

Analogue of Hayman's Theorem and its Application to Some System of Linear Partial Differential Equations

Andriy Bandura and Oleh Skaskiv

We used the analogue of known Hayman's theorem to study the boundedness of \mathbf{L} -index in joint variables of entire solutions of some linear higher-order systems of PDE's and found sufficient conditions providing the boundedness, where $\mathbf{L}(z) = (l_1(z), \dots, l_n(z))$, $l_j : \mathbb{C}^n \rightarrow \mathbb{R}_+$ is a continuous function $j \in \{1, \dots, n\}$. Growth estimates of these solutions are also obtained. We proposed the examples of systems of PDE's which prove the exactness of these estimates for entire solutions. The obtained results are new even for the one-dimensional case because of the weakened restrictions imposed on the positive continuous function l .

Key words: entire function, bounded \mathbf{L} -index in joint variables, linear higher-order systems of PDE, analytic theory of PDE, entire solution, linear higher-order differential equation.

Mathematical Subject Classification 2010: 32W50, 32A15, 32A22, 35G35, 32A40.

1. Introduction

W. K. Hayman [19] proved that if $f(z)$ is analytic in $|z| < 2p$, where it satisfies

$$|f^{(p)}(z)| \leq \max_{0 \leq \nu \leq p-1} |f^{(\nu)}(z)|, \quad (1.1)$$

then $f(z)$ cannot have more than $(p-1)$ zeros in $|z| < \frac{\sqrt{p}}{e\sqrt{20}}$. Q.I. Rahman, J. Stankiewicz, V. Singh, and R.M. Goel [25, 30] refined this result and enlarged the value $\frac{\sqrt{p}}{e\sqrt{20}}$.

On the other hand, an entire function f is called a function of bounded index [22] if there exists a nonnegative integer p_0 such that

$$\frac{|f^{(p)}(z)|}{p!} \leq \max_{0 \leq j \leq p_0} \frac{|f^{(j)}(z)|}{j!}$$

for all $z \in \mathbb{C}$ and for all $p \in \mathbb{Z}_+$. In order that an entire function f be of bounded index [19], it is necessary and sufficient that (1.1) be satisfied for all $z \in \mathbb{C}$.

In the theory of functions of bounded index this necessary and sufficient condition and its generalizations [1, 4, 15, 21, 28] are known as Hayman's Theorem. The criterion is very convenient [6, 11, 14] for studying the boundedness of index of entire solutions of ordinary or partial differential equations. The functions of this class have good properties: sharp growth estimates, uniform distribution of zeros in some sense, certain regular behavior of the solution, etc.

There are two approaches to introduce and study the index boundedness in \mathbb{C}^n . The first approach uses a slice function $g_{z_0}(t) := F(z^0 + t\mathbf{b})$, $t \in \mathbb{C}$, where $z^0 \in \mathbb{C}^n$ is an arbitrary fixed point, $n \geq 2$, $\mathbf{b} \in \mathbb{C}^n \setminus \{0\}$ is a given direction in \mathbb{C}^n , $F : \mathbb{C}^n \rightarrow \mathbb{C}$ is an entire function. Applying the slice function and directional derivative, we considered the functions of bounded L -index in direction (see the definition and properties in [5–7]). There were obtained sufficient conditions for the boundedness of L -index in direction of entire solutions of some linear PDE's [5–7]. The second approach is based on partial derivatives. They are a background for the concept of entire function of bounded index in joint variables (see the definition and inequality (1.2) below).

Let $\mathbf{L}(z) = (l_1(z), \dots, l_n(z))$, where $l_j(z)$ are positive continuous functions of $z \in \mathbb{C}^n$, $j \in \{1, 2, \dots, n\}$. An entire function $F(z)$, $z \in \mathbb{C}^n$, is called a function of bounded \mathbf{L} -index in joint variables [2] if there exists a number $m \in \mathbb{Z}_+$ such that for all $z \in \mathbb{C}^n$ and $J = (j_1, j_2, \dots, j_n) \in \mathbb{Z}_+^n$:

$$\frac{|F^{(J)}(z)|}{J! \mathbf{L}^J(z)} \leq \max \left\{ \frac{|F^{(K)}(z)|}{K! \mathbf{L}^K(z)} : K \in \mathbb{Z}_+^n, \|K\| \leq m \right\}, \tag{1.2}$$

where for partial derivatives of the entire function $F(z) = F(z_1, \dots, z_n)$ we use the notation $F^{(K)}(z) = \frac{\partial^{\|K\|} F}{\partial z^K} = \frac{\partial^{k_1 + \dots + k_n} F}{\partial z_1^{k_1} \dots \partial z_n^{k_n}}$, and $\mathbf{L}^K(z) = l_1^{k_1}(z) \dots l_n^{k_n}(z)$, $K! = k_1! \cdot \dots \cdot k_n!$, $\|K\| = k_1 + \dots + k_n$, $K = (k_1, \dots, k_n) \in \mathbb{Z}_+^n$.

If $l_j(z_j) = l_j(|z_j|)$ for every $j \in \{1, 2, \dots, n\}$, then we obtain a concept of entire functions of bounded \mathbf{L} -index in a sense of the definition given in [12, 15]. And if $l_j(z_j) \equiv 1$, then the entire function F is called a function of bounded index in joint variables [16–18, 20, 26]. The least integer m for which inequality (1.2) holds is called \mathbf{L} -index in joint variables of the function F and is denoted by $N(F, \mathbf{L})$.

There are many papers [1, 12, 15–18, 23, 24, 26] devoted to the class of entire functions of bounded index in joint variables. The recent ones are about analytic functions [3, 4, 8, 9] in a ball or a polydisc satisfying (1.2). However, linear higher-order systems of PDE were considered only in two theses [13, 27]. In particular, in [13], there was considered the system

$$a_j(z) f^{(K_j^0)}(z) + \sum_{\|K\| \leq s-1} g_{K,j}(z) f^{(K)}(z) = h_j(z), \quad j \in \{1, \dots, m\}, \tag{1.3}$$

where for all $j \in \{1, \dots, m\}$ $\|K_j^0\| = s$, $\{f^{(K_j^0)}(z) : j = 1, \dots, m\}$ is a set of all possible s -order partial derivatives of the function f , the entire functions $a_j, g_{K,j}$,

h_j are the functions with separable variables of the form

$$g(z) = \prod_{j=1}^n g_j(z_j). \quad (1.4)$$

The author stated the conditions providing the boundedness of \mathbf{L} -index in joint variables for every entire solution, where $\mathbf{L}(z) = (l_1(|z_1|), \dots, l_n(|z_n|))$ and each function $l_j : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is continuous. Obviously, restriction (1.4) is very strong. Earlier M. Salmassi [27] proved that every entire solution of the system

$$\begin{cases} a_0 f^{(n_1, 0)}(z) + a_1 f^{(n_1-1, 0)}(z) + \dots + a_{n_1} f(z) = g(z), & a_0 \neq 0, \\ b_0 f^{(0, n_2)}(z) + b_1 f^{(0, n_2-1)}(z) + \dots + b_{n_2} f(z) = h(z), & b_0 \neq 0, \end{cases} \quad z = (z_1, z_2), \quad (1.5)$$

is a function of bounded index in joint variables, where $a_j \in \mathbb{C}$, $b_i \in \mathbb{C}$, $h(z)$ and $g(z)$ are arbitrary entire functions in \mathbb{C}^2 of bounded index in joint variables. Unlike in [13], it was not assumed that $h(z)$ and $g(z)$ are functions with separable variables. Therefore, the following natural question arises: *Is it possible to deduce sufficient conditions of the boundedness of \mathbf{L} -index in joint variables for entire solutions of a linear higher-order system of PDE without assumption (1.4)?*

This paper gives a positive answer to the posed question for system (3.1) which is more general than (1.5). Theorems 3.1–3.4 are generalizations of Salmassi's results in the following directions:

- we do not assume that the coefficients in system (3.1) are constants;
- we consider a system that may also contain the mixed partial derivatives.

Theorems 3.1–3.4 are also improved analogs of the results from [13] for system (3.1) in the following directions:

- we do not assume that the coefficients in (3.1) are the functions with separable variables;
- the function $\mathbf{L}(z) = (l_1(z), \dots, l_n(z))$ is of more general form than $\mathbf{L}(z) = (l_1(|z_1|), \dots, l_n(|z_n|))$, where $z = (z_1, \dots, z_n) \in \mathbb{C}^n$;
- we obtain sharp, in general, growth estimates of entire solutions of the system.

Note that the growth estimates of solutions are not discussed at all in [13, 27].

Recently, it has been proved in [10] that if F is an entire function of bounded \mathbf{L} -index in joint variables $N(F, L)$ and the function \mathbf{L} satisfies some additional assumptions, then

$$\overline{\lim}_{|R| \rightarrow +\infty} \frac{\ln \max\{|F(z)| : z \in \mathbb{T}^n(\mathbf{0}, R)\}}{\max_{\Theta \in [0, 2\pi]^n} \int_0^1 \langle R, \mathbf{L}(\tau R e^{i\Theta}) \rangle d\tau} \leq N(F, L) + 1.$$

Thus, in the paper we will estimate the growth of the same fraction by some constants. The paper uses the methods from [1, 2, 7, 10, 14] to study the entire solutions of system (3.1).

2. Auxiliary notations and propositions

We need some standard notations. Let $\mathbb{R}_+ = [0, +\infty)$. Denote $\mathbf{0} = (0, \dots, 0) \in \mathbb{R}_+^n$, $\mathbf{1} = (1, \dots, 1) \in \mathbb{R}_+^n$, $\mathbf{1}_j = (0, \dots, 0, \underbrace{1}_{j\text{-th place}}, 0, \dots, 0) \in \mathbb{R}_+^n$.

For $A = (a_1, \dots, a_n) \in \mathbb{C}^n$, $B = (b_1, \dots, b_n) \in \mathbb{C}^n$, $z = (z_1, \dots, z_n) \in \mathbb{C}^n$, we will use formal notations without violation of the existence of these expressions:

$$\begin{aligned} A + B &= (a_1 + b_1, \dots, a_n + b_n), \\ AB &= (a_1 b_1, \dots, a_n b_n), \\ A/B &= (a_1/b_1, \dots, a_n/b_n), \\ A^B &= a_1^{b_1} a_2^{b_2} \dots a_n^{b_n} \quad (\text{for } B \in \mathbb{Z}^n), \quad |z| = \left(\sum_{j=1}^n |z_j|^2 \right)^{1/2}. \end{aligned}$$

If $A, B \in \mathbb{R}^n$, the notation $A < B$ means that $a_j < b_j$ ($j \in \{1, \dots, n\}$). The relation $A \leq B$ is defined in a similar way. For $z, w \in \mathbb{C}^n$, we define $\langle z, w \rangle = z_1 \bar{w}_1 + \dots + z_n \bar{w}_n$, where \bar{w}_k is the complex conjugate of w_k .

