since

$$\overline{\lim_{n\to\infty}}\left(\frac{\left|\left(V^T\right)^{(2n+1)}(0)\right|}{(2n+1)!}\right)^{\frac{1}{2n+1}}\le \lim_{n\to\infty}\left(\frac{W_{T^*}}{\pi\sqrt{2n+1}}\right)^{\frac{1}{2n+1}}\sqrt{\frac{2T^*e}{2n+1}}=0,$$

we can continue V^T to an odd entire function. Hence,

$$V^T(\sigma)=\sum_{n=0}^\infty\frac{\left(V^T\right)^{(2n+1)}(0)}{(2n+1)!}\sigma^{2n+1},\qquad \sigma\in\mathbb{R}. \tag{4.13}$$

Due to (3.8), we get

$$|V_N(\sigma,T)|\le\sqrt{\frac{2}{\pi}}L\frac{1-e^{-T\sigma^2}}{|\sigma|}. \tag{4.14}$$

Hence,

$$\begin{aligned}\int_a^\infty |V_N(\sigma,T)|^2\,d\sigma &\le \frac{2}{\pi}L^2\int_a^\infty\left|\frac{1-e^{-T\sigma^2}}{\sigma}\right|d\sigma\le\frac{8L^2}{\pi}\int_a^\infty\frac{d\sigma}{\sigma^2}\\ &=\frac{8L^2}{\pi a}\to 0\quad \text{as } a\to\infty.\end{aligned} \tag{4.15}$$

According to (3.7), we get

$$\begin{aligned}V_N(\sigma,T)&=-\sqrt{\frac{2}{\pi}}i\sigma\int_0^T e^{-\xi\sigma^2}v_N(\xi)d\xi\\ &=-i\sqrt{\frac{2}{\pi}}\sum_{n=0}^\infty\frac{(-1)^n}{n!}\sigma^{2n+1}\int_0^T\xi^n v_N(\xi)d\xi.\end{aligned} \tag{4.16}$$

Due to (4.8), we obtain

$$V^T(\sigma) - V_N(\sigma, T) = \sum_{n=N+1}^{\infty} \sigma^{2n+1} \left[\frac{\left(V^T\right)^{(2n+1)}(0)}{(2n+1)!} - i\sqrt{\frac{2}{\pi}} \frac{(-1)^{n+1}}{n!} \int_0^T \xi^n v_N(\xi) d\xi \right]. \quad (4.17)$$

With regard to (4.12) and using (4.11), we get

$$\left| \frac{\left(V^T\right)^{(2n+1)}(0)}{(2n+1)!} \right| \le \frac{W_{T^*}}{\pi\sqrt{2n+1}} \left(\frac{2T^* e}{2n+1} \right)^{\frac{2n+1}{2}} \le \frac{W_{T^*} e^{3/2}}{\pi n! \sqrt{2n+1}} \left(\frac{2T^* n}{2n+1} \right)^{\frac{2n+1}{2}} \le \frac{W_{T^*} e^{3/2}}{\pi n! \sqrt{2n+1}} \left(\sqrt{T^*} \right)^{2n+1}.$$

Therefore, for $|\sigma| \le a$,

$$\left| \sum_{n=N+1}^{\infty} \frac{\left(V^T\right)^{(2n+1)}(0)}{(2n+1)!} \sigma^{2n+1} \right| \le \frac{e^{3/2} W_{T^*}}{\pi} \sum_{n=N+1}^{\infty} \frac{\left(\sqrt{T^*} a\right)^{2n+1}}{n! \sqrt{2n+1}} \to 0 \text{ as } N \to \infty$$

and

$$\sqrt{\frac{2}{\pi}} \left| \sum_{n=N+1}^{\infty} \frac{(-1)^{n+1}}{n!} \sigma^{2n+1} \int_0^T \xi^n v_N(\xi) d\xi \right| \le \sqrt{\frac{2}{\pi}} L \sum_{n=N+1}^{\infty} \frac{a^{2n+1} T^{n+1}}{(n+1)!} \to 0$$

as $N \to \infty$. Taking into account (4.17), we get

$$S_N(a) = \sup_{\sigma \in [-a,a]} \left| V^T(\sigma) - V_N(\sigma, T) \right| \to 0 \quad \text{as } N \to \infty.$$

Therefore,

$$\int_{-a}^{a} \left| V^T(\sigma) - V_N(\sigma, T) \right|^2 d\sigma \le 2a \left(S_N(a) \right)^2 \to 0 \quad \text{as } N \to \infty. \quad (4.18)$$

With regard to (4.9), (4.15) and (4.18), we obtain

$$\left\| W^T(\sigma) - W_N(\sigma, T) \right\|^0 = \left\| V^T(\sigma) - V_N(\sigma, T) \right\|_0 \to 0 \quad \text{as } N \to \infty,$$

i.e., $W^T \in \overline{\mathcal{R}_T^L(0)}$. □

The last theorem is illustrated by the examples in Section 8 (see Examples 8.1 and 8.2).

5. Approximate reachability

Definition 5.1. For control system (3.1), (3.2), a state $W^T \in \widetilde{H}^0$ is said to be approximately reachable from a state $W^0 \in \widetilde{H}^0$ in a given time $T > 0$ if $W^T \in \overline{\mathcal{R}_T(W^0)}$, where the closure is considered in the space $\widetilde{H}^0$.

In other words, a state $W^T \in \widetilde{H}^0$ is approximately reachable from a state $W^0 \in \widetilde{H}^0$ in a given time $T > 0$ iff for each $\varepsilon > 0$ there exists $u_\varepsilon \in L^\infty(0,T)$ such that there exists a unique solution W to (3.1), (3.2) with $u = u_\varepsilon$ and $\left\| W(\cdot,T) - W^T \right\|^0 < \varepsilon$.

