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Invariant Subspaces on KPC-Hypergroups

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In this paper, we study translation invariant function spaces and spectral analysis on KPC-hypergroups and describe a correspondence between ideals in the algebra of compactly supported measures and varieties of continuous functions on a KPC-hypergroup.

Key words: DJS-hypergroup, KPC-hypergroup, spectral analysis, spectral synthesis.

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1. Introduction and preliminaries

Hypergroups, as extensions of locally compact groups, were introduced in a series of papers by Dunkl [3], Jewett [4], and Spector [6] in 70's (we refer to this concept as DJS-hypergroup). For more details about DJS-hypergroups refer to [1]. In 2010, Kalyuzhnyi, Podkolzin, and Chapovsky introduced new axioms for hypergroups which are extensions of DJS-hypergroups and also generalized normal hypercomplex systems [5] (see also [13–15]). We refer to this notion as a KPC-hypergroup. They studied harmonic analysis on KPC-hypergroups and showed that there is an example of a compact KPC-hypergroup related to the generalized Tchebycheff polynomials, which is not a DJS-hypergroup [5]. In this paper, we initiate spectral analysis on KPC-hypergroups and give a correspondence between ideals and invariant subspaces of $\mathcal{C}(Q)$, where Q is a KPC-hypergroup. Spectral analysis and spectral synthesis were studied on locally compact groups and DJS-hypergroups by Székelyhidi in [7–9, 11]. As the main result, we give a necessary and sufficient condition for the presence of spectral analysis of a given variety. Spectral synthesis problems on KPC-hypergroups will be treated in a subsequent paper.

Let Q be a locally compact Hausdorff space. We denote by $\mathcal{C}(Q)$ the space of all continuous complex-valued functions on Q, and by $\mathcal{M}_c(Q)$ the space of all compactly supported complex Radon measures on Q. For the sake of simplicity, functions in $\mathcal{C}(Q)$ will be called *functions*, and measures in $\mathcal{M}_c(Q)$ will be called *measures*.

Let Δ be a function from $\mathcal{C}(Q)$ into $\mathcal{C}(Q \times Q)$, $f, g \in \mathcal{C}(Q)$ and $p, q, r \in Q$. We write

$$[(\Delta \times \mathrm{id}) \circ \Delta](f)(p, q, r) := \Delta(\Delta f(p, \cdot))(q, r),$$

$$[(\mathrm{id} \times \Delta) \circ \Delta](f)(p, q, r) := \Delta(\Delta f(\cdot, q))(p, r)$$

and

$$(f \otimes g)(p, q) := f(p)g(q).$$

Definition 1.1. Let Q be a locally compact Hausdorff space with an involutive homeomorphism $\star : Q \longrightarrow Q$ satisfying the following conditions:

(H_1) there exists an element $e \in Q$ such that $e^\star = e$;

(H_2) there exists a $\mathbb{C}$-linear mapping $\Delta : \mathcal{C}(Q) \to \mathcal{C}(Q \times Q)$ such that

(i) Δ is coassociative, that is,

$$(\Delta \times \mathrm{id}) \circ \Delta = (\mathrm{id} \times \Delta) \circ \Delta; \tag{1.1}$$

(ii) Δ is positive, that is, for each non-negative function f, the function Δf is non-negative;

(iii) Δ preserves the identity, that is, $(\Delta 1)(p, q) = 1$ for all $p, q \in Q$;

(iv) for all compactly supported functions f, g, the functions $(1 \otimes f) \cdot (\Delta g)$ and $(f \otimes 1) \cdot (\Delta g)$ are compactly supported;

(H_3) we have $(\Delta f)(e, p) = (\Delta f)(p, e) = f(p)$, for all $f \in \mathcal{C}(Q)$ and p in Q;

(H_4) the function $\check{f}$ defined by $\check{f}(q) = f(q^\star)$ for each function f satisfies

$$(\Delta \check{f})(p, q) = (\Delta f)(q^\star, p^\star); \tag{1.2}$$

(H_5) there exists a non-zero positive Radon measure λ with support Q such that

$$\int_Q (\Delta f)(p, q)g(q)d\lambda(q) = \int_Q f(q)(\Delta g)(p^\star, q)d\lambda(q) \tag{1.3}$$

for each p in Q, whenever f, g are functions and at least one of them is compactly supported; the measure λ will be called a left Haar measure on Q.