For $R = (r_1, \dots, r_n)$, we denote by

$$\mathbb{D}^n(z^0, R) := \{z \in \mathbb{C}^n : |z_j - z_j^0| < r_j, j \in \{1, \dots, n\}\}$$

the polydisc, by

$$\mathbb{T}^n(z^0, R) := \{z \in \mathbb{C}^n : |z_j - z_j^0| = r_j, j \in \{1, \dots, n\}\}$$

its skeleton and by

$$\mathbb{D}^n[z^0, R] := \{z \in \mathbb{C}^n : |z_j - z_j^0| \leq r_j, j \in \{1, \dots, n\}\}$$

the closed polydisc.

By Q^n (in particular, $Q := Q^1$), we denote a class of positive continuous functions $\mathbf{L}(z) = (l_1(z), \dots, l_n(z))$ such that

$$\begin{aligned} \exists R_0 \in \mathbb{R}_+^n \exists C, c \in \mathbb{R}_+^n \quad (\mathbf{0} < c \leq C) \quad \forall z_0 \in \mathbb{C}^n \quad \forall z \in \mathbb{D}^n[z_0, R_0/L(z_0)] \\ c \leq \mathbf{L}(z)/\mathbf{L}(z_0) \leq C. \end{aligned} \quad (2.1)$$

Note that if (2.1) holds for some R_0 , then (2.1) is valid for all $R \in \mathbb{R}_+^n$. Besides, if for all $z \in \mathbb{C}^n$, $j, m \in \{1, 2, \dots, n\}$,

$$\left| \frac{\partial l_j(z)}{\partial z_m} \right| \leq P(c + |l_j(z)|), \quad P > 0,$$

then $\mathbf{L}^* \in Q^n$, where $\mathbf{L}^*(z) = (c + |l_1(z)|, \dots, c + |l_n(z)|)$, $c > 0$. It is proved in [10, Lemma 1]. Particularly, if $\mathbf{L}(z) = (l_1(R), \dots, l_n(R))$, $R = (|z_1|, \dots, |z_n|)$, for every $j \in \{1, \dots, n\}$ the function $l_j(R)$ is positive continuously differentiable and $|\nabla \ln l_j(R)| \leq P$ for all $R \in \mathbb{R}_+^n$, then $\mathbf{L} \in Q^n$, where $\nabla l_j(R) = \left(\frac{\partial l_j(R)}{\partial r_1}, \dots, \frac{\partial l_j(R)}{\partial r_n} \right)$.

Every function $\mathbf{L} \in Q^n$ has the property

$$\max_{\Theta \in [0, 2\pi]^n} \int_0^1 \langle R, \mathbf{L}(\tau R e^{i\Theta}) \rangle d\tau \rightarrow +\infty \quad \text{as } |R| \rightarrow +\infty. \quad (2.2)$$

We will need the following analog of Hayman's Theorem.

Theorem 2.1 ([1]). *Let $\mathbf{L} \in Q^n$. An entire function F has bounded \mathbf{L} -index in joint variables if and only if there exists $p \in \mathbb{Z}_+$, $c \in \mathbb{R}_+$ such that for each $z \in \mathbb{C}^n$:*

$$\max \left\{ \frac{|F^{(J)}(z)|}{\mathbf{L}^J(z)} : \|J\| = p + 1 \right\} \leq c \max \left\{ \frac{|F^{(K)}(z)|}{\mathbf{L}^K(z)} : \|K\| \leq p \right\}. \quad (2.3)$$

By \overline{G} , we denote the closure of a domain $G \subset \mathbb{C}^n$. Every entire function $F: \mathbb{C}^n \rightarrow \mathbb{C}$ is a function of bounded \mathbf{L} -index in joint variables with arbitrary continuous function $\mathbf{L}: \mathbb{C}^n \rightarrow \mathbb{R}_+$ in any bounded domain $G \subset \mathbb{C}^n$.

Theorem 2.2. *Let $F(z)$ be an entire function, G be a bounded domain in \mathbb{C}^n . If $\mathbf{L}: \mathbb{C}^n \rightarrow \mathbb{R}_+$ is a continuous function and $F(z)$ is an entire function, then there exists $m \in \mathbb{Z}_+$ such that for all $z \in \overline{G}$ and $J = (j_1, j_2, \dots, j_n) \in \mathbb{Z}_+^n$:*

$$\frac{|F^{(J)}(z)|}{J! \mathbf{L}^J(z)} \leq \max \left\{ \frac{|F^{(K)}(z)|}{K! \mathbf{L}^K(z)} : K \in \mathbb{Z}_+^n, \|K\| \leq m \right\}. \quad (2.4)$$

Proof. If $F(z) \equiv 0$, then (2.4) is obvious. Let $F(z) \not\equiv 0$. For every fixed $z^0 \in \mathbb{C}^n$, $\frac{|F^{(J)}(z^0)|}{J!}$ is the modulus of a coefficient of power series expansion of the function $F(z)$, $z \in T^n(z^0, \frac{1}{L(z^0)})$ in the neighborhood of the point z^0 . Since $F(z)$ is entire, $\frac{|F^{(J)}(z^0)|}{J! \mathbf{L}^J(z^0)} \rightarrow 0$ as $\|J\| \rightarrow \infty$ for every $z^0 \in G$, i.e., there exists $m_0 = m(z^0)$ for which inequality (2.4) holds.

Assume on the contrary that the set of values m_0 is not uniformly bounded in z^0 , i.e., $\sup_{z^0 \in G} m_0 = +\infty$. Hence, for every $m \in \mathbb{Z}_+$, there exists $z^m \in \overline{G}$ and $J^m \in \mathbb{Z}_+^n$:

$$\frac{|F^{(J^m)}(z^m)|}{J^m! \mathbf{L}^{J^m}(z^m)} > \max \left\{ \frac{|F^{(K)}(z^m)|}{K! \mathbf{L}^K(z^m)} : K \in \mathbb{Z}_+^n, \|K\| \leq m \right\}. \quad (2.5)$$

Since $z^m \in \overline{G}$, there exists the subsequence $z'^m \rightarrow z' \in \overline{G}$ as $m \rightarrow +\infty$. By Cauchy's integral formula, for any $J \in \mathbb{Z}_+^n$ we have

$$\frac{F^{(J)}(z^0)}{J!} = \frac{1}{(2\pi i)^n} \int_{\mathbb{T}^n(z^0, R)} \frac{F(z)}{(z - z^0)^{J+1}} dz.$$

We rewrite (2.5) in the form

$$\max \left\{ \frac{|F^{(K)}(z^m)|}{K! \mathbf{L}^K(z^m)} : K \in \mathbb{Z}_+^n, \|K\| \leq m \right\}$$

$$\begin{aligned} &\leq \frac{1}{(2\pi)^n \mathbf{L}^{J^m}(z^m)} \int_{\mathbb{T}^n(z^0, R/L(z^m))} \frac{|F(z)|}{|z - z^m|^{J^m+1}} |dz| \\ &\leq \frac{1}{R^{J^m}} \max\{|F(z)| : z \in G_R\}, \end{aligned} \tag{2.6}$$

where $G_R = \bigcup_{z^* \in \bar{G}} \mathbb{D}^n[z^*, R/\mathbf{L}(z^*)]$, $R = (r_1, \dots, r_n) > \mathbf{0}$. We choose $R > \mathbf{1}$. Taking the limit in (2.6) as $m \rightarrow \infty$, we deduce

$$\forall K \in \mathbb{Z}_+^n \quad \frac{|F^{(K)}(z')|}{K! \mathbf{L}^K(z')} \leq \lim_{m \rightarrow \infty} \frac{1}{R^{J^m}} \max\{|F(z)| : z \in G_R\} = 0$$

as $m \rightarrow +\infty$. Thus, all partial derivatives of the function F at the point z' equal 0. By a uniqueness theorem, $F(z) \equiv 0$, which is impossible. \square

Remark 2.3. A similar proposition to Theorem 2.2 holds for entire functions of bounded \mathbf{L} -index in a direction $\mathbf{b} \in \mathbb{C}^n \setminus \mathbf{0}$. This is valid under the additional assumption: $\forall z \in \bar{G} \quad g_z(t) := F(z + t\mathbf{b}) \neq 0, t \in \mathbb{C}$ (see [7]).

Using Theorems 2.1 and 2.2, we obtain the following corollary.

Corollary 2.4. *Let $\mathbf{L} \in Q^n$, F be an entire function, G be a bounded domain in \mathbb{C}^n . The function F is of bounded \mathbf{L} -index in joint variables if and only if there exists $p \in \mathbb{Z}_+$ and $c > 0$ such that for all $z \in \mathbb{C}^n \setminus G$ inequality (2.3) holds.*

Let us denote $a^+ = \max\{a, 0\}$, $A^+ = (a_1^+, \dots, a_n^+)$, where $a \in \mathbb{R}$, $A \in \mathbb{R}^n$.

Let $\mathbf{L}(z) = \mathbf{L}(Re^{i\Theta})$ be a positive continuously differentiable function in each variable $r_k, k \in \{1, \dots, n\}$, $\Theta \in [0, 2\pi]^n$, $R = (r_1, \dots, r_n) > \mathbf{0}$.

By QW^n , we denote the class of functions $\mathbf{L} \in Q^n$ such that

$$\left\langle \frac{\mathbf{1}}{R}, \left(\frac{d}{dt} \frac{\mathbf{1}}{\mathbf{L}(tRe^{i\Theta})} \Big|_{t=1} \right)^+ \right\rangle \rightarrow 0 \quad (|R| \rightarrow +\infty, R \geq R_0 > \mathbf{0})$$

uniformly in $\Theta \in [0, 2\pi]^n$. For simplicity, we write $QW := QW^1$.

