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Inverse Scattering Problems with the Potential Known on an Interior Subinterval

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The inverse scattering problem for one-dimensional Schrödinger operators on the line is considered when the potential is real valued and integrable and has a finite first moment. It is shown that the potential on the line is uniquely determined by the mixed scattering data consisting of the scattering matrix, known potential on a finite interval, and one nodal point on the known interval for each eigenfunction.

Key words: Schrödinger equation, inverse scattering problem, potential recovery with partial data.

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1. Introduction

In this paper we consider the inverse scattering problems for one-dimensional Schrödinger operators on the real line and study the unique recovery of the potential that is known a priori on a finite interval $[a, b]$. Let H be the self-adjoint Schrödinger operator

$$H := -\frac{d^2}{dx^2} + V(x) \tag{1.1}$$

on $L^2(\mathbb{R})$, where the potential V is real valued and belongs to $L^1_1(\mathbb{R})$, the class of measurable functions on the real axis $\mathbb{R}$ such that $\int_{-\infty}^{\infty}(1 + |x|)\,|V(x)|\,\mathrm{d}x$ is finite. It is known [14] that H has absolutely continuous spectrum $[0, \infty)$ and a finite number of simple negative eigenvalues (bound-state energies), denoted by $\{-\kappa_j^2\}_{j=1}^N$, where $\kappa_j > 0$. Moreover, for each eigenvalue $-\kappa_j^2$, the corresponding eigenfunction has $(j-1)$ zeros (nodal points) on $\mathbb{R}$ denoted by $\{x_j^i\}_{i=1}^{j-1}$.

The inverse scattering problem is about the construction of V in terms of the scattering data consisting of a reflection coefficient, the bound-state energies, and the bound-state norming constants (see (2.4) below). There are various methods to solve the inverse scattering problems, such as the Marchenko method [15], the trace method [7], and so on. However, the bound-state norming constants have no obvious physical meaning, which is not ideal from the physical point of view.

There are many results (see [1–3, 5, 9, 17, 20, 21] and references cited therein) related to inverse scattering problems for one-dimensional Schrödinger operators defined on the real line $\mathbb{R}$ with incomplete scattering data. These results show that if the potential is known on a half-line, then the norming constants and even the bound-state energies are not needed to recover the potential uniquely (some of these papers are limited to the case where V is assumed to vanish on a half-line). In 1994, Weder (cf., [3, p. 222]) raised a question whether one can uniquely reconstruct V by the mixed scattering data consisting of the bound-state energies, the reflection coefficient $L(k)$ (or $R(k)$) for $k \in \mathbb{R}$ and the known potential on a finite interval $[a, b]$, i.e., all the bound-state norming constants are missing. Aktosun and Weder [4] studied this inverse problem when only one norming constant is missing and proved that the missing norming constant in the data can cause at most a double nonuniqueness in the recovery. They also illustrated the nonuniqueness with some explicit examples. This enlighten us that, when the potential is known a priori on a finite interval and some norming constants are missing, we need an additional condition to obtain the uniqueness for this type of inverse scattering problems.

The aim of this paper is to study the uniqueness problem of recovering V on the real line $\mathbb{R}$ under the condition that the potential is known a priori on a finite interval $[a, b]$. More precisely, we prove that the potential, which further is a constant on a subinterval of the known interval $[a, b]$, is uniquely determined by the mixed scattering data consisting of the scattering matrix and additional information related to the zeros of eigenfunctions, which are just as experimentally observable as eigenvalues in some situations (see [6,10,11,16] and references cited therein). Consequently, all the bound-state energies and bound-state norming constants may be missing.