Theorem 5.2. *Each state $W^T \in \widetilde{H}^0$ is approximately reachable from the origin in a given time $T > 0$.*

First we consider an orthogonal basis in $L^2(\mathbb{R})$. Let $\psi_n(x) = \mathcal{H}_n(x)e^{-\frac{x^2}{2}}$, $x \in \mathbb{R}$, $n = \overline{0,\infty}$, where

$$\mathcal{H}_n(x) = (-1)^n e^{x^2} \left(\frac{d}{dx}\right)^n e^{-x^2} = n! \sum_{m=0}^{\left[\frac{n}{2}\right]} \frac{(-1)^m}{m!(n-2m)!}(2x)^{n-2m}$$

is the Hermite polynomial, $[\cdot]$ is the integer part of a real number. It is well known [7] that

$$\int_{-\infty}^{\infty} \psi_n(x)\psi_m(x)dx = \sqrt{\pi}2^n n!\delta_{mn}, \qquad 0 \le m < n < +\infty, \tag{5.1}$$

where δ_{mn} is the Kronecker delta, and $\{\psi_n\}_{n=0}^\infty$ is an orthogonal basis in $L^2(\mathbb{R})$. It is easy to see that

$$\mathcal{F}\psi_n = (-i)^n \psi_n, \qquad n = \overline{0,\infty}. \tag{5.2}$$

Define

$$\psi_n^T(x) = \psi_{2n+1}\left(\frac{x}{\sqrt{2T}}\right), \qquad x \in \mathbb{R},\ n = \overline{0,\infty},$$
$$\widehat{\psi}_n^T(\sigma) = \left(\mathcal{F}\psi_n^T\right)(\sigma) = (-1)^{n+1} i\sqrt{2T}\psi_{2n+1}(\sqrt{2T}\sigma), \qquad \sigma \in \mathbb{R},\ n = \overline{0,\infty}.$$

According to (5.1),we get

$$\langle \psi_n^T, \psi_m^T \rangle = \langle \widehat{\psi}_n^T, \widehat{\psi}_m^T \rangle = \sqrt{2\pi T}2^{2n+1}(2n+1)!\delta_{mn}, \qquad 0 \le m < n < +\infty. \tag{5.3}$$

Obviously, $\{\psi_n^T\}_{n=0}^\infty$ and $\{\widehat{\psi}_n^T\}_{n=0}^\infty$ are orthogonal bases in $\widetilde{H}^0$. Therefore, for $f \in \widetilde{H}^0$,

$$f = \sum_{n=0}^\infty f_n \psi_n^T, \quad \mathcal{F}f = \sum_{n=0}^\infty f_n \widehat{\psi}_n^T, \qquad \text{where } f_n = \frac{\langle f, \psi_n^T \rangle}{\langle \psi_n^T, \psi_n^T \rangle} = \frac{\langle \mathcal{F}f, \widehat{\psi}_n^T \rangle}{\langle \psi_n^T, \psi_n^T \rangle},$$

and

$$\sum_{n=0}^\infty |f_n|^2 \langle \widehat{\psi}_n^T, \widehat{\psi}_n^T \rangle = \sqrt{2\pi T} \sum_{n=0}^\infty |f_n|^2 2^{2n+1}(2n+1)!. \tag{5.4}$$

Consider also the operator $\Phi_T : L^2(\mathbb{R}) \to \widetilde{H}^0$ with the domain $D(\Phi_T) = \{g \in L^\infty(\mathbb{R}) : \operatorname{supp} g \subset [0,T]\}$, acting by the rule

$$\Phi_T g = \sqrt{\frac{2}{\pi}} \mathcal{F}^{-1} \left(i\sigma \int_{-\infty}^{\infty} e^{-\sigma^2(T-\xi)} g(\xi) d\xi \right), \quad g \in D(\Phi_T).$$

Evidently,

$$\|\mathcal{F}\Phi_T g\|_0 \le \|g\|_{L^\infty(\mathbb{R})} \left(\frac{2^5 T}{\pi} \right)^{\frac{1}{4}}.$$

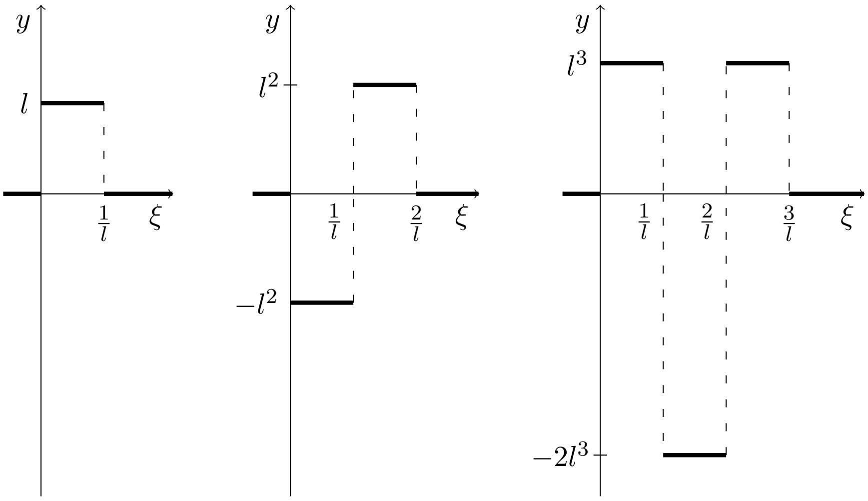
Taking into account (3.7), we obtain that $W^T \in \overline{\mathcal{R}_T(0)}$ iff

$$\exists \{u_n\}_{n=1}^\infty \subset L^\infty(0,T) \quad \left\| W^T + \Phi_T u_n \right\|^0 \to 0 \quad \text{as } n \to \infty. \tag{5.5}$$

Denote

$$\begin{aligned}
\varphi_n(\sigma) &= \sigma^{2n+1} e^{-T\sigma^2}, && \sigma \in \mathbb{R},\\
\varphi_n^l(\sigma) &= \sigma^{2n+1} e^{-T\sigma^2} \left(\frac{e^{\sigma^2/l} - 1}{\sigma^2/l} \right)^{n+1}, && \sigma \in \mathbb{R},\\
u_l^n(\xi) &= \begin{cases} (-1)^{n-j} \binom{n}{j} l^{n+1}, & \xi \in \left(\frac{j}{l}, \frac{j+1}{l} \right), \ j = \overline{0,n} \\ 0, & \xi \notin \left[0, \frac{n+1}{l}\right] \end{cases}, && l \in \mathbb{N}, \ n \in \mathbb{N} \cup \{0\}.
\end{aligned} \tag{5.6}$$