Let $\mathbf{L}(z) = \mathbf{L}(R)$ with $r_k = |z_k|, k \in \{1, \dots, n\}$, $R = (r_1, \dots, r_n) > \mathbf{0}$. Every function $\mathbf{L}(R)$ nondecreasing in each variable r_k belongs to the class QW^n . In particular, a polynomial

$$\mathbf{L}(R) = \sum_{\|J\| \leq p} a_J R^J$$

and an exponent

$$\mathbf{L}(R) = \exp \left(\sum_{\|J\| \leq p} a_J R^J \right)$$

belong to the same class with $a_J \in \mathbb{R}_+^n$. The function $\mathbf{L}(R) = (\frac{1}{\ln r_1}, \dots, \frac{1}{\ln r_n})$ ($r_j > 1$) is nonincreasing in each variable and also belongs to the class QW^n .

Lemma 2.5. *Let $\mathbf{L} \in QW^n$, F be an entire function. If there exists $R' \in \mathbb{R}_+^n$, $p \in \mathbb{Z}_+$, $c > 0$ such that for all $z \in \mathbb{C}^n \setminus \mathbb{D}^n(\mathbf{0}, R')$ inequality (2.3) holds, then*

$$\overline{\lim}_{|R| \rightarrow \infty} \frac{\ln \max\{|F(z)| : z \in \mathbb{T}^n(\mathbf{0}, R)\}}{\max_{\Theta \in [0, 2\pi]^n} \int_0^1 \langle R, \mathbf{L}(\tau R e^{i\Theta}) \rangle d\tau} \leq \max\{1, c\}. \quad (2.7)$$

Proof. Let $R = (r_1, \dots, r_n) \in \mathbb{R}^n \setminus \{\mathbf{0}\}$, $R > R'$, $\Theta \in [0, 2\pi]^n$. Denote $r^* = \max_{1 \leq j \leq n} r_j$, $\alpha_j = \frac{r_j}{r^*}$, $j \in \{1, \dots, n\}$ and $A = (\alpha_1, \dots, \alpha_n)$. We consider the function

$$g(t) = \max \left\{ \frac{|F^{(S)}(Ate^{i\Theta})|}{\mathbf{L}^S(Ate^{i\Theta})} : \|S\| \leq p \right\}, \quad (2.8)$$

where $Ate^{i\Theta} = (\alpha_1 t e^{i\theta_1}, \dots, \alpha_n t e^{i\theta_n})$ and $|At| > |R'|$, $t \in \mathbb{R}_+$.

Since the function $\frac{|F^{(S)}(Ate^{i\Theta})|}{\mathbf{L}^S(Ate^{i\Theta})}$ is continuously differentiable by real $t \in [0, +\infty)$ outside the zero set of the function $|F^{(S)}(Ate^{i\Theta})|$, the function $g(t)$ is a continuously differentiable function on $[0, +\infty)$ except, perhaps, for a countable set of points.

Let us denote $u_j(t) = u_j(t, R, \Theta) = l_j \left(\frac{tR}{r^*} e^{i\Theta} \right)$, where $t \in \mathbb{R}_+$, $j \in \{1, \dots, n\}$. Let $\mathbf{L}(Re^{i\Theta})$ be a positive function continuously differentiable in each variable r_k , $k \in \{1, \dots, n\}$, $\Theta \in [0, 2\pi]^n$. It is easy to check that $\mathbf{L} \in QW^n$ if

$$r^* \left(- (u_j(t, R, \Theta))'_{t=r^*} \right)^+ / (r_j l_j^2(Re^{i\Theta})) \rightarrow 0 \quad (2.9)$$

uniformly in $\Theta \in [0, 2\pi]^n$, $j \in \{1, \dots, n\}$ as $|R| \rightarrow \infty$, $R \geq R_0 > \mathbf{0}$.

Therefore, using the inequality $\frac{d}{dr}|g(r)| \leq |g'(r)|$ which holds except for the points $r = t$ such that $g(t) = 0$, we deduce

$$\begin{aligned} \frac{d}{dt} \left(\frac{|F^{(S)}(Ate^{i\Theta})|}{\mathbf{L}^S(Ate^{i\Theta})} \right) &= \frac{1}{\mathbf{L}^S(Ate^{i\Theta})} \frac{d}{dt} |F^{(S)}(Ate^{i\Theta})| \\ &\quad + |F^{(S)}(Ate^{i\Theta})| \frac{d}{dt} \frac{1}{\mathbf{L}^S(Ate^{i\Theta})} \\ &\leq \frac{1}{\mathbf{L}^S(Ate^{i\Theta})} \left| \sum_{j=1}^n F^{(S+1_j)}(Ate^{i\Theta}) \alpha_j e^{i\theta_j} \right| - \frac{|F^{(S)}(Ate^{i\Theta})|}{\mathbf{L}^S(Ate^{i\Theta})} \sum_{j=1}^n \frac{s_j u_j'(t)}{l_j(Ate^{i\Theta})} \\ &\leq \sum_{j=1}^n \frac{|F^{(S+1_j)}(Ate^{i\Theta})|}{\mathbf{L}^{S+1_j}(Ate^{i\Theta})} \alpha_j l_j(Ate^{i\Theta}) + \frac{|F^{(S)}(Ate^{i\Theta})|}{\mathbf{L}^S(Ate^{i\Theta})} \sum_{j=1}^n \frac{s_j (-u_j'(t))^+}{l_j(Ate^{i\Theta})}. \end{aligned} \quad (2.10)$$

For absolutely continuous functions h_1, h_2, \dots, h_k and $h(x) := \max\{h_j(z) : 1 \leq j \leq k\}$, $h'(x) \leq \max\{h_j'(x) : 1 \leq j \leq k\}$, $x \in [a, b]$ (see [29, Lemma 4.1, p. 81]). The function g is absolutely continuous, therefore, from (2.3) and (2.10) it follows that

$$g'(t) \leq \max \left\{ \frac{d}{dt} \left(\frac{|F^{(S)}(Ate^{i\Theta})|}{\mathbf{L}^S(Ate^{i\Theta})} \right) : \|S\| \leq p \right\}$$

$$\begin{aligned} &\leq \max_{\|S\| \leq p} \left\{ \sum_{j=1}^n \frac{\alpha_j l_j(Ate^{i\Theta}) |F^{(S+1_j)}(Ate^{i\Theta})|}{\mathbf{L}^{S+1_j}(Ate^{i\Theta})} \right. \\ &\quad \left. + \frac{|F^{(S)}(Ate^{i\Theta})|}{\mathbf{L}^S(Ate^{i\Theta})} \sum_{j=1}^n \frac{s_j(-u'_j(t))^+}{l_j(Ate^{i\Theta})} \right\} \\ &\leq g(t) \left(\max\{1, c\} \sum_{j=1}^n \alpha_j l_j(Ate^{i\Theta}) + \max_{\|S\| \leq p} \left\{ \sum_{j=1}^n \frac{s_j(-u'_j(t))^+}{l_j(Ate^{i\Theta})} \right\} \right) \\ &= g(t)(\beta(t) + \gamma(t)), \end{aligned}$$

where

$$\beta(t) = \max\{1, c\} \sum_{j=1}^n \alpha_j l_j(Ate^{i\Theta}), \quad \gamma(t) = \max_{\|S\| \leq p} \left\{ \sum_{j=1}^n \frac{s_j(-u'_j(t))^+}{l_j(Ate^{i\Theta})} \right\}.$$

Thus,

$$\frac{d}{dt} \ln g(t) \leq \beta(t) + \gamma(t) \quad \text{and} \quad g(t) \leq g(t_0) \exp \int_{t_0}^t (\beta(\tau) + \gamma(\tau)) d\tau,$$

where t_0 is chosen such that $g(t_0) \neq 0$. The condition $\mathbf{L} \in W^n$ gives

$$\frac{\gamma(t)}{\beta(t)} = \frac{\sum_{j=1}^n \frac{s_j(-u'_j(t))^+}{l_j(Ate^{i\Theta})}}{\max\{1, c\} \sum_{j=1}^n \alpha_j l_j(Ate^{i\Theta})} \leq p \sum_{j=1}^n \frac{(-u'_j(t))^+}{\alpha_j l_j^2(Ate^{i\Theta})} \leq p\varepsilon,$$

where $\varepsilon = \varepsilon(R) \rightarrow 0$ uniformly in $\Theta \in [0, 2\pi]^n$, $t = r^*$ as $|R| \rightarrow \infty$.

But

$$|F(Ate^{i\Theta})| \leq g(t) \leq g(t_0) \exp \int_{t_0}^t (\beta(\tau) + \gamma(\tau)) d\tau$$

and $r^*A = R$. Then we put $t = r^*$ and obtain

$$\begin{aligned} \ln \max\{|F(z): z \in \mathbb{T}^n(\mathbf{0}, R)\} &= \ln \max_{\Theta \in [0, 2\pi]^n} |F(Re^{i\Theta})| \leq \ln \max_{\Theta \in [0, 2\pi]^n} g(r^*) \\ &\leq \ln g(t_0) + \max_{\Theta \in [0, 2\pi]^n} \int_{t_0}^{r^*} (\beta(\tau) + \gamma(\tau)) d\tau \\ &\leq \ln g(t_0) + \max_{\Theta \in [0, 2\pi]^n} \int_{t_0}^{r^*} \max\{1, c\} \sum_{j=1}^n \alpha_j l_j(Ate^{i\Theta}) (1 + p\varepsilon) d\tau \\ &= \ln g(t_0) + \max\{1, c\} \max_{\Theta \in [0, 2\pi]^n} \int_{t_0}^{r^*} \sum_{j=1}^n \frac{r_j}{r^*} l_j\left(\frac{\tau}{r^*} Re^{i\Theta}\right) (1 + p\varepsilon) d\tau. \end{aligned}$$

This implies (2.7). □

Lemma 2.6. *Let $\mathbf{L} \in QW^n$, F be an entire function. If there exists $R' \in \mathbb{R}_+^n$, $p \in \mathbb{Z}_+$, $c > 0$ such that for all $z \in \mathbb{C}^n \setminus \mathbb{D}^n(\mathbf{0}, R')$ the inequality*

$$\max \left\{ \frac{|F^{(J)}(z)|}{J! \mathbf{L}^J(z)} : \|J\| = p + 1 \right\} \leq c \max \left\{ \frac{|F^{(K)}(z)|}{K! \mathbf{L}^K(z)} : \|K\| \leq p \right\} \quad (2.11)$$

holds, then

$$\overline{\lim}_{|R| \rightarrow \infty} \frac{\ln \max\{|F(z)| : z \in \mathbb{T}^n(\mathbf{0}, R)\}}{\max_{\Theta \in [0, 2\pi]^n} \int_0^1 \langle R, \mathbf{L}(\tau R e^{i\Theta}) \rangle d\tau} \leq (p + 1) \max\{1, c\}. \quad (2.12)$$

Proof. The proof of Lemma 2.6 is similar to that of Lemma 2.5. Let $R = (r_1, \dots, r_n) \in \mathbb{R}^n \setminus \{\mathbf{0}\}$, $R > R'$, $\Theta \in [0, 2\pi]^n$. As in the proof of Lemma 2.5, we denote $r^* = \max_{1 \leq j \leq n} r_j$, $\alpha_j = \frac{r_j}{r^*}$, $j \in \{1, \dots, n\}$ and $A = (\alpha_1, \dots, \alpha_n)$. We consider the function

$$g(t) = \max \left\{ \frac{|F^{(S)}(Ate^{i\Theta})|}{S! \mathbf{L}^S(Ate^{i\Theta})} : \|S\| \leq p \right\}, \quad (2.13)$$

where $Ate^{i\Theta} = (\alpha_1 t e^{i\theta_1}, \dots, \alpha_n t e^{i\theta_n})$, $|At| > |R'|$, $t \in \mathbb{R}_+$.