The strategy we use to prove our unique results is to establish a Vandermonde matrix equation associated with the unknown bound-state energies and the unknown bound-state norming constants. We find that if the known potential is a constant on a subinterval $[a_0, b_0] \subset [a, b]$, then the scattering matrix can determine the bound-state energies uniquely by a Vandermonde matrix equation (see (3.25) and (3.28) below). Note here that when the bound-state energies and either one of the reflection coefficients are given as the scattering data, the knowledge of the potential on a finite interval can not give enough information to determine the unspecified norming constants, which means the potential can not be constructed uniquely in generally. Therefore, we need additional information to deal with this uniqueness problem. We put forward nodal points $\{x_j^i\}$ as additional spectral data, and suppose there is one nodal point known on $[a, b]$ for each eigenfunction with $j = 2, \ldots, N$. Especially, because the eigenfunction corresponding to the first eigenvalue $-\kappa_1^2$ has no zeros on $\mathbb{R}$, we will further assume that the value $\int_{-\infty}^{a} V(t)\mathrm{d}t$ or $\int_{b}^{\infty} V(t)\mathrm{d}t$ is known a priori. Together with these data, we determine the norming constants uniquely, and finally obtain the uniqueness theorem.

The method we use is a generalization of that used by Wei and Xu [22], for which the basic idea is to relate our data to the Marchenko integral equations where both integral equations have generalized degeneracy (see [13, 18]) in the case that the part associated with the continuous spectrum being the same for two systems.

The paper is organized as follows. In Section 2, we state the main results of this paper. Section 3 contains the proofs of our main results.

2. Main results

In this section, we will give the main results of this paper, which are associated with the unique determination of the potential V on $\mathbb{R}$ under the condition that it is known a priori on a finite interval.

Consider the radial Schrödinger equation

$$-y''(k,x)+V(x)y(k,x)=k^2y(k,x),\quad x\in\mathbb{R}, \tag{2.1}$$

where k^2 is the energy, x is the space coordinate, and the prime denotes the derivative with respect to x. For the L_1^1-class potentials there are two linearly independent solutions of (2.1), $f_l(k,x)$ and $f_r(k,x)$, known as the Jost solutions from the left and from the right, respectively, satisfying the boundary conditions:

$$\begin{aligned} e^{-ikx}f_l(k,x)&=1+o(1), & e^{-ikx}f_l'(k,x)&=ik+o(1), & x&\to+\infty,\\ e^{ikx}f_r(k,x)&=1+o(1), & e^{ikx}f_r'(k,x)&=-ik+o(1), & x&\to-\infty. \end{aligned}$$

From the spatial asymptotics

$$f_l(k,x)=\frac{e^{ikx}}{T(k)}+\frac{L(k)}{T(k)}e^{-ikx}+o(1),\qquad x\to-\infty, \tag{2.2}$$

$$f_r(k,x)=\frac{e^{-ikx}}{T(k)}+\frac{R(k)}{T(k)}e^{ikx}+o(1),\qquad x\to+\infty, \tag{2.3}$$

we obtain the scattering coefficients, namely, T is the transmission coefficient, and L and R are the reflection coefficients from the left and right, respectively. The scattering matrix $S(k)$ associated with $V(x)$ is a 2×2 unitary matrix defined as

$$S(k)=\begin{pmatrix} T(k) & R(k)\\ L(k) & T(k)\end{pmatrix}.$$

It is known [8,14,15] that the potential V on the whole line is uniquely determined by the scattering data and consists of

$$\{L(k),k\in\mathbb{R}\}\cup\left\{\kappa_j,m_j^-\right\}_{j=1}^N\quad\text{or}\quad\{R(k),\ k\in\mathbb{R}\}\cup\left\{\kappa_j,m_j^+\right\}_{j=1}^N, \tag{2.4}$$

where $m_j^\pm$ are the bound-state norming constants corresponding to the bound-state energy $-\kappa_j^2$ defined as

$$m_j^-=\|f_r(i\kappa_j,\cdot)\|^{-2},\quad m_j^+=\|f_l(i\kappa_j,\cdot)\|^{-2}. \tag{2.5}$$

We state the main results of this paper through two cases. We first treat the case where all bound-state energies and bound-state norming constants are missing (see Theorem 2.1 below). The case where only norming constants are missing will be considered in Theorems 2.2 and 2.4.