Then, $\mathcal{F}\Phi_T u_l^n = \sqrt{\frac{2}{\pi}} i \varphi_n^l$. Figure 5.1 illustrates the functions u_l^n. If $l > \frac{2n+2}{T}$, we



(a) $y = u_l^0(\xi)$ (b) $y = u_l^1(\xi)$ (c) $y = u_l^2(\xi)$

Fig. 5.1: The functions u_l^n.

have

$$\left| \varphi_n^l(\sigma) \right| \le \sigma^{2n+1} e^{-T\sigma^2} e^{\frac{(n+1)\sigma^2}{l}} = \sigma^{2n+1} e^{-\sigma^2 \left(T - \frac{n+1}{l}\right)} \le \sigma^{2n+1} e^{\frac{-\sigma^2 T}{2}}$$

and $\varphi_n^l \to \varphi_n$ as $l \to \infty$ a.e. on $\mathbb{R}$. According to Lebesgue's dominated convergence theorem, we get

$$\left\|\varphi_n - \varphi_n^l\right\|_0 \to 0 \quad \text{as } l \to \infty, \quad n = \overline{0, \infty}\ .$$

Proof of Theorem 5.2. Let $W^T \in \widetilde{H}^0$. Denote $V^T = \mathcal{F}W^T$. Then,

$$W^T = \sum_{n=0}^{\infty} \omega_n \psi_n^T, \qquad V^T = \sum_{n=0}^{\infty} \omega_n \widehat{\psi}_n^T.$$

Due to (5.4), for each $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\sqrt{2\pi T} \sum_{n=N+1}^{\infty} |\omega_n|^2 2^{2n+1} (2n+1)! < \varepsilon^2. \tag{5.7}$$

We have

$$\sum_{n=0}^{N} \omega_n \widehat{\psi}_n^T = i \sum_{n=0}^{N} \omega_n \sum_{p=0}^{n} h_p^n \varphi_p = i \sum_{p=0}^{N} \varphi_p \sum_{n=p}^{N} \omega_n h_p^n,$$

where

$$h_p^n = \frac{(-1)^{p+1} 2^{2p+1} (2T)^{p+1}}{(n-p)!(2p+1)!} (2n+1)!. \tag{5.8}$$

For each $p = \overline{0, N}$, determine $l_p^N \in \mathbb{N}$ such that

$$\left\|\varphi_p - \varphi_p^{l_p^N}\right\|_0 < \left(\frac{\pi^3}{Te^2}\right)^{\frac{1}{4}} \frac{\varepsilon}{\|V^T\|_0 \sqrt{N+2} \cosh\left(2\sqrt{2T(N+2)}\right)}$$

and denote

$$V_N^T = i \sum_{p=0}^{N} \varphi_p^{l_p^N} \sum_{n=p}^{N} \omega_n h_p^n.$$

Then,

$$\left\|V^T - V_N^T\right\|_0 \le \varepsilon \left(1 + \frac{E_N \left(\frac{\pi^3}{Te^2}\right)^{1/4}}{\|V^T\|_0 \sqrt{N+2} \cosh\left(2\sqrt{2T(N+2)}\right)}\right), \tag{5.9}$$

where $E_N = \sum_{p=0}^{N} \sum_{n=p}^{N} \left|\omega_n h_p^n\right|$. Let us estimate E_N. For $p = \overline{0, N}$, we have

$$\begin{aligned} \sum_{n=p}^{N} \left|\omega_n h_p^n\right| &\le \left(\sum_{n=p}^{N} |\omega_n|^2 \sqrt{2\pi T} 2^{2n+1} (2n+1)!\right)^{\frac{1}{2}} \left(\sum_{n=p}^{N} \frac{\left|h_p^n\right|^2}{\sqrt{2\pi T} 2^{2n+1} (2n+1)!}\right)^{\frac{1}{2}} \\ &\le \left\|V^T\right\|_0 \left(\sum_{n=p}^{N} \frac{\left|h_p^n\right|^2}{\sqrt{2\pi T} 2^{2n+1} (2n+1)!}\right)^{\frac{1}{2}}. \end{aligned} \tag{5.10}$$

Taking into account (5.8), we get

$$\frac{\left|h_p^n\right|^2}{\sqrt{2\pi T}2^{2n+1}(2n+1)!} = \frac{1}{\sqrt{2\pi T}}\left(\frac{2^{2p+1}(2T)^{p+1}}{(2p+1)!}\right)^2 \frac{(2n+1)!}{2^{2n+1}\left((n-p)!\right)^2}. \tag{5.11}$$

By using (4.11), we obtain

$$\begin{aligned}\frac{(2n+1)!}{2^{2n+1}\left((n-p)!\right)^2} &\le \frac{e\sqrt{2n+1}}{2^{2n+2}\pi}\left(\frac{2n+1}{e}\right)^{2n+1}\frac{1}{n-p}\left(\frac{e}{n-p}\right)^{2(n-p)} \\ &\le \frac{\sqrt{2n+1}}{2\pi}\left(\frac{2n+1}{2(n-p)}\right)^{2(n-p)+1}\left(\frac{n+1}{e}\right)^{2p}.\end{aligned}$$

Since $\left(\frac{2n+1}{2(n-p)}\right)^{2(n-p)+1}$ is increasing with respect to n, we conclude that

$$\sup_{n\ge p}\left\{\left(\frac{2n+1}{2(n-p)}\right)^{2(n-p)+1}\right\} = \lim_{n\to\infty}\left(\frac{2n+1}{2(n-p)}\right)^{2(n-p)+1} = e^{2p+1}.$$

Therefore,

$$\frac{(2n+1)!}{2^{2n+1}\left((n-p)!\right)^2} \le \frac{\sqrt{2n+1}}{2\pi}e^{2p+1}\left(\frac{n+1}{e}\right)^{2p} \le \frac{e}{\sqrt{2}\pi}(n+1)^{2p+\frac{1}{2}}.$$

According to (5.11), we get

$$\frac{\left|h_p^n\right|^2}{\sqrt{2\pi T}2^{2n+1}(2n+1)!} \le \frac{1}{\sqrt{2\pi T}}\left(\frac{2^{2p+1}(2T)^{p+1}}{(2p+1)!}\right)^2 \frac{e}{\sqrt{2}\pi}(n+1)^{2p+\frac{1}{2}}.$$