As above, the function $\frac{|F^{(S)}(Ate^{i\Theta})|}{S! \mathbf{L}^S(Ate^{i\Theta})}$ is continuously differentiable by real $t \in [0, +\infty)$ outside the zero set of the function $|F^{(S)}(Ate^{i\Theta})|$, the function $g(t)$ is a continuously differentiable function on $[0, +\infty)$ except, perhaps, for a countable set of points.

Therefore, using the inequality $\frac{d}{dr}|g(r)| \leq |g'(r)|$, which holds except for the points $r = t$ such that $g(t) = 0$, we deduce

$$\begin{aligned} & \frac{d}{dt} \left(\frac{|F^{(S)}(Ate^{i\Theta})|}{S! \mathbf{L}^S(Ate^{i\Theta})} \right) \\ &= \frac{1}{S! \mathbf{L}^S(Ate^{i\Theta})} \frac{d}{dt} |F^{(S)}(Ate^{i\Theta})| + |F^{(S)}(Ate^{i\Theta})| \frac{d}{dt} \frac{1}{S! \mathbf{L}^S(Ate^{i\Theta})} \\ &\leq \frac{1}{S! \mathbf{L}^S(Ate^{i\Theta})} \left| \sum_{j=1}^n F^{(S+\mathbf{1}_j)}(Ate^{i\Theta}) \alpha_j e^{i\theta_j} \right| - \frac{|F^{(S)}(Ate^{i\Theta})|}{S! \mathbf{L}^S(Ate^{i\Theta})} \sum_{j=1}^n \frac{s_j u_j'(t)}{l_j(Ate^{i\Theta})} \\ &\leq \sum_{j=1}^n \frac{|F^{(S+\mathbf{1}_j)}(Ate^{i\Theta})|}{(S+\mathbf{1}_j)! \mathbf{L}^{S+\mathbf{1}_j}(Ate^{i\Theta})} \alpha_j (s_j + 1) l_j(Ate^{i\Theta}) \\ &\quad + \frac{|F^{(S)}(Ate^{i\Theta})|}{S! \mathbf{L}^S(Ate^{i\Theta})} \sum_{j=1}^n \frac{s_j (-u_j'(t))^+}{l_j(Ate^{i\Theta})}. \end{aligned} \quad (2.14)$$

For absolutely continuous functions h_1, h_2, \dots, h_k and $h(x) := \max\{h_j(x) : 1 \leq j \leq k\}$, $h'(x) \leq \max\{h_j'(x) : 1 \leq j \leq k\}$, $x \in [a, b]$ (see [29, Lemma 4.1, p. 81]). The function g is absolutely continuous. Therefore, (2.3) and (2.14) yield

$$g'(t) \leq \max \left\{ \frac{d}{dt} \left(\frac{|F^{(S)}(Ate^{i\Theta})|}{S! \mathbf{L}^S(Ate^{i\Theta})} \right) : \|S\| \leq N \right\}$$

$$\begin{aligned}
 &\leq \max_{\|S\| \leq p} \left\{ \sum_{j=1}^n \frac{\alpha_j(s_j+1)l_j(Ate^{i\Theta}) |F^{(S+\mathbf{1}_j)}(Ate^{i\Theta})|}{(S+\mathbf{1}_j)! \mathbf{L}^{S+\mathbf{1}_j}(Ate^{i\Theta})} \right. \\
 &\quad \left. + \frac{|F^{(S)}(Ate^{i\Theta})| s_j(-u'_j(t))^+}{S! \mathbf{L}^S(Ate^{i\Theta}) l_j(Ate^{i\Theta})} \right\} \\
 &\leq g(t) \left(\max\{1, c\} \max_{\|S\| \leq p} \left\{ \sum_{j=1}^n \alpha_j(s_j+1)l_j(Ate^{i\Theta}) \right\} \right. \\
 &\quad \left. + \max_{\|S\| \leq p} \left\{ \sum_{j=1}^n \frac{s_j(-u'_j(t))^+}{l_j(Ate^{i\Theta})} \right\} \right) \\
 &= g(t)(\beta(t) + \gamma(t)),
 \end{aligned}$$

where

$$\begin{aligned}
 \beta(t) &= \max\{1, c\} \max_{\|S\| \leq p} \left\{ \sum_{j=1}^n \alpha_j(s_j+1)l_j(Ate^{i\Theta}) \right\}, \\
 \gamma(t) &= \max_{\|S\| \leq p} \left\{ \sum_{j=1}^n \frac{s_j(-u'_j(t))^+}{l_j(Ate^{i\Theta})} \right\}.
 \end{aligned}$$

Thus,

$$\frac{d}{dt} \ln g(t) \leq \beta(t) + \gamma(t) \quad \text{and} \quad g(t) \leq g(t_0) \exp \int_{t_0}^t (\beta(\tau) + \gamma(\tau)) d\tau,$$

where t_0 is chosen such that $g(t_0) \neq 0$. Denote $\tilde{\beta}(t) = \sum_{j=1}^n \alpha_j l_j(Ate^{i\Theta})$. Since $\mathbf{L} \in W^n$, for some S^* , $\|S^*\| \leq p$ and \tilde{S} , $\|\tilde{S}\| \leq p$, we obtain

$$\frac{\gamma(t)}{\tilde{\beta}(t)} = \frac{\sum_{j=1}^n \frac{s_j^*(-u'_j(t))^+}{l_j(Ate^{i\Theta})}}{\sum_{j=1}^n \alpha_j l_j(Ate^{i\Theta})} \leq \sum_{j=1}^n s_j^* \frac{(-u'_j(t))^+}{\alpha_j l_j^2(Ate^{i\Theta})} \leq p \sum_{j=1}^n \frac{(-u'_j(t))^+}{\alpha_j l_j^2(Ate^{i\Theta})} \leq p\varepsilon$$

and

$$\begin{aligned}
 \frac{\beta(t)}{\tilde{\beta}(t)} &= \frac{\max\{1, c\} \sum_{j=1}^n \alpha_j(\tilde{s}_j+1)l_j(Ate^{i\Theta})}{\sum_{j=1}^n \alpha_j l_j(Ate^{i\Theta})} \\
 &= \max\{1, c\} \left(1 + \frac{\sum_{j=1}^n \alpha_j \tilde{s}_j l_j(Ate^{i\Theta})}{\sum_{j=1}^n \alpha_j l_j(Ate^{i\Theta})} \right) \\
 &\leq \max\{1, c\} \left(1 + \sum_{j=1}^n \tilde{s}_j \right) \leq (1+p) \max\{1, c\},
 \end{aligned}$$

where $\varepsilon = \varepsilon(R) \rightarrow 0$ uniformly in $\Theta \in [0, 2\pi]^n$, $t = r^*$ as $|R| \rightarrow \infty$.

But

$$|F(Ate^{i\Theta})| \leq g(t) \leq g(t_0) \exp \int_{t_0}^t (\beta(\tau) + \gamma(\tau)) d\tau$$

and $r^*A = R$. Then we put $t = r^*$ and obtain

$$\begin{aligned} \ln \max\{|F(z): z \in \mathbb{T}^n(\mathbf{0}, R)\} &= \ln \max_{\Theta \in [0, 2\pi]^n} |F(Re^{i\Theta})| \leq \ln \max_{\Theta \in [0, 2\pi]^n} g(r^*) \\ &\leq \ln g(t_0) + \max_{\Theta \in [0, 2\pi]^n} \int_{t_0}^{r^*} (\beta(\tau) + \gamma(\tau)) d\tau \\ &\leq \ln g(t_0) + \max_{\Theta \in [0, 2\pi]^n} \int_{t_0}^{r^*} \sum_{j=1}^n \alpha_j l_j(A\tau e^{i\Theta}) (\max\{1, c\}(1+p) + p\varepsilon) d\tau \\ &= \ln g(t_0) + \max_{\Theta \in [0, 2\pi]^n} \int_{t_0}^{r^*} \sum_{j=1}^n \frac{r_j}{r^*} l_j\left(\frac{\tau}{r^*} Re^{i\Theta}\right) (\max\{1, c\}(1+p) + p\varepsilon) d\tau. \end{aligned}$$

This implies (2.12). \square

Remark 2.7. Note that condition (2.11) means that

$$\begin{aligned} \max \left\{ \frac{|F^{(J)}(z)|}{\mathbf{L}^J(z)} : \|J\| = p+1 \right\} &\leq \max \left\{ \frac{|F^{(J)}(z)|}{J! \mathbf{L}^J(z)} : \|J\| = p+1 \right\} \max_{\|J\|=p+1} J! \\ &\leq c(p+1)! \max \left\{ \frac{|F^{(K)}(z)|}{K! \mathbf{L}^K(z)} : \|K\| \leq p \right\} \\ &\leq \frac{c(p+1)!}{\min_{\|K\| \leq p} K!} \max \left\{ \frac{|F^{(K)}(z)|}{\mathbf{L}^K(z)} : \|K\| \leq p \right\} \\ &\leq c(p+1)! \max \left\{ \frac{|F^{(K)}(z)|}{\mathbf{L}^K(z)} : \|K\| \leq p \right\}. \end{aligned}$$

Hence, by Lemma 2.5, we have

$$\overline{\lim}_{|R| \rightarrow \infty} \frac{\ln \max\{|F(z)|: z \in \mathbb{T}^n(\mathbf{0}, R)\}}{\max_{\Theta \in [0, 2\pi]^n} \int_0^1 \langle R, \mathbf{L}(\tau Re^{i\Theta}) \rangle d\tau} \leq \max\{1, c(p+1)!\}.$$

Since $c(p+1)! > c(p+1)$ for $p > 1$, we see that Lemma 2.6 does not imply Lemma 2.5. Clearly, Lemma 2.5 does not imply Lemma 2.6 as well. Therefore we need both Lemma 2.5 and Lemma 2.6.