Theorem 2.1. *Let V be a real-valued potential belonging to $L_1^1(\mathbb{R})$. Suppose the following conditions are satisfied:*

(i) *the potential V is known on a finite interval $[a,b]$ and is a constant C on a subinterval $[a_0,b_0] \subset [a,b]$;*

(ii) *for each j with $j = 2,\dots,N$, the eigenfunction $f_l(i\kappa_j, x)$ has one known nodal point x_j' satisfying $x_j' \in [a,b]$;*

(iii) *the value $\int_{-\infty}^{a} V(t)\mathrm{d}t$ or $\int_{b}^{\infty} V(t)\mathrm{d}t$ is known a priori.*

Then V on the whole line is uniquely determined by the scattering matrix $S(k)$ for $k \in \mathbb{R}$.

For the case where two or more norming constants are missing, we have the following result.

Theorem 2.2. *Let V be a real-valued potential belonging to $L_1^1(\mathbb{R})$. Suppose the following conditions are satisfied:*

(i) *the potential V is known on a finite interval $[a,b]$ and is a constant C on a subinterval $[a_0,b_0] \subset [a,b]$;*

(ii) *the norming constants $\{m_{l_s}^-\}_{s=1}^n$ are known with $0 \le l_n \le N-2$, and for each j with $j \notin \{l_s\}_{s=1}^n \cup \{1\}$, the eigenfunction $f_l(i\kappa_j, x)$ has one known nodal point x_j' satisfying $x_j' \in [a,b]$;*

(iii) *the value $\int_{-\infty}^{a} V(t)\mathrm{d}t$ is known a priori.*

Then V on the whole line is uniquely determined by $\{\kappa_j\}_{j=1}^N$ and the refection coefficient $L(k)$ for $k \in \mathbb{R}$.

Remark 2.3. In fact, the condition (iii) of Theorem 2.2 is not needed in the case of $l_1 = 1$.

For the case where only one norming constant is missing, we have the following result.

Theorem 2.4. *Let V be a real-valued potential belonging to $L_1^1(\mathbb{R})$. Suppose the following conditions are satisfied:*

(i) *the potential V is known on a finite interval $[a,b]$;*

(ii) *the norming constants $\{m_j^-\}_{j=1, j\neq j_0}^N$ are known with $1 \le j_0 \le N$;*

(iii) *the value $\int_{-\infty}^{a} V(t)\mathrm{d}t$ is known a priori.*

Then V on the whole line is uniquely determined by $\{\kappa_j\}_{j=1}^N$ and the refection coefficient $L(k)$ for $k \in \mathbb{R}$.

3. The proofs

In order to prove our main results, we need the following lemmas.

Lemma 3.1. *Let $f_1(k,x), f_2(k,x)$ and $f_3(k,x)$ be three nontrivial solutions of the equation*

$$y''(k,x) = k^2 y(k,x), \qquad x \in [0,1], \tag{3.1}$$

where $k^2 \neq 0$ is fixed.

(i) *If the two solutions $f_2(k,x)$ and $f_3(k,x)$ are linearly independent, then there exists at most one non-zero real constant c and at most finitely many zeros, denoted as $x_0 \in [0,1]$, such that*

$$[f_1(f_2 + cf_3)]'(k,x_0) = 0.$$

(ii) *If the two solutions $f_1(k,x)$ and $f_2(k,x)$ have a finite number of zeros on $[0,1]$ respectively, then either*

$$[f_1 f_2]'(k,x) \equiv 0,$$

or there exist at most finitely many zeros, denoted as $x_0 \in [0,1]$, such that

$$[f_1 f_2]'(k,x_0) = 0.$$

Proof. It is easy to see that equation (3.1) has the system of basic solutions e^{ikx} and e^{-ikx} for the fixed $k^2 \neq 0$. So, there exist constants a_j and b_j such that $f_j(k,x) = a_j e^{ikx} + b_j e^{-ikx}$ for $j = 1,2,3$.

(i) For the constant c $(c \neq 0)$, we have

$$[f_1(f_2 + cf_3)]'(k,x) = 2ikA_1 e^{2ikx} - 2ikB_1 e^{-2ikx}, \tag{3.2}$$

where

$$A_1 = a_1(a_2 + ca_3), \quad B_1 = b_1(b_2 + cb_3).$$

Obviously, the function $[f_1(f_2 + cf_3)]'(k,x)$ has a finite number of zeros on $[0,1]$ provided that $|A_1|^2 + |B_1|^2 \neq 0$.