Taking into account (5.10), we have

$$\sum_{n=p}^{N}\left|\omega_n h_p^n\right| \le \left\|V^T\right\|_0\left(\frac{1}{4\pi T}\right)^{1/4}\sqrt{\frac{e}{\pi}}\frac{2^{2p+1}(2T)^{p+1}}{(2p+1)!}\left(\sum_{n=p}^{N}(n+1)^{2p+\frac{1}{2}}\right)^{\frac{1}{2}}. \tag{5.12}$$

Since

$$\sum_{n=p}^{N}(n+1)^{2p+\frac{1}{2}} \le \int_p^{N+1}(x+1)^{2p+\frac{1}{2}}dx,$$

we obtain

$$\begin{aligned}\sum_{n=p}^{N}\left|\omega_n h_p^n\right| &\le \left\|V^T\right\|_0\left(\frac{1}{4\pi T}\right)^{\frac{1}{4}}\sqrt{\frac{e}{\pi}}\frac{2^{2p+1}(2T)^{p+1}}{(2p+1)!}(N+2)^{p+1} \\ &= \left\|V^T\right\|_0\left(\frac{Te^2}{\pi^3}\right)^{\frac{1}{4}}\frac{\left(2\sqrt{2T(N+2)}\right)^{2p+1}}{(2p+1)!}\sqrt{N+2}.\end{aligned}$$

Hence,

$$E_N \le \left\|V^T\right\|_0 \left(\frac{Te^2}{\pi^3}\right)^{\frac{1}{4}} \sqrt{N+2} \sum_{p=0}^{N} \frac{\left(2\sqrt{2T(N+2)}\right)^{2p+1}}{(2p+1)!}$$
$$= \left\|V^T\right\|_0 \left(\frac{Te^2}{\pi^3}\right)^{\frac{1}{4}} \sqrt{N+2} \cosh\left(2\sqrt{2T(N+2)}\right).$$

Taking into account (5.9), we conclude that

$$\left\|V^T - V_N^T\right\|_0 \le 2\varepsilon. \tag{5.13}$$

Put $u_N = -\sqrt{\frac{\pi}{2}} \sum_{p=0}^{N} u^p_{l_p^N} \sum_{n=p}^{N} \omega_n h_p^n$. With regard to (5.13) and (5.5), we get

$$\left\|W^T + \Phi_T u_N\right\|^0 \le 2\varepsilon. \qquad \square$$

Remark 5.3. The controls

$$u_N = -\sqrt{\frac{\pi}{2}} \sum_{p=0}^{N} u^p_{l_p^N} \sum_{n=p}^{N} \omega_n h_p^n, \quad N \in \mathbb{N}, \tag{5.14}$$

found in the proof of Theorem 5.2 solve the approximate reachability problem for system (3.1), (3.2). Here $u^p_{l_p^N}$ is defined by (5.6), h_p^n is defined by (5.8) and ω_n, $n = \overline{0,\infty}$, are the coefficients of decomposition of W^T with respect to the basis $\{\psi_n^T\}_{n=0}^{\infty}$.

Corollary 5.4. *Each state $W^T \in \widetilde{H}^0$ is approximately reachable from any state $W^0 \in \widetilde{H}^0$ in a given time $T > 0$.*

6. Controllability

Definition 6.1. For control system (3.1), (3.2), a state $W^0 \in \widetilde{H}^0$ is said to be null-controllable in a given time $T > 0$ if $0 \in \mathcal{R}_T(W^0)$.

In other words, the state $W^0 \in \widetilde{H}^0$ is null-controllable in a given time $T > 0$ iff there exists $u \in L^\infty(0,T)$ such that there exists a unique solution W to (3.1), (3.2) and $W(\cdot,T) = 0$.

Theorem 6.2. *If a state $W^0 \in \widetilde{H}^0$ is null-controllable in a time $T > 0$, then $W^0 = 0$.*

Proof. Find $u \in L^\infty(0,T)$ such that there exists a unique solution W to (3.1), (3.2) and $W(\cdot,T) = 0$. Denote $V^0 = \mathcal{F}W^0$, $V(\cdot,t) = \mathcal{F}_{x\to\sigma}W(\cdot,t)$, $t \in [0,T]$. Taking into account (3.7), we obtain

$$V^0(\sigma) = \sqrt{\frac{2}{\pi}} i\sigma \int_0^T e^{\xi\sigma^2} u(\xi)\, d\xi, \quad \sigma \in \mathbb{R}. \tag{6.1}$$

Let $T^* > T$ be fixed. Then,

$$\sum_{m=0}^{\infty} \nu_m \frac{\widehat{\psi}_m^{T^*}}{\left(\left\|\widehat{\psi}_m^{T^*}\right\|\right)^2} = \sum_{m=0}^{\infty} \int_0^T \mu_m(\xi) u(\xi)\, d\xi \frac{\widehat{\psi}_m^{T^*}}{\left(\left\|\widehat{\psi}_m^{T^*}\right\|\right)^2},$$

where

$$\nu_m = 2\int_0^\infty V^0(\sigma)\widehat{\psi}_m^{T^*}(\sigma)\, d\sigma, \tag{6.2}$$

$$\mu_m(\xi) = 2i\sqrt{\frac{2}{\pi}} \int_0^\infty \sigma e^{\xi\sigma^2} \widehat{\psi}_m^{T^*}(\sigma)\, d\sigma. \tag{6.3}$$

Therefore,

$$\int_0^T \mu_m(\xi) u(\xi)\, d\xi = \nu_m, \quad m = \overline{0,\infty}. \tag{6.4}$$

Let $m = \overline{0,\infty}$ be fixed. We have (see (5.8))

$$\begin{aligned} \mu_m(\xi) &= -2\sqrt{\frac{2}{\pi}} \sum_{p=0}^{m} h_p^m \int_0^\infty \sigma^{2p+2} e^{-(T^*-\xi)\sigma^2}\, d\sigma \\ &= (2m+1)! \frac{2\sqrt{2}T^*}{(T^*-\xi)^{3/2}} \sum_{p=0}^{m} \frac{(-1)^p}{(m-p)!p!} \left(\frac{2T^*}{T^*-\xi}\right)^p \\ &= (-1)^m \frac{(2m+1)!}{m!} \frac{2\sqrt{2}T^*}{(T^*-\xi)^{3/2}} \left(\frac{T^*+\xi}{T^*-\xi}\right)^m. \end{aligned} \tag{6.5}$$