Then $(Q, \star, e, \Delta, m)$, or simply Q, is called a locally compact KPC-hypergroup.

A KPC-hypergroup Q is called cocommutative if for all $f \in \mathcal{C}(Q)$ and all $p, q \in Q$, $\Delta f(p, q) = \Delta f(q, p)$. Throughout this paper Q always denotes a locally compact cocommutative KPC-hypergroup.

Definition 1.2. For measures μ, ν, the convolution $\mu * \nu$ is defined by

$$(\mu * \nu)(f) = \int_Q \int_Q (\Delta f)(p, q)d\mu(p)d\nu(q), \tag{1.4}$$

whenever f is a function in $\mathcal{C}(Q)$.

2. Varieties and ideals

In the sequel, the space $\mathcal{C}(Q)$ is equipped with the *topology of compact convergence*. This topology is defined by the family of seminorms (p_C), where C runs through the family of all compact subsets of Q, and for each $f \in \mathcal{C}(Q)$, $p_C(f)$ is the uniform norm of the restriction of f to C. This family of seminorms is separating (by Tietze's Extension Theorem [2, p. 87]), and so the generated topology is a Hausdorff topology on $\mathcal{C}(Q)$, hence the latter is a locally convex topological vector space. In the topology of compact convergence, the topological dual of $\mathcal{C}(Q)$ (equipped with this topology) is the space $\mathcal{M}_c(Q)$, and if $\mathcal{M}_c(Q)$ is equipped with the weak*-topology (the topology induced by $\mathcal{C}(Q)$), then its topological dual identifies with $\mathcal{C}(Q)$.

Definition 2.1. For each x, y in Q, the *translate* of a function f by y is defined by $(\tau_y f)(x) := \Delta f(x, y)$.

By Definition 1.1, we have $\tau_e f(x) = \Delta f(x, e) = f(x)$. So, $\tau_e = I$, the identity operator on $\mathcal{C}(Q)$.

Definition 2.2. Let Q be a KPC-hypergroup. For each measure μ and function f, we define

$$(\mu * f)(x) := \int_Q \Delta f(x, y^\star) d\mu(y) \tag{2.1}$$

where x is in Q. Clearly, $\mu * f$ and $f * \mu$ are functions. In particular, we have $\tau_y f(x) = (\delta_{y^\star} * f)(x)$.

It is easy to check that $\mathcal{M}_c(Q)$ is a (topological) unital algebra and $\mathcal{C}(Q)$ is a (topological) vector module over $\mathcal{M}_c(Q)$.

Definition 2.3. A closed linear subspace E of $\mathcal{C}(Q)$ is called a *variety* on Q if for all f in E and y in Q the translate $\tau_y f$ is in E. The smallest variety containing a given function f is denoted by $\tau(f)$, and it is called the *variety generated by* f or the *variety of* f. Clearly, the functions which are non-zero scalar multiples, or translates of each other generate the same variety.

For any set V in $\mathcal{C}(Q)$, its orthogonal complement $V^\perp$ in $M_c(Q)$ is the set of all measures in $\mathcal{M}_c(Q)$ which vanish on V. Clearly, $V^\perp$ is a closed linear subspace of $\mathcal{M}_c(Q)$. We have also the dual correspondence: the orthogonal complement of any subset I of $\mathcal{M}_c(Q)$ is $I^\perp$, the set of all functions in $\mathcal{C}(Q)$, which belong to the kernel of all linear functionals in I. Clearly, it is a closed linear subspace of $\mathcal{C}(Q)$. By the Hahn–Banach theorem, we have the obvious relations $V = V^{\perp\perp}$ and also $I = I^{\perp\perp}$ for any closed linear subspace V of $\mathcal{C}(Q)$ and for any closed linear subspace I of $\mathcal{M}_c(Q)$. In the case of varieties, the annihilators can be characterized.