Basing on the fact that $f_2(k,x)$ and $f_3(k,x)$ are linearly independent, there are three cases to consider. For the case $|a_1| = 0$, $|b_1| \neq 0$. Only if $c = -b_2/b_3$ for the cases of $b_2 \neq 0$ and $b_3 \neq 0$, we will have $|A_1|^2 + |B_1|^2 = 0$. Otherwise we all have $|A_1|^2 + |B_1|^2 \neq 0$ for any real constant c $(c \neq 0)$. The other cases for $|a_1| \neq 0$, $|b_1| \neq 0$ and $|a_1| \neq 0$, $|b_1| = 0$ can be treated in a similar way. In all, there exists at most one non-zero real constant c and at most finitely many zeros, denoted as $x_0 \in [0,1]$, such that $[f_1(f_2 + cf_3)]'(k,x_0) = 0$.

(ii) It gives that

$$[f_1 f_2]'(k,x) = 2ikA_2 e^{2ikx} - 2ikB_2 e^{-2ikx}, \tag{3.3}$$

where

$$A_2 = a_1 a_2, \quad B_2 = b_1 b_2.$$

Thus, if $|A_2|^2 + |B_2|^2 = 0$, then we have $[f_1 f_2]'(k,x) \equiv 0$. Otherwise, if $|A_2|^2 + |B_2|^2 \neq 0$, it is clear that the function $[f_1 f_2]'(k,x)$ has a finite number of zeros on $[0,1]$. The proof is completed. □

For a finite number of different values $k = k_s$ with $s = 1, \dots, n$, Lemma 3.1 implies that there exist common constants c and $x' \in [0, 1]$ such that $[f_1(f_2 + cf_3)]'(k_s, x') \neq 0$ for all k_s with $s = 1, \dots, n$.

The following lemma can be derived from [22, Lemma 3.1].

Lemma 3.2. *Let $\lambda_1 < \lambda_2 < \cdots < \lambda_n$ and $\widetilde{\lambda}_1 < \widetilde{\lambda}_2 < \cdots < \widetilde{\lambda}_{\widetilde{n}}$ with $n \geq \widetilde{n}$. Denote the $m \times n$ Vandermonde matrix associated with entries $\{\lambda_j\}_{j=1}^n$ by $V_{m\times n}[\lambda_j]_{j=1}^n$, that is,*

$$V_{m\times n}[\lambda_j]_{j=1}^n = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \vdots & \vdots & & \vdots \\ \lambda_1^{m-1} & \lambda_2^{m-1} & \cdots & \lambda_n^{m-1} \end{pmatrix}. \tag{3.4}$$

If there exists $m' \leq \widetilde{n}$ satisfying $\lambda_{l_j} = \widetilde{\lambda}_{l_j}$ for $j = 1, \dots, m'$, and $m := n + \widetilde{n} - m'$,

$$V_{m\times n}[\lambda_j]_{j=1}^n A = V_{m\times \widetilde{n}}[\widetilde{\lambda}_j]_{j=1}^{\widetilde{n}} \widetilde{A},$$

where $A = [a_1, \dots, a_n]^T \in \mathbb{R}^n$ and $\widetilde{A} = [\widetilde{a}_1, \dots, \widetilde{a}_{\widetilde{n}}]^T \in \mathbb{R}^{\widetilde{n}}$ are such that $\widetilde{a}_j \neq 0$ and $\widetilde{a}_j \neq 0$ for all $1 \leq j \leq \widetilde{n}$. Then $\lambda_j = \widetilde{\lambda}_j$, $a_j = \widetilde{a}_j$ for all $j = 1, 2, \dots, \widetilde{n}$ and $a_j = 0$ for $j = \widetilde{n} + 1, \dots, n$. In particular, in the case where $m' = 0$, the result still holds true.

For the purpose of this paper, together with the Schrödinger operator H defined by (1.1), we consider another operator $\widetilde{H}$ of the same form but with different coefficient $\widetilde{V}$, i.e., we consider another Schrödinger equation

$$-\widetilde{y}''(k, x) + \widetilde{V}(x)\widetilde{y}(k, x) = k^2 \widetilde{y}(k, x), \quad x \in \mathbb{R}. \tag{3.5}$$

We agree that everywhere below if the symbol ν denotes an object related to H, then $\widetilde{\nu}$ denotes the analogous object related to $\widetilde{H}$.