Replacing $\frac{T^*+\xi}{T^*-\xi}$ by e^s, we get

$$\int_0^T \frac{T^*}{(T^*-\xi)^{3/2}} \left(\frac{T^*+\xi}{T^*-\xi}\right)^m u(\xi)\, d\xi = \sqrt{\frac{T^*}{2}} \int_0^{\overline{T}} e^{ms} u\left(\frac{T^*(e^s-1)}{e^s+1}\right) \frac{e^s}{\sqrt{e^s+1}}\, ds,$$

where $\overline{T} = \ln\left(\frac{T^*+T}{T^*-T}\right)$. Denoting $U^*(s) = u\left(\frac{T^*(e^s-1)}{e^s+1}\right)\frac{e^s}{\sqrt{e^s+1}}$, $s \in (0, \overline{T})$, $\nu_m^* = \frac{(-1)^m m!}{2\sqrt{T^*}(2m+1)!}\nu_m$, $m = \overline{0,\infty}$ and taking into account (6.4), (6.5), we obtain

$$\int_0^{\overline{T}} U^*(s) e^{ms} = \nu_m^*, \quad m = \overline{0,\infty}. \tag{6.6}$$

Since

$$|\nu_m| \le \left\|V^0\right\|_0 \left\|\widehat{\psi}_m^{T^*}\right\|_0, \quad m = \overline{0,\infty},$$

then, taking into account (5.3) and the Stirling formula (4.11), we obtain

$$|\nu_m^*| \le \left\|V^0\right\|_0 \left(\frac{\pi}{T^*}\right)^{1/4} \frac{2^{m-1/4} m!}{\sqrt{(2m+1)!}} \sim \left(\frac{\pi^2}{2^3 T^*}\right)^{1/4} \frac{\left\|V^0\right\|_0}{(2m+1)^{1/4}} \quad \text{as } m \to \infty.$$

Therefore, for all $\delta > 0$ there exists $C_\delta > 0$ such that

$$|\nu_m^*| \le C_\delta e^{m\delta}, \quad m = \overline{0,\infty}. \tag{6.7}$$

We have

$$\int_0^{\overline{T}} |U^*(s)|^2\, ds = \int_0^T |u(\xi)|^2 \frac{T^*+\xi}{(T^*-\xi)^2}\, d\xi \le \left(\|u\|_{L^\infty(0,T)}\right)^2 \int_0^T \frac{T^*+\xi}{(T^*-\xi)^2}\, d\xi$$
$$= \left(\|u\|_{L^\infty(0,T)}\right)^2 \left(\frac{2T}{T^*-T} - \ln\left(1+\frac{T}{T^*-T}\right)\right). \qquad (6.8)$$

Thus, $U^* \in L^2(0,T_*)$ and (6.6), (6.7) hold. Due to [9, Theorem 3.1, b)], we obtain $\nu_m^* = 0$, $m = \overline{0,\infty}$, i.e., $V^0 = W^0 = 0$. □

7. Approximate controllability

Definition 7.1. For control system (3.1), (3.2), a state $W^0 \in \widetilde{H}^0$ is said to be approximately controllable to a target state $W^T \in \widetilde{H}^0$ in a given time $T>0$ if $W^T \in \overline{\mathcal{R}_T(W^0)}$, where the closure is considered in the space $\widetilde{H}^0$.

In other words, the state $W^0 \in \widetilde{H}^0$ is approximately controllable to a target state $W^T \in \widetilde{H}^0$ in a given time $T>0$ iff for each $\varepsilon>0$ there exists $u_\varepsilon \in L^\infty(0,T)$ such that there exists a unique solution W to (3.1), (3.2) with $u=u_\varepsilon$ and $\left\|W(\cdot,T)-W^T\right\|^0 < \varepsilon$.

Taking into account Theorem 5.2, we get the following theorem.

Theorem 7.2. *Each state $W^0 \in \widetilde{H}^0$ is approximately controllable to any target state $W^T \in \widetilde{H}^0$ in a given time $T>0$.*

8. Examples

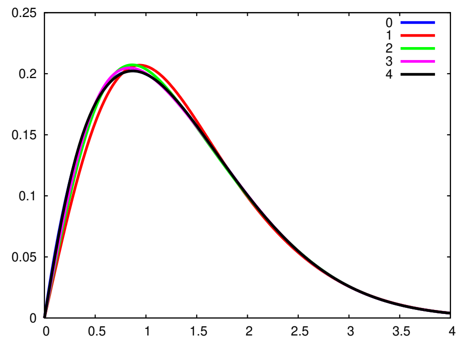
The following two examples illustrate the results of Theorem 4.5.

Example 8.1. Let $T=1$, $W^T(x) = \sqrt{\frac{2}{\pi}}x\int_0^T e^{-\frac{x^2}{4\xi}}\frac{d\xi}{2(2\xi)^{1/2}}$. Let us find the controls $u_N(\xi) = v_N(T-\xi)$, $\xi\in[0,T]$, where v_N is the solution to (4.8) for $N = 2P-1$, $P\in\mathbb{N}$. We use the algorithm given in [6] to find v_N in the form

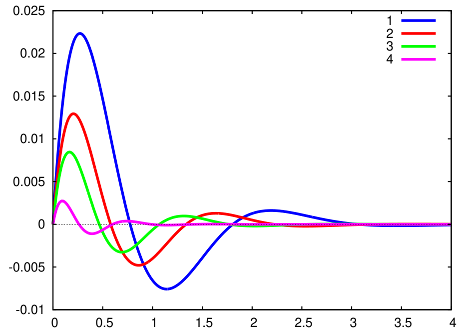
$$v_N(\xi) = \begin{cases} 1 & \text{if } \xi\in[\nu_{2p-1},\nu_{2p}],\ p=\overline{1,P}, \\ 0 & \text{if } \xi\in[\nu_{2p},\nu_{2p+1}],\ p=\overline{0,P}, \end{cases} \qquad (8.1)$$

where $0=\nu_0\le\nu_1\le\nu_2\le\nu_3\le\cdots\le\nu_{2P-1}\le\nu_{2P}\le\nu_{2P+1}=T$. By W_N, we denote the value at $t=T$ of the solution to (3.1), (3.2) with the control $u=u_N$. The influence of the controls u_N, $N=3,5,7,15$, on the end states of solutions W_N is given in Figure 8.1.