Proposition 2.4. *Let Q be a cocommutative KPC-hypergroup, V be a variety in $\mathcal{C}(Q)$ and I be a closed ideal in $\mathcal{M}_c(Q)$. Then $V^\perp$ is a closed ideal in $\mathcal{M}_c(Q)$ and $I^\perp$ is a variety in $\mathcal{C}(Q)$.*

Proof. Let V be a variety in $\mathcal{C}(Q)$, $f \in V$, $\mu \in V^{\perp}$, and $\nu \in \mathcal{M}_c(Q)$. Then

$$(\mu * \nu)(f) = \int_Q \int_Q \Delta f(x,y) d\mu(x) d\nu(y) = \int_Q \left(\int_Q \tau_y f(x) d\mu(x) \right) d\nu(y) = 0.$$

Hence, $\mu * \nu \in V^{\perp}$, and $V^{\perp}$ is a closed ideal in $\mathcal{M}_c(Q)$. Now, let I be a closed ideal in $\mathcal{M}_c(Q)$. Then for any $\mu \in I$, $f \in I^{\perp}$ and $\nu \in \mathcal{M}_c(Q)$, we have

$$\begin{aligned} 0 = \int_Q f d(\mu * \nu) &= \int_Q \int_Q \Delta f(x,y) d\mu(x) d\nu(y) \\ &= \int_Q \mu(\tau_y f) d\nu(y), \end{aligned}$$

that is, the function $y \mapsto \mu(\tau_y f)$ annihilates $\mathcal{M}_c(Q)$. Therefore, by Hahn-Banach theorem, for all $y \in Q$, $\tau_y f$ is in $I^{\perp}$ and so $I^{\perp}$ is a variety in $\mathcal{C}(Q)$. □

In particular, we have a one-to-one correspondence between the closed ideals of $\mathcal{M}_c(Q)$ and the closed translation invariant subspaces of $\mathcal{C}(Q)$.

Another important concept is the annihilator.

Definition 2.5. Given a subset V in $\mathcal{C}(Q)$, its annihilator $\operatorname{Ann} V$ is the set of all measures in $\mathcal{M}_c(Q)$ satisfying $\mu * f = 0$ for each f in V. The dual concept is the annihilator $\operatorname{Ann} I$ of a subset I in $\mathcal{M}_c(Q)$, which is the set of all functions f satisfying $\mu * f = 0$ for each μ in I.

It is obvious that $\operatorname{Ann} V$ is a closed linear subspace in $\mathcal{M}_c(Q)$ and $\operatorname{Ann} I$ is a closed linear subspace in $\mathcal{C}(Q)$. With the notation

$$\check{V} = \{\check{f} : f \in V\}, \ \ \check{I} = \{\check{\mu} : \ \ \mu \in I\}$$

we have $\operatorname{Ann} V = (\check{V})^{\perp}$ and $\operatorname{Ann} I = (\check{I})^{\perp}$. Here $\check{\mu}$ is the measure defined by $\check{\mu}(f) = \mu(\check{f})$ whenever μ is a measure and f is a function.

Proposition 2.6. *For each variety V in $\mathcal{C}(Q)$, its annihilator* $\operatorname{Ann} V$ *is a closed ideal in $\mathcal{M}_c(Q)$, and* $\operatorname{Ann} V = (\check{V})^{\perp}$. *Similarly, for each ideal I in $\mathcal{M}_c(Q)$, its annihilator* $\operatorname{Ann} I$ *is a variety in $\mathcal{C}(Q)$, and* $\operatorname{Ann} I = (\check{I})^{\perp}$.