The following lemma is crucial for the proofs of our main results.

Lemma 3.3. *Consider two Schrödinger operators H and $\widetilde{H}$. Suppose $V(x) = C = \widetilde{V}(x)$ for a.e. $x \in [a_0, b_0]$.*

(i) *If $L(k) = \widetilde{L}(k)$ for $k \in \mathbb{R}$, then*

$$\sum_{j=1}^{N} (\kappa_j^2)^l m_j^- (f_r \widetilde{f}_r)'(i\kappa_j, x) = \sum_{j=1}^{\widetilde{N}} (\widetilde{\kappa}_j^2)^l \widetilde{m}_j^- (f_r \widetilde{f}_r)'(i\widetilde{\kappa}_j, x) \tag{3.6}$$

for $x \in [a_0, b_0]$ and $l = 0, 1, \dots, 2M - 1$ with $M = N + \widetilde{N}$.

(ii) *If $R(k) = \widetilde{R}(k)$ for $k \in \mathbb{R}$, then*

$$\sum_{j=1}^{N} (\kappa_j^2)^l m_j^+ (f_l \widetilde{f}_l)'(i\kappa_j, x) = \sum_{j=1}^{\widetilde{N}} (\widetilde{\kappa}_j^2)^l \widetilde{m}_j^+ (f_l \widetilde{f}_l)'(i\widetilde{\kappa}_j, x) \tag{3.7}$$

for $x \in [a_0, b_0]$ and $l = 0, 1, \dots, 2M - 1$ with $M = N + \widetilde{N}$.

Proof. We only consider the case for $L(k) = \widetilde{L}(k)$, the other case for $R(k) = \widetilde{R}(k)$ can be treated in a similar way. It is known [14, pp. 132–133] that the Marchenko integral equation when used in inverse scattering problems associated with the two operators H and $\widetilde{H}$ can be written as

$$B^-(x,y) + \Phi^-(x,y) + \int_{-\infty}^{x} B^-(x,t)\Phi^-(t,y)\mathrm{d}t = 0, \tag{3.8}$$

where $y < x$ and the function $\Phi^-(x,y)$ has the form

$$\Phi^-(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty}[L(k) - \widetilde{L}(k)]\widetilde{f}_r(k,x)\widetilde{f}_r(k,y)\mathrm{d}k \\ + \sum_{j=1}^{N} m_j^- \widetilde{f}_r(i\kappa_j,x)\widetilde{f}_r(i\kappa_j,y) - \sum_{j=1}^{\widetilde{N}} \widetilde{m}_j^- \widetilde{f}_r(i\widetilde{\kappa}_j,x)\widetilde{f}_r(i\widetilde{\kappa}_j,y). \tag{3.9}$$

Here $\widetilde{f}_r(k,x)$ is the Jost solution of (3.5) from the left and $\widetilde{m}_j^-$ is the bound-state norming constant defined by (2.5) corresponding to the bound-state energy $-\widetilde{\kappa}_j^2$. Furthermore, the function $B^-(x,y)$ satisfies the differential equation

$$\frac{\partial^2 B^-}{\partial x^2} - V(x)B^- = \frac{\partial^2 B^-}{\partial y^2} - \widetilde{V}(y)B^- \tag{3.10}$$

and the condition

$$B^-(x,x) = \frac{1}{2}\int_{-\infty}^{x}[V(t) - \widetilde{V}(t)]\mathrm{d}t. \tag{3.11}$$

As a transformation operator, we have

$$f_r(k,x) = \widetilde{f}_r(k,x) + \int_{-\infty}^{x} B^-(x,t)\widetilde{f}_r(k,t)\mathrm{d}t. \tag{3.12}$$