Example 8.2. Let $T=1$, $W^T(x) = \sqrt{\frac{2}{\pi}}x\int_0^T e^{-\frac{x^2}{4\xi}}\frac{1-\xi}{(2\xi)^{3/2}}d\xi$. Let us find the controls $u_N(\xi)=v_N(T-\xi)$, $\xi\in[0,T]$, where v_N is the solution to (4.8) for $N = 2P-1$, $P\in\mathbb{N}$. We use the algorithm given in [6] to find v_N in the form (8.1). By W_N, we denote the value at $t=T$ of the solution to (3.1), (3.2) with the control $u=u_N$. The influence of the controls u_N, $N=3,5,7,15$, on the end states of solutions W_N is given in Figure 8.2.

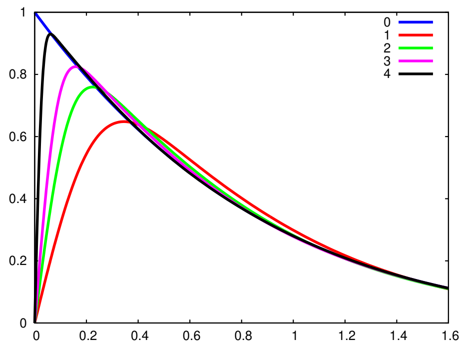


(a) The influence of the control u_N on the end state W_N in the cases: ⓪ $u = 0$, ① $N = 3$, ② $N = 5$, ③ $N = 7$, ④ $N = 15$.

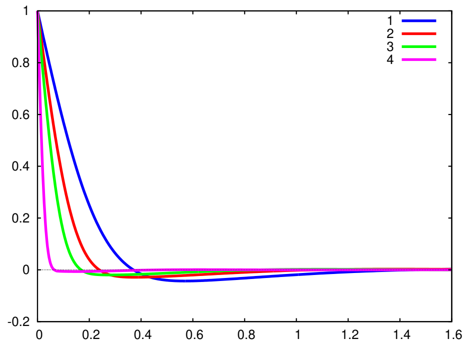


(b) The difference $W^T - W_N$ in the cases: ① $N = 3$, ② $N = 5$, ③ $N = 7$, ④ $N = 15$.

Fig. 8.1: The influence of the control u_N on the end state of the solution to (3.1), (3.2) with $u = u_N$ and $W^T(x) = \sqrt{\frac{2}{\pi}} x \int_0^T e^{-\frac{x^2}{4\xi}} \frac{d\xi}{2(2\xi)^{1/2}}$.



(a) The influence of the control u_N on the end state W_N in the cases: ⓪ $u = 0$, ① $N = 3$, ② $N = 5$, ③ $N = 7$, ④ $N = 15$.



(b) The difference $W^T - W_N$ in the cases: ① $N = 3$, ② $N = 5$, ③ $N = 7$, ④ $N = 15$.

Fig. 8.2: The influence of the control u_N on the end state of the solution to (3.1), (3.2) with $u = u_N$ and $W^T(x) = \sqrt{\frac{2}{\pi}} x \int_0^T e^{-\frac{x^2}{4\xi}} \frac{1-\xi}{(2\xi)^{3/2}} d\xi$.

The following example illustrates the result of Theorem 5.2.

Example 8.3. Let $W^T(x) = 2\sqrt{\frac{2}{\pi}} e^{\frac{1}{4}} e^{-\frac{x^2}{4T}} \sin \frac{x}{\sqrt{2T}}$. Consider the reachability problem for system (3.1), (3.2) with $W^0 = 0$. Denote $V^T = \mathcal{F}W^T$. Then $V^T(\sigma) = -4i\sqrt{\frac{T}{\pi}} e^{-\frac{1}{4}} e^{-T\sigma^2} \sinh \sqrt{2T}\sigma$. Since $V^T = \sum_{n=0}^{\infty} \omega_n \widehat{\psi}_n^T$, then it is easy to see that $V^T(\sigma) = ie^{-T\sigma^2} \sum_{p=0}^{\infty} \sigma^{2p+1} \sum_{n=p}^{\infty} \omega_n h_p^n$, where h_p^n is defined by (5.8) and $\omega_n = \sqrt{\frac{2}{\pi}} \frac{(-1)^n}{2^{2n}(2n+1)!}$.

For each $N \in \mathbb{N}$, denote $g_p^N = \sum_{n=p}^N \omega_n h_p^n$. Denote also

$$V_N(\sigma) = i \sum_{p=0}^N g_p^N \varphi_p(\sigma) = ie^{-T\sigma^2} \sum_{p=0}^N g_p^N \sigma^{2p+1},$$
$$V_N^l(\sigma) = i \sum_{p=0}^N g_p^N \varphi_p^l(\sigma) = ie^{-T\sigma^2} \sum_{p=0}^N g_p^N \sigma^{2p+1} \left(\frac{e^{\sigma^2/l} - 1}{\sigma^2/l} \right)^{p+1}.$$

Then,

$$\left\| V^T - V_N^l \right\|_0 \le \left\| V^T - V_N \right\|_0 + \left\| V_N^l - V_N \right\|_0. \tag{8.2}$$

Using (5.3), we get

$$\begin{aligned} \left\| V^T - V_N \right\|_0 &= \sqrt{\frac{2}{\pi}} \left(\sum_{n=N+1}^\infty \left(\frac{(-1)^n}{2^{2n}(2n+1)!} \right)^2 \sqrt{2\pi T} 2^{2n+1} (2n+1)! \right)^{\frac{1}{2}} \\ &\le \sqrt{8} \left(\frac{2T}{\pi} \right)^{\frac{1}{4}} \sqrt{\frac{\cosh \frac{1}{2}}{2^{2N+3}(2N+3)!}}. \end{aligned} \tag{8.3}$$

We have

$$\left\| V_N^l - V_N \right\|_0 \le \sum_{p=0}^N \left| g_p^N \right| \left\| \varphi_p^l - \varphi_p \right\|_0. \tag{8.4}$$