Proof. Let $\nu \in \mathcal{M}_c(Q)$ and $\mu \in \operatorname{Ann} V$. Then for each $f \in V$, $(\mu * \nu) * f = \nu * (\mu * f) = 0$. Hence, $\operatorname{Ann} V$ is a closed ideal of $\mathcal{M}_c(Q)$. Let $\mu \in \operatorname{Ann} V$. For each $f \in V$, we have

$$\mu(\check{f}) = \int_Q \check{f}(x) d\mu(x) = \int_Q \Delta f(x^\star, e) d\mu(x) = \int_Q \Delta f(e, x^\star) d\mu(x) = (\mu * f)(e) = 0,$$

and so $\mu \in (\check{V})^{\perp}$. Conversely, let $\mu \in (\check{V})^{\perp}$, $f \in V$ and $x \in Q$. Then

$$(\mu * f)(x) = \int_Q \Delta f(x, y^\star) d\mu(y) = \int_Q (\tau_x f)\check{}(y) d\mu(y) = \mu((\tau_x f)\check{}) = 0$$

since $\tau_x f \in V$. The second part is proved in a similar way. □

Proposition 2.7. *For each variety* $V \subseteq W$ *in* $\mathcal{C}(Q)$ *we have* $\operatorname{Ann} V \supseteq \operatorname{Ann} W$*, and for each ideal* $I \subseteq J$ *in* $\mathcal{M}_c(Q)$ *we have* $\operatorname{Ann} I \supseteq \operatorname{Ann} J$*. In addition, we have* $\operatorname{Ann}(\operatorname{Ann} V) = V$ *and* $\operatorname{Ann}(\operatorname{Ann} I) \supseteq I$*. In particular,* $V \neq W$ *implies* $\operatorname{Ann} V \neq \operatorname{Ann} W$*.*

Proof. The proof is straightforward. □

Corollary 2.8. *The varieties in* $\mathcal{C}(Q)$ *are exactly the closed vector submodules of the vector module* $\mathcal{C}(Q)$*.*

Proof. Clearly, every closed vector submodule is a variety. Conversely, we have to show that if f is in V, then $\mu * f$ is in $V = \operatorname{Ann}\operatorname{Ann} V$ for each μ in $\mathcal{M}_c(Q)$, that is, $\mu * f$ is annihilated by any element of $\operatorname{Ann} V$. Let ν be in $\operatorname{Ann} V$. As $\operatorname{Ann} V$ is an ideal, we have $\nu * \mu$ is in $\operatorname{Ann} V$, hence

$$\nu * (\mu * f) = (\nu * \mu) * f = 0,$$

that is, $\mu * f$ is in $\operatorname{Ann}\operatorname{Ann} V$. □

Proposition 2.9. *Let* $(V_i)_{i\in I}$ *be a family of varieties in* $\mathcal{C}(Q)$*, and* $(I_i)_{i\in I}$ *be a family of closed ideals in* $\mathcal{M}_c(Q)$*. Then*

$$\Big(\sum_{i\in I} V_i\Big)^{\perp} = \bigcap_{i\in I} V_i^{\perp}, \qquad \Big(\sum_{i\in I} I_i\Big)^{\perp} = \bigcap_{i\in I} I_i^{\perp}.$$

Proof. Let $\mu \in \big(\sum_{i\in I} V_i\big)^{\perp}$. Then for each $i \in I$ we have $\mu \in V_i^{\perp}$ since $V_i \subseteq \sum_{i\in I} V_i$, and so $\mu \in \bigcap_{i\in I} V_i^{\perp}$. Conversely, let $\mu \in \bigcap_{i\in I} V_i^{\perp}$. Then μ annihilates any finite sum of the elements in $\bigcup_{i\in I} V_i$. So, by continuity, we have $\mu \in \big(\sum_{i\in I} V_i\big)^{\perp}$.

Now let f be in $\big(\sum_{i\in I} I_i\big)^{\perp}$. Then f is annihilated by finite sums of measures taken from the I_i's. Hence, in particular, f is annihilated by each I_i, hence f is in $I_i^{\perp}$ for each i and thus f is in $\bigcap_{i\in I} I_i^{\perp}$. The reverse inclusion is equally obvious. □

Proposition 2.10. *Let* $(V_i)_{i\in I}$ *be a family of varieties in* $\mathcal{C}(Q)$*, and* $(I_i)_{i\in I}$ *be a family of closed ideals in* $\mathcal{M}_c(Q)$*. Then*

$$\Big(\bigcap_{i\in I} V_i\Big)^{\perp} = \sum_{i\in I} V_i^{\perp}, \qquad \Big(\bigcap_{i\in I} I_i\Big)^{\perp} = \sum_{i\in I} I_i^{\perp}.$$