3. Growth and boundedness of L-index in joint variables of entire solutions of system of PDE's

Using the proved lemmas, we will formulate and prove the propositions that provide the growth estimates of entire solutions of the following system of partial differential equations:

$$G_{p_j \mathbf{1}_j}(z) F^{(p_j \mathbf{1}_j)}(z) + \sum_{\|S_j\| \leq p_j - 1} G_{S_j}(z) F^{(S_j)}(z) = H_j(z), \quad j \in \{1, \dots, n\}, \quad (3.1)$$

$p_j \in \mathbb{N}$, $S_j \in \mathbb{Z}_+^n$, H_j and G_{S_j} are entire functions. Note if $\mathbf{L} \in \mathcal{Q}^n$, then \mathbf{L} satisfies (2.2). It is proved in (Lemma 2, [10]).

We will say that nonhomogeneous system of PDE's (3.1) belongs to the class $\mathcal{A}(\mathbf{G}, \mathbf{H}, \mathbf{L})$, if $\mathbf{L} \in \mathcal{QW}^n$, for all $z \in \mathbb{C}^n$ and for every $j \in \{1, \dots, n\}$ the entire functions H_j and G_{S_j} satisfy the following conditions:

1) for every $\|S_j\| \leq p_j - 1$ and $M \in \mathbb{Z}_+^n$,

$$\|M\| \leq 1 + \sum_{\substack{k=1 \\ k \neq j}}^n p_k, \quad \left| G_{S_j}^{(M)}(z) \right| \mathbf{L}^{S_j - M}(z) \leq B_{S_j, M} l_j^{p_j}(z) |G_{p_j \mathbf{1}_j}(z)|,$$

and

$$\left| G_{p_j \mathbf{1}_j}^{(M)}(z) \right| \leq B_{p_j \mathbf{1}_j, M} \mathbf{L}^M(z) |G_{p_j \mathbf{1}_j}(z)|,$$

2) for every $I \in \mathbb{Z}_+^n$,

$$\|I\| = 1 + \sum_{k=1, k \neq j}^n p_k, \quad \left| H_j^{(I)}(z) \right| \leq D_{I, j} \mathbf{L}^I(z) |H_j(z)|,$$

3) $G_{p_j \mathbf{1}_j}(z) \neq 0$,

where $B_{S_j, M}$, $D_{I, j}$, $B_{p_j \mathbf{1}_j, M}$ are positive constants, $\mathbf{H}(z) = (H_1(z), \dots, H_n(z))$, $\mathbf{G}(z)$ is a matrix consisting of the coefficients $G_{S_j}(z)$ of system (3.1).

A homogeneous system of PDE's (3.1) belongs to the class $\mathcal{A}(\mathbf{G}, \mathbf{0}, \mathbf{L})$ if condition 1) holds for $M \in \mathbb{Z}_+^n$ such that $\|M\| \leq \sum_{k=1, k \neq j}^n p_k$ and $G_{p_j \mathbf{1}_j}(z) \neq 0$. Condition 2) is not required.

Instead of the condition $G_{p_j \mathbf{1}_j}(z) \neq 0$, we can require the validity of conditions 1) and 2) for all $z \in \mathbb{C}^n \setminus \mathbb{D}^n(\mathbf{0}, R')$. It is possible in view of Theorem 2.2. If for some $M \in \mathbb{Z}_+^n$ $G_{S_j}^{(M)}(z) \equiv 0$ or $H_j^{(M)}(z) \equiv 0$, then we suppose $B_{S_j, M} = 0$ or $D_{M, j} = 0$, respectively.

Theorem 3.1. *If nonhomogeneous system of PDE's (3.1) belongs to the class $\mathcal{A}(\mathbf{G}, \mathbf{H}, \mathbf{L})$ and an entire function $F(z)$ satisfies (3.1), then F has bounded \mathbf{L} -index in joint variables, and*

$$\overline{\lim}_{|R| \rightarrow \infty} \frac{\ln \max\{|F(z)| : z \in \mathbb{T}^n(\mathbf{0}, R)\}}{\max_{\Theta \in [0, 2\pi]^n} \int_0^1 \langle R, \mathbf{L}(\tau R e^{i\Theta}) \rangle d\tau} \leq \max\{1, c\}, \tag{3.2}$$

where c is defined in (3.8).

Proof. Taking into account that the function $F(z)$ satisfies system (3.1), we calculate the partial derivative $I \in \mathbb{Z}_+^n$ in each equation of the system

$$\sum_{\mathbf{0} \leq M \leq I} C_I^M \left(G_{p_j \mathbf{1}_j}^{(M)}(z) F^{(p_j \mathbf{1}_j + I - M)}(z) \right)$$

$$+ \sum_{\|S_j\| \leq p_j - 1} G_{S_j}^{(M)}(z) F^{(S_j+I-M)}(z) \Big) = H_j^{(I)}(z), \quad (3.3)$$

where $C_I^M = \frac{i_1! \dots i_n!}{m_1!(i_1-m_1)! \dots m_n!(i_n-m_n)!}$ and $\|I\| = 1 - p_j + \sum_{k=1}^n p_k = 1 + \sum_{k=1, k \neq j}^n p_k$. Using condition 2) of the theorem, we obtain

$$\begin{aligned} |H_j^{(I)}(z)| &\leq D_{I,j} \mathbf{L}^I(z) |H_j(z)| \\ &\leq D_{I,j} \mathbf{L}^I(z) \left(|G_{p_j \mathbf{1}_j}(z)| |F^{(p_j \mathbf{1}_j)}(z)| + \sum_{\|S_j\| \leq p_j - 1} |G_{S_j}(z)| |F^{(S_j)}(z)| \right). \end{aligned} \quad (3.4)$$

Equation (3.3) yields

$$\begin{aligned} F^{(p_j \mathbf{1}_j+I)}(z) &= \frac{1}{G_{p_j \mathbf{1}_j}(z)} \left(H_j^{(I)}(z) - \sum_{\mathbf{0} \leq M \leq I, M \neq \mathbf{0}} C_I^M G_{p_j \mathbf{1}_j}^{(M)}(z) F^{(p_j \mathbf{1}_j+I-M)}(z) \right. \\ &\quad \left. - \sum_{\mathbf{0} \leq M \leq I} C_I^M \sum_{\|S_j\| \leq p_j - 1} G_{S_j}^{(M)}(z) F^{(S_j+I-M)}(z) \right). \end{aligned} \quad (3.5)$$

From (3.5) and condition 2) it follows that

$$\begin{aligned} |F^{(p_j \mathbf{1}_j+I)}(z)| &\leq \left(D_{I,j} \mathbf{L}^I(z) \left(|F^{(p_j \mathbf{1}_j)}(z)| + \sum_{\|S_j\| \leq p_j - 1} \frac{|G_{S_j}(z)|}{|G_{p_j \mathbf{1}_j}(z)|} |F^{(S_j)}(z)| \right) \right. \\ &\quad + \sum_{\mathbf{0} \leq M \leq I, M \neq \mathbf{0}} C_I^M \frac{|G_{p_j \mathbf{1}_j}^{(M)}(z)|}{|G_{p_j \mathbf{1}_j}(z)|} |F^{(p_j \mathbf{1}_j+I-M)}(z)| \\ &\quad \left. + \sum_{\mathbf{0} \leq M \leq I} C_I^M \sum_{\|S_j\| \leq p_j - 1} \frac{|G_{S_j}^{(M)}(z)|}{|G_{p_j \mathbf{1}_j}(z)|} |F^{(S_j+I-M)}(z)| \right) \\ &\leq D_{I,j} \mathbf{L}^I(z) \left(|F^{(p_j \mathbf{1}_j)}(z)| + \sum_{\|S_j\| \leq p_j - 1} B_{S_j, \mathbf{0}} l_j^{p_j}(z) L^{-S_j}(z) |F^{(S_j)}(z)| \right) \\ &\quad + \sum_{\mathbf{0} \leq M \leq I, M \neq \mathbf{0}} C_I^M B_{p_j \mathbf{1}_j, M} \mathbf{L}^M(z) |F^{(p_j \mathbf{1}_j+I-M)}(z)| \\ &\quad + \sum_{\mathbf{0} \leq M \leq I} C_I^M \sum_{\|S_j\| \leq p_j - 1} B_{S_j, M} l_j^{p_j}(z) \mathbf{L}^{M-S_j}(z) |F^{(S_j+I-M)}(z)|. \end{aligned} \quad (3.6)$$

Dividing this inequality by $l_j^{p_j}(z) \mathbf{L}^I(z)$, we obtain that for every I , $\|I\| = 1 + \sum_{k=1, k \neq j}^n p_k$ and $j \in \{1, \dots, n\}$:

$$\frac{|F^{(p_j \mathbf{1}_j+I)}(z)|}{\mathbf{L}^{p_j \mathbf{1}_j+I}(z)} \leq D_{I,j} \left(\frac{|F^{(p_j \mathbf{1}_j)}(z)|}{l_j^{p_j}(z)} + \sum_{\|S_j\| \leq p_j - 1} B_{S_j, \mathbf{0}} \frac{|F^{(S_j)}(z)|}{\mathbf{L}^{S_j}(z)} \right)$$

$$\begin{aligned}
 & + \sum_{\mathbf{0} \leq M \leq I, M \neq \mathbf{0}} C_I^M B_{p_j \mathbf{1}_j, M} \frac{|F^{(p_j \mathbf{1}_j + I - M)}(z)|}{\mathbf{L}^{p_j \mathbf{1}_j + I - M}(z)} \\
 & + \sum_{\mathbf{0} \leq M \leq I} C_I^M \sum_{\|S_j\| \leq p_j - 1} B_{S_j, M} \frac{|F^{(S_j + I - M)}(z)|}{\mathbf{L}^{S_j + I - M}(z)} \\
 & \leq \left(D_{I, j} \left(1 + \sum_{\|S_j\| \leq p_j - 1} B_{S_j, \mathbf{0}} \right) + \sum_{\mathbf{0} \leq M \leq I, M \neq \mathbf{0}} C_I^M B_{p_j \mathbf{1}_j, M} \right. \\
 & \left. + \sum_{\mathbf{0} \leq M \leq I} C_I^M \sum_{\|S_j\| \leq p_j - 1} B_{S_j, M} \right) \max \left\{ \frac{|F^{(S)}(z)|}{\mathbf{L}^S(z)} : \|S\| \leq \sum_{j=1}^n p_j \right\}.
 \end{aligned}$$