Substituting h_p^n and ω_n in g_p^N, we obtain

$$\begin{aligned} \left| g_p^N \right| &= \sqrt{\frac{2}{\pi}} \left| \sum_{n=p}^N \frac{(-1)^{n+p+1} 2^{2p+1} (2T)^{p+1} (2n+1)!}{2^{2n}(2n+1)!(n-p)!(2p+1)!} \right| \\ &= 2\sqrt{\frac{2}{\pi}} \frac{(2T)^{p+1}}{(2p+1)!} \left| \sum_{n=p}^N \frac{(-1)^{n-p}}{2^{2(n-p)}(n-p)!} \right| \le 2\sqrt{\frac{2}{\pi}} \frac{(2T)^{p+1}}{(2p+1)!} e^{-\frac{1}{4}}. \end{aligned} \tag{8.5}$$

Evidently, the following three estimates hold:

$$\begin{aligned} &\left| (y+1)^{p+1} - 1 \right| \le (p+1)(y+1)^p y, && y > 0, \\ &\frac{e^z - 1}{z} \le e^z, \quad \frac{e^z - 1}{z} - 1 \le \frac{1}{2} z e^z, && z > 0. \end{aligned}$$

Therefore,

$$\left| \left(\frac{e^{\sigma^2/l} - 1}{\sigma^2/l} \right)^{p+1} - 1 \right| \le (p+1) \left(\frac{e^{\sigma^2/l} - 1}{\sigma^2/l} \right)^p \left(\frac{e^{\sigma^2/l} - 1}{\sigma^2/l} - 1 \right) \le \frac{p+1}{2l} \sigma^2 e^{\frac{(p+1)}{l}\sigma^2}.$$

From here, it follows that

$$\left\| \varphi_p^l - \varphi_p \right\|_0 = \left(2 \int_0^\infty \left(\sigma^{2p+1} e^{-T\sigma^2} \left| \left(\frac{e^{\sigma^2/l} - 1}{\sigma^2/l} \right)^{p+1} - 1 \right| \right)^2 d\sigma \right)^{\frac{1}{2}}$$

$$\leq \left(\frac{(p+1)^2}{2l^2} \int_0^\infty \left(\sigma^{2p+3} e^{-\sigma^2 (T-(p+1)/l)} \right)^2 d\sigma \right)^{\frac{1}{2}}$$

$$\leq \left(\frac{(p+1)^2}{2l^2} \int_0^\infty \left(\sigma^{2p+3} e^{-\frac{3}{4} T \sigma^2} \right)^2 d\sigma \right)^{\frac{1}{2}}, \tag{8.6}$$

if $\frac{p+1}{l} < \frac{T}{4}$. Since $\max_{\sigma > 0} \sigma^{2p+3} e^{-T\sigma^2/2} = \left(\frac{2p+3}{T} \right)^{p+3/2} e^{-(2p+3)/2}$, then we get

$$\left\| \varphi_p^l - \varphi_p \right\|_0 \leq \left(\frac{(p+1)^2}{2l^2} \left(\frac{2p+3}{T} \right)^{2p+3} e^{-(2p+3)} \int_0^\infty e^{-\frac{T\sigma^2}{2}} d\sigma \right)^{\frac{1}{2}}$$

$$\leq \left(\frac{2\pi}{T} \right)^{\frac{1}{4}} \frac{p+1}{l} \frac{2^{p+1/2}}{T^{p+3/2}} \left(\frac{p+2}{e} \right)^{p+2}.$$

From here, using the Stirling formula (4.11), we obtain

$$\left\| \varphi_p^l - \varphi_p \right\|_0 \leq \left(\frac{1}{2\pi T} \right)^{\frac{1}{4}} \frac{\sqrt{p+2}}{l} \frac{2^{p+1/2}}{T^{p+3/2}} (p+2)!. \tag{8.7}$$

According to (8.5), (8.7) and continuing (8.4), we have

$$\left\| V_N^l - V_N \right\|_0 \leq \sum_{p=0}^N 2 \sqrt{\frac{2}{\pi} \frac{(2T)^{p+1}}{(2p+1)!}} e^{-\frac{1}{4}} \left(\frac{1}{2\pi T} \right)^{\frac{1}{4}} \frac{\sqrt{p+2}}{l} \frac{2^{p+1/2}}{T^{p+3/2}} (p+2)!$$

$$= \frac{2^{\frac{11}{4}}}{l} \left(\frac{1}{T^3 \pi^3 e} \right)^{\frac{1}{4}} \sum_{p=0}^N \frac{2^{2p} \sqrt{p+2} (p+2)!}{(2p+1)!}. \tag{8.8}$$

From (8.2), taking into account (8.3) and (8.8), we get

$$\left\| V^T - V_N^l \right\|_0 \leq \sqrt{8} \left(\frac{2T}{\pi} \right)^{\frac{1}{4}} \sqrt{\frac{\cosh \frac{1}{2}}{2^{2N+3} (2N+3)!}}$$

$$+ \frac{2^{\frac{11}{4}}}{l} \left(\frac{1}{T^3 \pi^3 e} \right)^{\frac{1}{4}} \sum_{p=0}^N \frac{2^{2p} \sqrt{p+2} (p+2)!}{(2p+1)!}. \tag{8.9}$$

For the last sum, we have

$$\sum_{p=0}^N \frac{2^{2p} \sqrt{p+2} (p+2)!}{(2p+1)!} \leq \sum_{p=0}^N \frac{(p+1)(p+2)^{3/2}}{p!} \leq 26 + 8e.$$

Therefore, (8.9) takes the form

$$\left\| V^T - V_N^l \right\|_0 \leq \sqrt{8} \left(\frac{2T}{\pi} \right)^{\frac{1}{4}} \sqrt{\frac{\cosh \frac{1}{2}}{2^{2N+3} (2N+3)!}} + 2^{\frac{11}{4}} \left(\frac{1}{T^3 \pi^3 e} \right)^{\frac{1}{4}} \frac{1}{l} (26 + 8e).$$