Proof. The statements are immediate consequences of the previous result and of the relations $V^{\perp\perp} = V$ and $I^{\perp\perp} = I$ for each variety V and closed ideal I. □

We note that, by the relations $\operatorname{Ann} V = (\check{V})^{\perp}$ and $\operatorname{Ann} I = (\check{I})^{\perp}$, we have similar statements about the annihilators of sums and intersections.

Definition 2.11. A variety $V \subseteq \mathcal{C}(Q)$ is called *decomposable* if there are two proper subvarieties whose algebraic sum is dense in V. Otherwise it is called *indecomposable*.

We recall that an ideal in a commutative ring is called *irreducible* if it is not the intersection of two ideals different from it.

Corollary 2.12. *A variety V is indecomposable if and only if $V^{\perp}$ is irreducible.*

Proof. This is a consequence of Proposition 2.9. □

3. Exponentials and spectral analysis

Definition 3.1. A non-zero function f is called an exponential (on Q) if for all $p, q \in Q$ we have

$$\Delta f(p, q) = f(p)f(q).$$

Proposition 3.2. *A variety on a KPC-hypergroup is one-dimensional if and only if it is generated by an exponential.*

Proof. If u is an exponential function, then, by definition, every $\tau_y u$ is a constant multiple of u, Hence, all $\tau_y u$'s form a one-dimensional vector space in $\mathcal{C}(Q)$ and thus $\tau(u)$ is one-dimensional. Conversely, by assumption, every $\tau_y u$ is a constant multiple of u, that is, for each y in Q there is a complex number $\alpha(y)$ such that

$$\Delta u(x, y) = \alpha(y)u(x)$$

for each x in Q. It follows that $u(y) = \Delta u(e, y) = u(e)\alpha(y)$. Hence,

$$\Delta\alpha(x, y) = \alpha(y)\alpha(x),$$

and thus $\alpha \neq 0$ is an exponential. On the other hand, α and u generate the same variety, hence $\tau(u)$ is generated by the exponential α. □

Definition 3.3. Let f be a function, and let y be in Q. Then the modified difference $D_{f;y}$ is defined by

$$D_{f;y} := \delta_{y^\star} - f(y)\delta_e.$$

For any positive integer n and $y_1, y_2, \ldots, y_n \in Q$, we write

$$D_{f;y_1,y_2,\ldots,y_n} = \prod_{i=1}^{n}[\delta_{y_i^\star} - f(y_i)\delta_e].$$

Theorem 3.4. *Let f be a function on Q, and M_f denote the closed ideal generated by all modified differences $D_{f;y}$ where $y \in Q$. Then the followings are equivalent:*

(1) *f is an exponential,*

(2) *the ideal* M_f *is proper and* $f(e) = 1$,

(3) *the ideal* M_f *is maximal and* $f(e) = 1$,

(4) $M_f = \operatorname{Ann} \tau(f)$ *and* $f(e) = 1$.

Proof. (1) ⇒ (2): Let f be an exponential. Then $f(e)f(e) = \Delta f(e, e) = f(e)$. Since $f \neq 0$, we have $f(e) = 1$. For each $y \in Q$,

$$(D_{f;y} * f)(x) = \Delta f(x, y) - f(y)f(x) = 0.$$

It follows that f is in $\operatorname{Ann} M_f$, hence $\operatorname{Ann} M_f \neq 0$, and, by $M_f = \operatorname{Ann} \operatorname{Ann} M_f$, M_f is proper.

(2) ⇒ (3): Let M_f be a proper ideal and $f(e) = 1$. Then there is $g \neq 0$ in $\operatorname{Ann} M_f$, and we have

$$\Delta g(x, y) - f(y)g(x) = D_{f;y} * g(x) = 0,$$

where $x, y \in Q$. By Definition 1.1, $g = g(e)f$. It follows that $\operatorname{Ann} M_f$ is one-dimensional, hence $M_f = \operatorname{Ann} \operatorname{Ann} M_f$ is a maximal ideal.