Since $L(k) = \widetilde{L}(k)$ for $k \in \mathbb{R}$, it follows from (3.9) that

$$\Phi^-(x,y) = \sum_{j=1}^{N} m_j^- \widetilde{f}_r(i\kappa_j,x)\widetilde{f}_r(i\kappa_j,y) - \sum_{j=1}^{\widetilde{N}} \widetilde{m}_j^- \widetilde{f}_r(i\widetilde{\kappa}_j,x)\widetilde{f}_r(i\widetilde{\kappa}_j,y), \tag{3.13}$$

which together with (3.8) and (3.12) yields

$$B^-(x,y) = -\Phi^-(x,y) - \int_{-\infty}^{x} B^-(x,t)\Phi^-(t,y)\mathrm{d}t \\ = \sum_{j=1}^{\widetilde{N}} \widetilde{m}_j^- \widetilde{f}_r(i\widetilde{\kappa}_j,x)\widetilde{f}_r(i\widetilde{\kappa}_j,y) + \sum_{j=1}^{\widetilde{N}} \widetilde{m}_j^- \widetilde{f}_r(i\widetilde{\kappa}_j,y)\int_{-\infty}^{x} B^-(x,t)\widetilde{f}_r(i\widetilde{\kappa}_j,t)\mathrm{d}t \\ - \sum_{j=1}^{N} m_j^- \widetilde{f}_r(i\kappa_j,x)\widetilde{f}_r(i\kappa_j,y) - \sum_{j=1}^{N} m_j^- \widetilde{f}_r(i\kappa_j,y)\int_{-\infty}^{x} B^-(x,t)\widetilde{f}_r(i\kappa_j,t)\mathrm{d}t$$

$$= \sum_{j=1}^{\widetilde{N}} \widetilde{m}_j^- f_r(i\widetilde{\kappa}_j, x)\widetilde{f}_r(i\widetilde{\kappa}_j, y) - \sum_{j=1}^{N} m_j^- f_r(i\kappa_j, x)\widetilde{f}_r(i\kappa_j, y). \tag{3.14}$$

It can be checked from [12, Theorem 4.15(b)] that the solution $B(x, y)$ of boundary value problem (3.10), (3.11) is a continuous function on $\Omega = \{(x, y) \in \mathbb{R}^2 : y \leq x\}$. By (3.11) and (3.14), we have for $x \in \mathbb{R}$ that

$$\sum_{j=1}^{\widetilde{N}} \widetilde{m}_j^- f_r(i\widetilde{\kappa}_j, x)\widetilde{f}_r(i\widetilde{\kappa}_j, x) - \sum_{j=1}^{N} m_j^- f_r(i\kappa_j, x)\widetilde{f}_r(i\kappa_j, x) = \frac{1}{2}\int_{-\infty}^{x} [V(t) - \widetilde{V}(t)]\mathrm{d}t,$$

which together with the condition $V(x) = C = \widetilde{V}(x)$ for $x \in [a_0, b_0]$ yields that for all $x \in [a_0, b_0]$

$$\begin{aligned} &\sum_{j=1}^{\widetilde{N}} \widetilde{m}_j^- f_r(i\widetilde{\kappa}_j, x)\widetilde{f}_r(i\widetilde{\kappa}_j, x) - \sum_{j=1}^{N} m_j^- f_r(i\kappa_j, x)\widetilde{f}_r(i\kappa_j, x) \\ &\qquad = \frac{1}{2}\int_{-\infty}^{x} [V(t) - \widetilde{V}(t)]\mathrm{d}t = \frac{1}{2}\int_{-\infty}^{a_0} [\widetilde{V}(t) - V(t)]\mathrm{d}t =: C'. \end{aligned} \tag{3.15}$$

Differentiating the identity (3.15) with respect to x, we infer for $x \in [a_0, b_0]$ that

$$\sum_{j=1}^{\widetilde{N}} \widetilde{m}_j^- (f_r\widetilde{f}_r)'(i\widetilde{\kappa}_j, x) - \sum_{j=1}^{N} m_j^- (f_r\widetilde{f}_r)'(i\kappa_j, x) = 0. \tag{3.16}$$