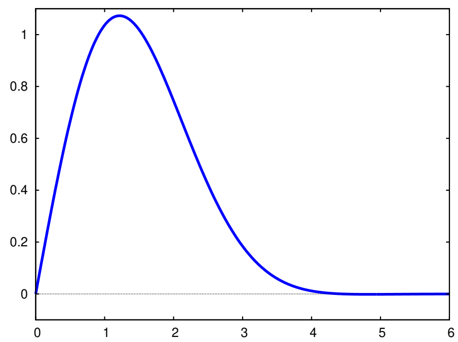
Due to Theorem (5.2), we obtain $W_N^l = -\Phi_T u_N$. With regard to (5.14), we get

$$W_N^l(x) = -x \int_0^T \frac{u_N^l(\xi)}{(2(T-\xi))^{3/2}} e^{-\frac{x^2}{4(T-\xi)}} d\xi,$$

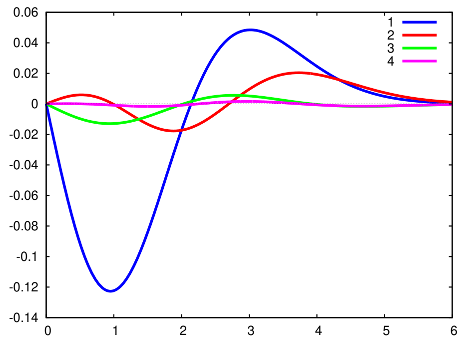
where $u_N^l = \sum_{p=0}^N g_p^N u_l^p$. Some estimates for $\left\|W^T - W_N^l\right\|^0$ are given in Table 8.1 and the influence of the control u_N^l on the end state W_N^l of solution to (3.1), (3.2) with the control $u = u_N^l$ and the target state W^T is shown in Figure 8.3.

	ε_1	ε_2	ε
$N=1,\ l=10$	0.0433	2.1662	2.2095
$N=1,\ l=100$	0.0433	0.2167	0.2600
$N=2,\ l=100$	0.0034	0.3588	0.3622
$N=2,\ l=1000$	0.0034	0.0359	0.0393

Table 8.1: The estimates for $\left\|W^T - W_N^l\right\|^0$, $\varepsilon_1 = \sqrt{8}\left(\frac{2T}{\pi}\right)^{\frac14}\sqrt{\frac{\cosh\frac12}{2^{2N+3}(2N+3)!}}$, $\varepsilon_2 = 2^{\frac{11}{4}}\left(\frac{1}{T^3\pi^3 e}\right)^{\frac14}\frac1l\sum_{p=0}^N \frac{2^{2p}\sqrt{p+2}(p+2)!}{(2p+1)!}$, $\varepsilon = \varepsilon_1 + \varepsilon_2$ (see (8.9)).



(a) The given $W^T(x)$.



(b) The difference $W^T - W_N^l$ in the cases: ① $N=1$, $l=10$; ② $N=1$, $l=100$; ③ $N=2$, $l=100$; ④ $N=2$, $l=1000$.

Fig. 8.3: The influence of the control u_N^l on the end state W_N^l of the solution to (3.1), (3.2) with $u = u_N^l$ and $W^T(x) = \frac{4}{\sqrt{2\pi}} e^{\frac14} e^{-\frac{x^2}{4T}} \sin\frac{x}{\sqrt{2T}}$.

References

[1] V.R. Cabanillas, S.B. De Menezes, and E. Zuazua, *Null controllability in unbounded domains for the semilinear heat equation with nonlinearities involving gradient terms,* J. Optim. Theory Appl. **110** (2001), 245–264.

[2] P. Cannarsa, P. Martinez, and J. Vancostenoble, *Null controllability of the heat equation in unbounded domains by a finite measure control region,* ESAIM Control Optim. Calc. Var. **10** (2004), 381–408.

[3] J. Darde and S. Ervedoza, *On the reachable set for the one-dimensional heat equation,* SIAM J. Control Optim. **56** (2018), 1692–1715.

[4] L.V. Fardigola, *Transformation Operators and Influence Operators in Control Problems,* Thesis (Dr. Hab.), Kharkiv, 2016 (Ukrainian).

[5] S.G. Gindikin and L.R. Volevich, *Distributions and Convolution Equations,* Gordon and Breach Sci. Publ., Philadelphia, 1992.

[6] L. Gosse and O. Runberg, Resolution of the finite Markov moment problem, Comptes Rendus Mathematiques, **341** (2005), 775–789.

[7] Handbook of Mathematical Functions with Formulas Graphs and Mathematical Tables, *Eds. M. Abramowitz and I.A. Stegun,* National Bureau of Standards, Applied Mathematics Series, **55**, Washington, DC, 1972.

[8] S. Micu and E. Zuazua, *On the lack of null controllability of the heat equation on the half-space,* Port. Math. (N.S.) **58** (2001), No. 1, 1–24.

[9] S. Micu and E. Zuazua, *On the lack of null controllability of the heat equation on the half-line,* Trans. Amer. Math. Soc. **353** (2001), No. 4, 1635–1659.

[10] A. Munch and P. Pedregal, *Numerical null controllability of the heat equation through a least squares and variational approach,* European J. Appl. Math. **25** (2014), 277–306.

[11] L. de Teresa and E. Zuazua, *Approximate controllability of a semilinear heat equation in unbounded domains,* Nonlinear Anal. **37** (1999), No. 8, 1059–1090.

[12] E. Zuazua, *Some problems and results on the controllability of partial differential equations,* Proceedings of the Second European Congress of Mathematics, Budapest, July 1996, Progress in Mathematics, **169**, Birkhäuser Verlag, Basel, 276–311.

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Проблеми досяжності та керованості для рівняння теплопровідності на півосі

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У роботі досліджено проблеми керованості, наближеної керованості, досяжності та наближеної досяжності для керованої системи $w_t = w_{xx}$, $w(0,\cdot) = u$, $x > 0$, $t \in (0,T)$, де $u \in L^\infty(0,T)$ є керуванням. Доведено, що кожний кінцевий стан цієї системи є наближено досяжним за заданий час T, та кожний її початковий стан є наближено керованим за заданий час T. Одержано необхідну і достатню умову досяжності за заданий час T в термінах розв'язності степеневої проблеми моментів Маркова.

Показано також, що не існує початкових даних, які є 0-керованими за заданий час T. Результати проілюстровано прикладами.

Ключові слова: рівняння теплопровідності, керованість, наближена керованість, досяжність, наближена досяжність, степенева проблема моментів Маркова.