Obviously, $\|p_j \mathbf{1}_j + I\| = 1 + \sum_{j=1}^n p_j$. This implies

$$\begin{aligned}
 & \max \left\{ \frac{|F^{(K)}(z)|}{\mathbf{L}^K(z)} : \|K\| = 1 + \sum_{j=1}^n p_j \right\} \\
 & \leq \max\{1, c\} \max \left\{ \frac{|F^{(S)}(z)|}{\mathbf{L}^S(z)} : \|S\| \leq \sum_{j=1}^n p_j \right\}, \tag{3.7}
 \end{aligned}$$

where

$$\begin{aligned}
 c = & \max_{\substack{\|I\| = 1 - p_j + \sum_{k=1}^n p_k, \\ j \in \{1, \dots, n\}}} \left(D_{I, j} \left(1 + \sum_{\|S_j\| \leq p_j - 1} B_{S_j, \mathbf{0}} \right) \right. \\
 & \left. + \sum_{\mathbf{0} \leq M \leq I, M \neq \mathbf{0}} C_I^M B_{p_j \mathbf{1}_j, M} + \sum_{\mathbf{0} \leq M \leq I} C_I^M \sum_{\|S_j\| \leq p_j - 1} B_{S_j, M} \right) \tag{3.8}
 \end{aligned}$$

for all $z \in \mathbb{C}^n \setminus \mathbb{D}^n(\mathbf{0}, R')$. Thus, by Lemma 2.5, estimate (3.2) holds, and by Corollary 2.4, the entire function F has bounded \mathbf{L} -index in joint variables. \square

If system (3.1) is homogeneous ($H_j(z) \equiv 0$), the previous theorem can be simplified.

Theorem 3.2. *If homogeneous system of PDE's (3.1) belongs to the class $\mathcal{A}(\mathbf{G}, \mathbf{0}, \mathbf{L})$ and an entire function F is a solution of the system, then F has bounded \mathbf{L} -index in joint variables, and*

$$\overline{\lim}_{|R| \rightarrow \infty} \frac{\ln \max\{|F(z)| : z \in \mathbb{T}^n(\mathbf{0}, R)\}}{\max_{\Theta \in [0, 2\pi]^n} \int_0^1 \langle R, \mathbf{L}(\tau R e^{i\Theta}) \rangle d\tau} \leq \max\{1, c\}, \tag{3.9}$$

where c is defined in (3.8) with $D_{I, j} = 0$ and $\|I\| = -p_j + \sum_{k=1}^n p_k$ instead of $\|I\| = 1 - p_j + \sum_{k=1}^n p_k$.

Proof. If $H_j(z) \equiv 0$, then (3.5) implies

$$F^{(p_j \mathbf{1}_j + I)}(z) = \frac{1}{G_{p_j \mathbf{1}_j}(z)} \left(- \sum_{\mathbf{0} \leq M \leq I, M \neq \mathbf{0}} C_I^M G_{p_j \mathbf{1}_j}^{(M)}(z) F^{(p_j \mathbf{1}_j + I - M)}(z) \right. \\ \left. - \sum_{\mathbf{0} \leq M \leq I} C_I^M \sum_{\|S_j\| \leq p_j - 1} G_{S_j}^{(M)}(z) F^{(S_j + I - M)}(z) \right). \quad (3.10)$$

Hence we obtain

$$\left| F^{(p_j \mathbf{1}_j + I)}(z) \right| \leq \frac{1}{|G_{p_j \mathbf{1}_j}(z)|} \left(\sum_{\mathbf{0} \leq M \leq I, M \neq \mathbf{0}} C_I^M \left| G_{p_j \mathbf{1}_j}^{(M)}(z) \right| \left| F^{(p_j \mathbf{1}_j + I - M)}(z) \right| \right. \\ \left. + \sum_{\mathbf{0} \leq M \leq I} C_I^M \sum_{\|S_j\| \leq p_j - 1} \left| G_{S_j}^{(M)}(z) \right| \left| F^{(S_j + I - M)}(z) \right| \right).$$

Dividing the obtained inequality by $\mathbf{L}^{p_j \mathbf{1}_j + I}(z)$ and using the assumptions of the theorem on the functions G_{S_j} , we deduce

$$\frac{|F^{(p_j \mathbf{1}_j + I)}(z)|}{\mathbf{L}^{p_j \mathbf{1}_j + I}(z)} \leq \sum_{\mathbf{0} \leq M \leq I, M \neq \mathbf{0}} C_I^M B_{p_j \mathbf{1}_j, M} \frac{|F^{(p_j \mathbf{1}_j + I - M)}(z)|}{\mathbf{L}^{p_j \mathbf{1}_j + I - M}(z)} \\ + \sum_{\mathbf{0} \leq M \leq I} C_I^M \sum_{\|S_j\| \leq p_j - 1} B_{S_j, M} \frac{|F^{(S_j + I - M)}(z)|}{\mathbf{L}^{S_j + I - M}(z)} \\ \leq \left(\sum_{\mathbf{0} \leq M \leq I, M \neq \mathbf{0}} C_I^M B_{p_j \mathbf{1}_j, M} + \sum_{\mathbf{0} \leq M \leq I} C_I^M \sum_{\|S_j\| \leq p_j - 1} B_{S_j, M} \right) \\ \times \max \left\{ \frac{|F^{(S)}(z)|}{\mathbf{L}^S(z)} : \|S\| \leq -1 + \sum_{j=1}^n p_j \right\}.$$

Obviously, $\|p_j \mathbf{1}_j + I\| = \sum_{j=1}^n p_j$. Therefore,

$$\max \left\{ \frac{|F^{(K)}(z)|}{\mathbf{L}^K(z)} : \|K\| = \sum_{j=1}^n p_j \right\} \\ \leq \max\{1, c\} \max \left\{ \frac{|F^{(S)}(z)|}{\mathbf{L}^S(z)} : \|S\| \leq -1 + \sum_{j=1}^n p_j \right\}$$

for all $z \in \mathbb{C}^n \setminus \mathbb{D}^n(\mathbf{0}, R')$. Thus, all conditions of Corollary 2.4 are satisfied. Hence the function F has a bounded \mathbf{L} -index in joint variables and, by Lemma 2.5, estimate (3.9) holds. \square

Note that estimates (3.2) and (3.9) cannot be improved (see examples for $n = 1$ in [14]). Let us consider the case $n = 2$. For example, the entire function $w = 1 + e^{-z_1 - z_2}$ is a solution of the nonhomogeneous system

$$\begin{cases} w'_{z_1} + w = 1, \\ w'_{z_2} + w = 1. \end{cases} \tag{3.11}$$

In view of (3.1) and (3.11), we have $H_1(z) = H_2(z) = 1$, $G_{1,1}(z) = G_{1,2} \equiv 1$, $G_{2,1}(z) = G_{2,2} \equiv 1$, where $G_{i,j}$ is the coefficient in the i -th equation of system (3.11) at j -th order partial derivative in variable z_i of the function w . Obviously, the entire function w has bounded index in joint variables, that is, $\mathbf{L}(z_1, z_2) = (1, 1)$. Validating the assumptions of Theorem 3.1, we deduce $B_{i,j,M} \equiv 0$ for $M \neq \mathbf{0}$ and $B_{i,j,\mathbf{0}} = 1$. Also, all second order partial derivatives of H_1 and H_2 equal 0. Then $D_{I,1} = D_{I,2} = 0$, where $I \in \mathcal{I} := \{(2, 0), (1, 1), (0, 2)\}$. Hence, in view of (3.8), $c = 1$. Thus, by Theorem 3.1,

$$\overline{\lim}_{|R| \rightarrow \infty} \frac{\ln \max\{|w(z)| : z \in \mathbb{T}^2(\mathbf{0}, R)\}}{\int_0^1 \sum_{j=1}^2 r_j d\tau} \leq 1.$$

But

$$\overline{\lim}_{|R| \rightarrow \infty} \frac{\ln \max\{|1 + \exp(-z_1 - z_2)| : (z_1, z_2) \in \mathbb{T}^2(\mathbf{0}, R)\}}{\int_0^1 (r_1 + r_2) d\tau} = \overline{\lim}_{|R| \rightarrow \infty} \frac{r_1 + r_2}{(r_1 + r_2)\tau} \Big|_0^1 = 1.$$

Therefore, estimate (3.2) is sharp.

By analogy, we can consider a homogeneous system

$$\begin{cases} w'_{z_1} - z_2 w = 0, \\ w'_{z_2} - z_1 w = 0. \end{cases}$$

The function $w(z_1, z_2) = \exp(z_1 z_2)$ is its entire solution of bounded \mathbf{L} -index in joint variables, where $\mathbf{L}(z_1, z_2) = (|z_2|, |z_1|)$. Using Theorem 3.2, it is easy to show that

$$\overline{\lim}_{|R| \rightarrow \infty} \frac{\ln \max\{|w(z)| : z \in \mathbb{T}^2(\mathbf{0}, R)\}}{\int_0^1 (r_1 r_2 \tau + r_2 r_1 \tau) d\tau} \leq 1.$$

Direct calculations prove that this estimate is exact too. Thus, (3.9) is non-improvable.

Moreover, using Corollary 2.4 and Lemma 2.6, we can supplement two previous Theorems 3.1 and 3.2 with the propositions that contain the estimates $\max\{|F(z)| : z \in \mathbb{T}^n(\mathbf{0}, R)\}$ which sometimes can be better than (3.9) and (3.2).

Two following theorems have the proofs similar to those of Theorems 3.1 and 3.2.

Theorem 3.3. *If nonhomogeneous system of PDE's (3.1) belongs to the class $\mathcal{A}(\mathbf{G}, \mathbf{H}, \mathbf{L})$ and an entire function $F(z)$ satisfies (3.1), then F has bounded \mathbf{L} -index in joint variables, and*

$$\overline{\lim}_{|R| \rightarrow \infty} \frac{\ln \max\{|F(z)| : z \in \mathbb{T}^n(\mathbf{0}, R)\}}{\max_{\Theta \in [0, 2\pi]^n} \int_0^1 \langle R, \mathbf{L}(\tau R e^{i\Theta}) \rangle d\tau} \leq \left(1 + \sum_{j=1}^n p_j \right) \max\{1, c'\}, \tag{3.12}$$

where c' is defined in (3.13).