Differentiating the identity (3.15) twice with respect to x, basing on the condition $V(x) = C = \widetilde{V}(x)$ a.e. for $x \in [a_0, b_0]$ and the equation

$$(f_r\widetilde{f}_r)''(k, x) = 2(C - k^2)(f_r\widetilde{f}_r)(k, x) + 2(f_r'\widetilde{f}_r')(k, x) \quad \text{a.e. on } [a_0, b_0], \tag{3.17}$$

we derive from (3.15) that

$$\begin{aligned} &\sum_{j=1}^{\widetilde{N}} \widetilde{m}_j^- [\widetilde{\kappa}_j^2(f_r\widetilde{f}_r) + (f_r'\widetilde{f}_r')](i\widetilde{\kappa}_j, x) - \sum_{j=1}^{N} m_j^- [\kappa_j^2(f_r\widetilde{f}_r) + (f_r'\widetilde{f}_r')](i\kappa_j, x) \\ &\qquad = -C\left[\sum_{j=1}^{\widetilde{N}} \widetilde{m}_j^- \widetilde{V}(x)(f_r\widetilde{f}_r)(i\widetilde{\kappa}_j, x) - \sum_{j=1}^{N} m_j^- (f_r\widetilde{f}_r)(i\kappa_j, x)\right] \\ &\qquad = -CC' \quad \text{a.e. on } [a_0, b_0]. \end{aligned}$$

Differentiating again equality (3.15) with respect to x for the third time, using the condition $V(x) = C = \widetilde{V}(x)$ a.e. for $x \in [a_0, b_0]$ and the fact that

$$(f_r'\widetilde{f}_r')'(k, x) = (C - k^2)(f_r\widetilde{f}_r)'(k, x), \quad \text{a.e. on } [a_0, b_0], \tag{3.18}$$

we have from (3.16) that

$$\sum_{j=1}^{\widetilde{N}} \widetilde{m}_j^- \widetilde{\kappa}_j^2 (f_r\widetilde{f}_r)'(i\widetilde{\kappa}_j, x) - \sum_{j=1}^{N} m_j^- \kappa_j^2 (f_r\widetilde{f}_r)'(i\kappa_j, x)$$

$$= -C\left[\sum_{j=1}^{\widetilde{N}}\widetilde{m}_j^-(f_r\widetilde{f}_r)'(i\widetilde{\kappa}_j,x)-\sum_{j=1}^{N}m_j^-(f_r\widetilde{f}_r)'(i\kappa_j,x)\right]=0. \tag{3.19}$$

Proceeding by induction, differentiating (3.15) $(2l+1)$ times with respect to x, by virtue of the condition $V(x)=C=\widetilde{V}(x)$ a.e. for $x\in[a_0,b_0]$, and the fact that (3.17), (3.18) are analogous to (3.16) and (3.19), we find that (3.6) holds. The proof is completed. □

Basing on the above lemmas, we are now in a position to give the proof of Theorem 2.1.

Proof of Theorem 2.1. According to the hypothesis of Theorem 2.1, for two operators H and $\widetilde{H}$, we have that $L(k)=\widetilde{L}(k)$, $R(k)=\widetilde{R}(k)$ for $k\in\mathbb{R}$, $V(x)=\widetilde{V}(x)$ for $x\in[a,b]$, $V(x)=C=\widetilde{V}(x)$ for $x\in[a_0,b_0]\subset[a,b]$, $x_j'=\widetilde{x}_j'\in[a,b]$ with $2\le j\le N$, and $\int_{-\infty}^{a}V(t)\mathrm{d}t=\int_{-\infty}^{a}\widetilde{V}(t)\mathrm{d}t$ (or $\int_b^{\infty}V(t)\mathrm{d}t=\int_b^{\infty}\widetilde{V}(t)\mathrm{d}t$). Moreover, from Lemma 3.3 we also have that (3.6) and (3.7) hold. Our purpose here is to prove $V=\widetilde{V}$ a.e. on $\mathbb{R}$.

Step 1. We show that $N=\widetilde{N}$ and $\kappa_j=\widetilde{\kappa}_j$ for $j=1,\dots,N$ by virtue of $V(x)=C=\widetilde{V}(x)$ for $x\in[a_0,b_0]$, $L(k)=\widetilde{L}(k)$ and $R(k)=\widetilde{R}(k)$ for $k\in\mathbb{R}$. We assume, contrary to what we want to show, that $N>\widetilde{N}$.