(3) ⇒ (4): Let M_f be a maximal ideal and $f(e) = 1$. If $g \neq 0$ is in $\operatorname{Ann} M_f$, then, in the same way as above, we have $g = g(e)f$. In particular, $g(e) \neq 0$, hence $f \in \operatorname{Ann} M_f$. Therefore $\tau(f) \subseteq \operatorname{Ann} M_f$, and $M_f = \operatorname{Ann} \operatorname{Ann} M_f \subseteq \operatorname{Ann} \tau(f)$. But $\operatorname{Ann} \tau(f)$ is a proper ideal and, by the maximality of M_f, we have $M_f = \operatorname{Ann} \tau(f)$.

(4) ⇒ (1): Let $M_f = \operatorname{Ann} \tau(f)$ and $f(e) = 1$. We have $f \in \tau(f) = \operatorname{Ann} M_f$. Then for each $x, y \in Q$,

$$0 = D_{f;y} * f(x) = \Delta f(x, y) - f(y)f(x),$$

that is, f is an exponential. □

Definition 3.5. The maximal ideal M in $\mathcal{M}_c(Q)$ is called an exponential maximal ideal if $M = M_m$ for some exponential $m : Q \to \mathbb{C}$.

Theorem 3.6. *The maximal ideal* M *in* $\mathcal{M}_c(Q)$ *is exponential if and only if the residue ring* $\mathcal{M}_c(Q)/M$ *is topologically isomorphic to the complex field.*

Proof. Suppose that M is a maximal ideal in $\mathcal{M}_c(Q)$, and $\Phi : \mathcal{M}_c(Q)/M \to \mathbb{C}$ is a topological isomorphism. Then the mapping $\Psi : \mathcal{M}_c(Q) \to \mathbb{C}$, defined by $\Psi(\mu) := \Phi(\mu + M)$ for $\mu \in \mathcal{M}_c(Q)$, is a multiplicative linear functional on $\mathcal{M}_c(Q)$. Since $\mathcal{M}_c(Q)^* = C(Q)$, there exists a function $f \in \mathcal{C}(Q)$ such that for all $\mu \in \mathcal{M}_c(Q)$, $\Psi(\mu) = \mu(\check{f})$. In particular, for each $x \in Q$, we have $\Psi(\delta_{x^*}) = f(x)$. Hence $f(e) = \Psi(\delta_e) = 1$ and for each $x, y \in Q$,

$$\Delta f(x, y) = \Psi((\delta_x * \delta_y)\check{}) = \Psi(\delta_{y^*} * \delta_{x^*}) = \Psi(\delta_{y^*})\Psi(\delta_{x^*}) = f(y)f(x).$$

This implies that f is an exponential. For each $y \in Q$, we have

$$\Psi(\delta_{y^*} - f(y)\delta_e) = \Psi(\delta_{y^*}) + f(y)\Psi(\delta_e) = f(y) - f(y) = 0.$$

Thus, for each $y \in Q$, $D_{f;y}$ is in $\ker \Psi = M$. Therefore, $M_f \subseteq M$, and since M is a maximal ideal, we have $M_f = M$, i.e., M is an exponential maximal ideal. Conversely, let $M = M_f$ for some exponential function f. Then the mapping $\Psi : \mathcal{M}_c(Q) \to \mathbb{C}$ defined by $\Psi(\mu) := \mu(\check{f})$ is a multiplicative linear functional with $\ker \Psi = M$. Therefore $\mathcal{M}_c(Q)/M$ is topologically isomorphic to $\mathbb{C}$. □

Definition 3.7. Let Q be a cocommutative KPC-hypergroup. We say that spectral analysis holds for a variety E on Q if every non-zero subvariety of E contains an exponential.