Proof. As in the proof of Theorem 3.1, dividing (3.6) by $(p_j \mathbf{1}_j + I)! \mathbf{L}^{p_j \mathbf{1}_j + I}(z)$, we obtain that for every $\|I\| = 1 + \sum_{\substack{k=1 \\ k \neq j}}^n p_k$ and $j \in \{1, \dots, n\}$:

$$\begin{aligned}
\frac{|F^{(p_j \mathbf{1}_j + I)}(z)|}{(p_j \mathbf{1}_j + I)! \mathbf{L}^{p_j \mathbf{1}_j + I}(z)} &\leq \frac{D_{I,j} |F^{(p_j \mathbf{1}_j)}(z)|}{(p_j \mathbf{1}_j + I)! \mathbf{L}^{p_j \mathbf{1}_j}(z)} + \sum_{\|S_j\| \leq p_j - 1} \frac{D_{I,j} B_{S_j, \mathbf{0}} |F^{(S_j)}(z)|}{(p_j \mathbf{1}_j + I)! \mathbf{L}^{p_j \mathbf{1}_j - S_j}(z)} \\
&+ \sum_{\mathbf{0} \leq M \leq I, M \neq \mathbf{0}} C_I^M B_{p_j \mathbf{1}_j, M} \frac{|F^{(p_j \mathbf{1}_j + I - M)}(z)|}{(p_j \mathbf{1}_j + I)! \mathbf{L}^{p_j \mathbf{1}_j + I - M}(z)} \\
&+ \sum_{\mathbf{0} \leq M \leq I} C_I^M \sum_{\|S_j\| \leq p_j - 1} B_{S_j, M} \frac{|F^{(S_j + I - M)}(z)|}{(p_j \mathbf{1}_j + I)! \mathbf{L}^{S_j + I - M}(z)} \\
&\leq D_{I,j} \left(\frac{|F^{(p_j \mathbf{1}_j)}(z)|}{(p_j \mathbf{1}_j + I)! \mathbf{L}^{p_j \mathbf{1}_j}(z)} \right. \\
&+ B \sum_{\|S_j\| \leq p_j - 1} \frac{|F^{(S_j)}(z)|}{(p_j \mathbf{1}_j + I)! \mathbf{L}^{p_j \mathbf{1}_j - S_j}(z)} \Big) \\
&+ \sum_{\mathbf{0} \leq M \leq I, M \neq \mathbf{0}} \frac{C_I^M B |F^{(p_j \mathbf{1}_j + I - M)}(z)|}{(p_j \mathbf{1}_j + I)! \mathbf{L}^{p_j \mathbf{1}_j + I - M}(z)} \\
&+ \sum_{\mathbf{0} \leq M \leq I} C_I^M \sum_{\|S_j\| \leq p_j - 1} \frac{B |F^{(S_j + I - M)}(z)|}{(p_j \mathbf{1}_j + I)! \mathbf{L}^{S_j + I - M}(z)} \\
&\leq \left(D_{I,j} \left(\frac{p_j!}{(p_j \mathbf{1}_j + I)!} + B \sum_{\|S_j\| \leq p_j - 1} \frac{(p_j \mathbf{1}_j - S_j)!}{(p_j \mathbf{1}_j + I)!} \right) \right. \\
&+ B \sum_{\substack{\mathbf{0} \leq M \leq I \\ M \neq \mathbf{0}}} C_I^M \frac{(p_j \mathbf{1}_j + I - M)!}{(p_j \mathbf{1}_j + I)!} \\
&+ B \sum_{\mathbf{0} \leq M \leq I} C_I^M \sum_{\|S_j\| \leq p_j - 1} \frac{(S_j + I - M)!}{(p_j \mathbf{1}_j + I)!} \Big) \\
&\quad \times \max \left\{ \frac{|F^{(S)}(z)|}{\mathbf{L}^S(z)} : \|S\| \leq \sum_{j=1}^n p_j \right\},
\end{aligned}$$

where

$$B = \max \left\{ B_{S_j, M}, B_{p_j \mathbf{1}_j, M} : j \in \{1, \dots, n\}, \mathbf{0} \leq M \leq I, \|I\| = 1 + \sum_{k=1, k \neq j}^n p_k \right\}.$$

Obviously, $\|p_j \mathbf{1}_j + I\| = 1 + \sum_{j=1}^n p_j$. For all $z \in \mathbb{C}^n \setminus \mathbb{D}^n(\mathbf{0}, R')$, it implies

$$\begin{aligned} \max \left\{ \frac{|F^{(K)}(z)|}{K! \mathbf{L}^K(z)} : \|K\| = 1 + \sum_{j=1}^n p_j \right\} \\ \leq \max\{1, c'\} \max \left\{ \frac{|F^{(S)}(z)|}{S! \mathbf{L}^S(z)} : \|S\| \leq \sum_{j=1}^n p_j \right\}, \end{aligned}$$

where

$$\begin{aligned} c' = \max_{\substack{\|I\|=1-p_j+\sum_{k=1}^n p_k, \\ j \in \{1, \dots, n\}}} \left(D_{I,j} \left(\frac{p_j!}{(p_j \mathbf{1}_j + I)!} + B \sum_{\|S_j\| \leq p_j - 1} \frac{(p_j \mathbf{1}_j - S_j)!}{(p_j \mathbf{1}_j + I)!} \right) \right. \\ \left. + B \sum_{\mathbf{0} \leq M \leq I, M \neq \mathbf{0}} C_I^M \frac{(p_j \mathbf{1}_j + I - M)!}{(p_j \mathbf{1}_j + I)!} \right. \\ \left. + B \sum_{\mathbf{0} \leq M \leq I} C_I^M \sum_{\|S_j\| \leq p_j - 1} \frac{(S_j + I - M)!}{(p_j \mathbf{1}_j + I)!} \right). \quad (3.13) \end{aligned}$$

In view of Theorem 2.2, the entire function F has bounded \mathbf{L} -index in joint variables. And by Lemma 2.6, estimate (3.12) holds. \square

By analogy to the proofs of Theorems 3.2 and 3.3, the following assertion can be proved.

Theorem 3.4. *If homogeneous system of PDE's (3.1) belongs to the class $\mathcal{A}(\mathbf{G}, \mathbf{0}, \mathbf{L})$ and an entire function F is a solution of the system, then F has bounded \mathbf{L} -index in joint variables, and*

$$\overline{\lim}_{|R| \rightarrow \infty} \frac{\ln \max\{|F(z)| : z \in \mathbb{T}^n(\mathbf{0}, R)\}}{\max_{\Theta \in [0, 2\pi]^n} \int_0^1 \langle R, \mathbf{L}(\tau R e^{i\Theta}) \rangle d\tau} \leq \sum_{j=1}^n p_j \max\{1, c\},$$

where c' is defined in (3.13) with $D_{I,j} = 0$ and $\|I\| = -p_j + \sum_{k=1}^n p_k$ instead of $\|I\| = 1 - p_j + \sum_{k=1}^n p_k$.

If $\mathbf{L}(z) \equiv \mathbf{1}$, then Theorem 3.1 implies the following corollary.

Corollary 3.5. *For all $z \in \mathbb{C}^n \setminus \mathbb{D}^n(\mathbf{0}, R')$ and for every $j \in \{1, \dots, n\}$, the entire functions H_j and G_{S_j} satisfy the following conditions:*

- 1) for every $\|S_j\| \leq p_j - 1$ and for each $M \in \mathbb{Z}_+^n$, $\|M\| \leq 1 + \sum_{\substack{k=1 \\ k \neq j}}^n p_k$,

$$\left| G_{S_j}^{(M)}(z) \right| \leq B_{S_j, M} |G_{p_j \mathbf{1}_j}(z)| \text{ and } \left| G_{p_j \mathbf{1}_j}^{(M)}(z) \right| \leq B_{p_j \mathbf{1}_j, M} |G_{p_j \mathbf{1}_j}(z)|,$$
- 2) for every $I \in \mathbb{Z}_+^n$, $\|I\| = 1 + \sum_{\substack{k=1 \\ k \neq j}}^n p_k$, $\left| H_j^{(I)}(z) \right| \leq D_{I,j} |H_j(z)|,$
- 3) $G_{p_j \mathbf{1}_j}(z) \neq 0,$

where $B_{S_j, M}$, $D_{I, j}$, $B_{p_j \mathbf{1}_j, M}$ are positive constants. If an entire function $F(z)$ satisfies (3.1), then F has bounded index in joint variables, and

$$\overline{\lim}_{|R| \rightarrow \infty} \frac{\ln \max\{|F(z)| : z \in \mathbb{T}^n(\mathbf{0}, R)\}}{r_1 + \dots + r_n} \leq \max\{1, c\},$$

where c is defined in (3.8).

Similar corollaries also can be obtained from Theorems 3.2–3.4.

Suppose that all coefficients in (3.1) are polynomials. Let $l_0(z) = \max_{1 \leq j \leq n} |z_j|$, $\deg G_{S_j}(z)$ be the degree of a polynomial $G_{S_j}(z)$ that is the highest degree of its terms. Then Theorem 3.1 implies the following corollary.

Corollary 3.6. *Let $H_j(z)$ be a monomial, $G_{S_j}(z)$ be a polynomial, $G_{p_j \mathbf{1}_j}(z) \equiv 1$ in (3.1) for every $j \in \{1, \dots, n\}$ and for each $\|S_j\| \leq p_j - 1$, $s = \max_{j, S_j} \frac{\deg G_{S_j}(z)}{\|p_j \mathbf{1}_j - S_j\|}$, $\mathbf{L}(z) = (l_0^s(z), \dots, l_0^s(z))$. Then every entire solution F of (3.1) has bounded \mathbf{L} -index in joint variables, and the inequality*

$$\overline{\lim}_{|R| \rightarrow \infty} \frac{\ln \max\{|F(z)| : z \in \mathbb{T}^n(\mathbf{0}, R)\}}{\frac{(r^*)^s}{s+1} \sum_{j=1}^n r_j} \leq \max\{1, c\}$$

holds, where c is defined in (3.8).

Remark 3.7. We should like to note that the obtained propositions are improvements of the corresponding theorems for $n = 1$ in [14]. Indeed, the author considered a positive continuous function $l = l(|z|)$ such that $l'(t) = o(l^2(t))$ as $t \rightarrow +\infty$. But our restrictions on the function l are weaker. We study a positive continuous function $l = l(z)$ such that $(-u(r, \theta))'_r / l^2(r, \theta) \rightarrow 0$ uniformly in $\theta \in [0, 2\pi]$ as $r \rightarrow \infty$, where $u(r, \theta) = l(re^{i\theta})$.

For example, if $n = 1$, then system (3.1) reduces to the following equation:

$$g_p(z)f^{(p)}(z) + \sum_{j=0}^{p-1} g_j(z)f^{(j)}(z) = h(z), \quad (3.14)$$

where h and g_j are entire functions. Theorem 3.1 implies the corollary for $n = 1$.

Corollary 3.8. *Let $l \in QW$ and for all $z \in \mathbb{C}$ such that $|z| > r'$ entire functions h and g_j satisfy the following conditions:*

- 1) $|g_j^{(m)}(z)| \leq B_{j, m} l^{p-j+m}(z) |g_p(z)|$ for every $j \in \{0, \dots, p\}$, $m \in \{0, 1\}$,
- 2) $|h'(z)| \leq D l(z) |h(z)|$,
- 3) $g_p(z) \neq 0$,

where $B_{j,m}$ and D are positive constants. If an entire function f satisfies (3.14), then f has bounded l -index in joint variables, and

$$\overline{\lim}_{r \rightarrow \infty} \frac{\ln \max\{|f(z)| : |z| = r\}}{\max_{\theta \in [0, 2\pi]} \int_0^r l(\tau e^{i\theta}) d\tau} \leq \max\{1, c\},$$

where $c = D(1 + \sum_{j=0}^{p-1} B_{j,0}) + \sum_{j=0}^p B_{j,1} + \sum_{j=0}^{p-1} B_{j,0}$.

Similar corollaries can be obtained from Theorems 3.2–3.4 for $n = 1$. Particularly, in a corollary from Theorem 3.2 the constant c equals $\sum_{j=0}^{p-1} B_{j,0}$, but in a corollary from Theorem 3.4 the constant c' equals $\max_j\{p, 2, 2B_{j,0}\}$, where p is the order of differential equation. Each of these constants may be greater than or lesser than the other [14].

Suppose that all coefficients in (3.14) are polynomials.

Corollary 3.9. *Let $h(z), g_j(z)$ be polynomials for every $j \in \{0, \dots, p-1\}$, $g_p(z) \equiv 1$ in (3.14), $s = \max_{0 \leq j \leq p-1} \frac{\deg g_j(z)}{p-j}$, $l(z) = |z|^s$, $z \in \mathbb{C}$. Then every entire solution f of (3.14) has bounded l -index, and*

$$\overline{\lim}_{r \rightarrow \infty} \frac{\ln \max\{|f(z)| : |z| = r\}}{r^{s+1}/(s+1)} \leq \max\{1, c\},$$

where $\mathcal{J} = \{j : \frac{\deg g_j}{p-j} = s, 0 \leq j \leq p-1\}$, $g_j(z) = b_{j,m}z^m + \dots + b_{j,0}$, $c = \sum_{j \in \mathcal{J}} |b_{j, \deg g_j}|$.

Note that Theorems 3.1–3.4 are proved for system (3.1). Perhaps, the following conjecture is true.

Conjecture 3.10 (O.B. Skaskiv). *The counterparts of Theorems 3.1–3.4 for entire solutions of system (1.3) are valid.*

References

- [1] A. Bandura, *New criteria of boundedness of \mathbf{L} -index in joint variables for entire functions*, Mat. Visn. Nauk. Tov. Shevchenka **13** (2016), 58–67 (Ukrainian).
- [2] A.I. Bandura, M.T. Bordulyak, and O.B. Skaskiv, *Sufficient conditions of boundedness of L -index in joint variables*, Mat. Stud. **45** (2016), No. 1, 12–26.
- [3] A.I. Bandura, N.V. Petrechko, and O.B. Skaskiv, *Analytic functions in a polydisc of bounded \mathbf{L} -index in joint variables*, Mat. Stud. **46** (2016), No. 1, 72–80.
- [4] A. Bandura, N. Petrechko, and O. Skaskiv, *Maximum modulus in a bidisc of analytic functions of bounded \mathbf{L} -index and an analogue of Hayman's theorem*, Math. Bohem. **143** (2018), No. 4, 339–354.
- [5] A.I. Bandura and O.B. Skaskiv, *Entire functions of bounded L -index in direction*, Mat. Stud. **27** (2007), No. 1, 30–52 (Ukrainian).
- [6] A.I. Bandura and O.B. Skaskiv, *Sufficient sets for boundedness L -index in direction for entire functions*, Mat. Stud. **30** (2008), No. 2, 177–182.

-
- [7] A. Bandura and O. Skaskiv, *Entire Functions of Several Variables of Bounded Index*, Publisher I.E. Chyzykykov, Chyslo, Lviv, 2016.
- [8] A. Bandura and O. Skaskiv, *Analytic in an unit ball functions of bounded L -index in joint variables*, Ukr. Mat. Visn. **14** (2017), No. 1, 1–15. Engl. transl.: J. Math. Sci. (N.Y.) **227** (2017), No. 1, 1–12.
- [9] A. Bandura and O. Skaskiv, *Analytic Functions in the Unit Ball. Bounded L -Index in Joint Variables and Solutions of Systems of PDE's*, LAP Lambert Academic Publishing, Beau-Bassin, 2017.
- [10] A. Bandura and O. Skaskiv, *Asymptotic estimates of entire functions of bounded L -index in joint variables*, Novi Sad J. Math. **48** (2018), No. 1, 103–116.
- [11] A. Bandura, O. Skaskiv, and P. Filevych, *Properties of entire solutions of some linear PDE's*, J. Appl. Math. Comput. Mech. **16** (2017), No. 2, 17–28.
- [12] M.T. Bordulyak, *The space of entire in \mathbb{C}^n functions of bounded L -index*, Mat. Stud. **4** (1995), 53–58 (Ukrainian).
- [13] M.T. Bordulyak, *Boundedness of L -Index of Entire Functions of Several Complex Variables*: Ph.D thesis, Lviv University, Lviv, 1995 (Ukrainian).
- [14] M.T. Bordulyak, *On the growth of entire solutions of linear differential equations*, Mat. Stud. **13** (2000), No. 2, 219–223.
- [15] M.T. Bordulyak and M.M. Sheremeta, *Boundedness of the L -index of an entire function of several variables*, Dopov. Nats. Akad. Nauk Ukr. (1993), No. 9, 10–13 (Ukrainian).
- [16] B.C. Chakraborty and R. Chanda, *A class of entire functions of bounded index in several variables*, J. Pure Math. **12** (1995), 16–21.
- [17] B.C. Chakraborty and T.K. Samanta, *On entire functions of bounded index in several variables*, J. Pure Math. **17** (2000), 53–71.
- [18] B.C. Chakraborty and T.K. Samanta, *On entire functions of L -bounded index*, J. Pure Math. **18** (2001), 53–64.
- [19] W.K. Hayman, *Differential inequalities and local valency*, Pacific J. Math. **44** (1973), No. 1, 117–137.
- [20] G.J. Krishna and S.M. Shah, *Functions of bounded indices in one and several complex variables*, In: Mathematical essays dedicated to A.J. Macintyre, Ohio Univ. Press, Athens, Ohio, 1970, 223–235.
- [21] V.O. Kushnir, *Analogue of Hayman theorem for analytic functions of bounded l -index*, Visn. Lviv Un-ty. Ser. Mekh. Math. **53** (1999), 48–51 (Ukrainian).
- [22] B. Lepson, *Differential Equations of Infinite Order, Hyperdirichlet Series and Entire Functions of Bounded Index*, Entire Functions and Related Parts of Analysis (Proc. Sympos. Pure Math., LaJolla, Calif., 1966), Amer. Math. Soc., Providence, R.I., 1968, 298–307.
- [23] F. Nuray and R.F. Patterson, *Entire bivariate functions of exponential type*, Bull. Math. Sci., **5** (2015), No. 2, 171–177.
- [24] F. Nuray and R.F. Patterson, *Multivalence of bivariate functions of bounded index*, Matematiche (Catania) **70** (2015), No. 2, 225–233.

- [25] Q.I. Rahman and J. Stankiewicz, *Differential inequalities and local valency*, Pacific J. Math. **54** (1974), No. 2, 165–181.
- [26] M. Salmassi, *Functions of bounded indices in several variables*, Indian J. Math. **31** (1989), No. 3, 249–257.
- [27] M. Salmassi, *Some Classes of Entire Functions of Exponential Type in One and Several Complex Variables*, Ph.D thesis, University of Kentucky, Lexington, KY, 1978.
- [28] M.N. Sheremeta, *Entire functions and Dirichlet series of bounded l -index*, Izv. Vyssh. Uchebn. Zaved. Mat. (1992), No. 9, 81–87 (Russian); Engl. transl.: Russian Math. (Iz. VUZ). **36** (1992), No. 9, 76–82.
- [29] M. Sheremeta, *Analytic Functions of Bounded Index*, VNTL Publishers, Lviv (1999).
- [30] V. Singh and R.M. Goel, *Differential inequalities and local valency*, Houston J. Math. **18** (1992), No. 2, 215–233.

Received October 28, 2017, revised November 6, 2017.

Andriy Bandura,

Ivano-Frankivsk National Technical University of Oil and Gas, 15 Karpatska Str., Ivano-Frankivsk, 76019, Ukraine,
E-mail: andriykopanytsia@gmail.com

Oleh Skaskiv,

Ivan Franko National University of Lviv, 1 Universytetska Str., Lviv, 79000, Ukraine,
E-mail: olskask@gmail.com

Аналог теореми Хеймана та його застосування до однієї системи лінійних рівнянь з частинними похідними

Andriy Bandura and Oleh Skaskiv

Аналог відомої теореми Хеймана застосовується до дослідження обмеженості L -індексу за сукупністю змінних цілих розв'язків деяких лінійних систем рівнянь з частинними похідними вищих порядків та знайдено достатні умови, які гарантують цю обмеженість, де $L(z) = (l_1(z), \dots, l_n(z))$, $l_j : \mathbb{C}^n \rightarrow \mathbb{R}_+$ — неперервна функція, $j \in \{1, \dots, n\}$. Також отримано оцінки зростання цих розв'язків. Наведено приклади систем РЧП, які доводять точність встановлених оцінок для цілих розв'язків. Отримані результати також є новими в одновимірному випадку, бо послаблено додаткові умови на додатну неперервну функцію l .

Ключові слова: ціла функція, обмежений L -індекс за сукупністю змінних, лінійна система РЧП вищих порядків, аналітична теорія РЧП, цілий розв'язок, лінійне диференціальне рівняння вищого порядку.