Theorem 3.8. *Spectral analysis holds for the variety E if and only if every maximal ideal containing $\operatorname{Ann} E$ is exponential, or equivalently, every maximal ideal of the residue ring $\mathcal{M}_c(Q)/\operatorname{Ann} E$ is exponential.*

Proof. Let spectral analysis hold for the variety E, and M be a maximal ideal containing $\operatorname{Ann} E$. Then, by Proposition 2.6, $\operatorname{Ann} M$ is a subvariety of E. Thus there is an exponential f in $\operatorname{Ann} M$, and the mapping $\Psi : \mathcal{M}_c(Q) \to \mathbb{C}$, defined by $\Psi(\mu) := \mu(\check{f})$, is a multiplicative linear functional with $M \subseteq \ker \Psi = M_f$. This implies that $M = M_f$ since M_f is a proper ideal (Theorem 3.4). Conversely, let every maximal ideal containing $\operatorname{Ann} E$ be exponential, and V be a non-zero subvariety of E. For a maximal ideal M containing $\operatorname{Ann} V$, $\operatorname{Ann} E \subseteq \operatorname{Ann} V \subseteq M$. Thus, there exists an exponential f such that $M = M_f = \operatorname{Ann} \tau(f)$. Then $f \in \tau(f) = \operatorname{Ann} \operatorname{Ann} \tau(f) \subseteq \operatorname{Ann} \operatorname{Ann} V = V$. □

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References

[1] W.R. Bloom and H. Heyer, *Harmonic Analysis of Probability Measures on Hypergroups*, De Gruyter, Berlin, 1995.

[2] J.B. Conway, *A Course in Point Set Topology*, Springer, Cham, 2014.

[3] C.F. Dunkl, *The measure algebra of a locally compact hypergroup*, Trans. Amer. Math. Soc. **179** (1973), 331–348.

[4] R.I. Jewett, *Spaces with an abstract convolution of measures*, Adv. Math. **18** (1975), 1–101.

[5] A.A. Kalyuzhnyi, G.B. Podkolzin, and Yu.A. Chapovski, *Harmonic analysis on a locally compact hypergroup*, Methods Funct. Anal. Topology **16** (2010), 304–332.

[6] R. Spector, *Measures invariantes sur les hypergroups*, Trans. Amer. Math. Soc. **239** (1978), 147–165.

[7] L. Székelyhidi, *Spectral analysis and spectral synthesis on polynomial hypergroups*, Monats. Math. **141** (2004), 33–43.

[8] L. Székelyhidi, *Spectral synthesis on multivariate polynomial*, Monats. Math. **153** (2008), 145–152.

[9] L. Székelyhidi, *Spectral synthesis problems on locally compact groups*, Monats. Math. **161** (2010), 223–232.

[10] L. Székelyhidi, *Exponential polynomials on commutative hypergroups*, Arch. Math. **101** (2013), No. 4, 341–347.

[11] L. Székelyhidi, *Spectral synthesis problems on hypergroups*, Annales Univ. Sci. Budapest., Sect. Comp. **39** (2013), 439–447.

[12] L. Székelyhidi, *Characterization of exponential polynomials on hypergroups*, Ann. Funct. Anal. **5** (2014), No. 2, 53–60.

[13] S.M. Tabatabaie and F. Haghighifar, *The weighted KPC–hypergroups*, Gen. Math. Notes **34** (2016), 29–38.

[14] S.M. Tabatabaie and F. Haghighifar, *The associated measure on locally compact cocommutative KPC-hypergroups*, Bull. Iran. Math. Soc. **43** (2017), 1–15.

[15] S.M. Tabatabaie and F. Haghighifar, *Translation invariant mappings on KPC-hypergroups*, Int. J. Nonlinear Anal. Appl. **8** (2017), No. 2, 99–107.

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Інваріантні підпростори на KPC-гіпергрупах

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У даній роботі ми вивчаємо простори функцій, інваріантних відносно зсувів, і спектральний аналіз на KPC-гіпергрупах та описуємо відповідність між ідеалами в алгебрі мір з компактними носіями і многовидами неперервних функцій на KPC-гіпергрупі.

Ключові слова: DJS-гіпергрупа, KPC-гіпергрупа, спектральний аналіз, спектральний синтез.