It follows from (3.6) and (3.7) that for any non-zero real constant c and $x\in[a_0,b_0]$,

$$\sum_{j=1}^{N}(\kappa_j^2)^l[m_j^-(f_r\widetilde{f}_r)+cm_j^+(f_l\widetilde{f}_l)]'(i\kappa_j,x)=\sum_{j=1}^{\widetilde{N}}(\widetilde{\kappa}_j^2)^l[\widetilde{m}_j^-(f_r\widetilde{f}_r)+c\widetilde{m}_j^+(f_l\widetilde{f}_l)]'(i\widetilde{\kappa}_j,x).$$

Note that for each bound-state energy $-\kappa_j^2$, the Jost solutions $f_l(i\kappa_j,x)$ and $f_r(i\kappa_j,x)$ become linearly dependent, i.e., there exists a nonzero real constant d_j such that $f_l(i\kappa_j,x)=d_jf_r(i\kappa_j,x)$. Analogous results are also valid for the operator $\widetilde{H}$. Thus the above equation deduces

$$\sum_{j=1}^{N}(\kappa_j^2)^l[f_r(m_j^-\widetilde{f}_r+cm_j^+d_j\widetilde{f}_l)]'(i\kappa_j,x)=\sum_{j=1}^{\widetilde{N}}(\widetilde{\kappa}_j^2)^l[\widetilde{f}_r(\widetilde{m}_j^-f_r+c\widetilde{m}_j^+\widetilde{d}_jf_l)]'(i\widetilde{\kappa}_j,x). \tag{3.20}$$

Denote

$$a_j(x)=[f_r(m_j^-\widetilde{f}_r+cm_j^+d_j\widetilde{f}_l)]'(i\kappa_j,x) \tag{3.21}$$

and

$$\widetilde{a}_j(x)=[\widetilde{f}_r(\widetilde{m}_j^-f_r+c\widetilde{m}_j^+\widetilde{d}_jf_l)]'(i\widetilde{\kappa}_j,x). \tag{3.22}$$

Here $f_r(i\kappa_j,x)$, $\widetilde{f}_r(i\kappa_j,x)$ and $\widetilde{f}_l(i\kappa_j,x)$ are three nontrivial solutions of the equation $y''(i\kappa_j,x)=(C+\kappa_j^2)y(i\kappa_j,x)$ for $x\in[a_0,b_0]$. Similar conclusions are also valid for the functions $\widetilde{f}_r(i\widetilde{\kappa}_j,x)$, $f_r(i\widetilde{\kappa}_j,x)$ and $f_l(i\widetilde{\kappa}_j,x)$. If $\kappa_j\neq\widetilde{\kappa}_j$, then $\widetilde{f}_r(i\kappa_j,x)$ and $\widetilde{f}_l(i\kappa_j,x)$ (resp. $f_r(i\widetilde{\kappa}_j,x)$ and $f_l(i\widetilde{\kappa}_j,x)$) are linearly independent.

We have from Lemma 3.1(i) that $a_j(x)$ and $\widetilde{a}_j(x)$ have at most finitely many zeros on $[a_0, b_0]$. If $\kappa_j = \widetilde{\kappa}_j$, then $\widetilde{f}_r(i\kappa_j, x)$ and $\widetilde{f}_l(i\kappa_j, x)$ (resp. $f_r(i\widetilde{\kappa}_j, x)$ and $f_l(i\widetilde{\kappa}_j, x)$) are linearly dependent, which means $a_j(x)$ and $\widetilde{a}_j(x)$ defined by (3.21) and (3.22) can be rewritten as

$$a_j(x) = (m_j^- + c m_j^+ d_j \widetilde{d}_j)[f_r \widetilde{f}_r]'(i\kappa_j, x) \text{ and } \widetilde{a}_j(x) = (\widetilde{m}_j^- + c\widetilde{m}_j^+ d_j \widetilde{d}_j)[f_r \widetilde{f}_r]'(i\kappa_j, x).$$

We have from Lemma 3.1(ii) that either $a_j(x) \equiv 0 \equiv \widetilde{a}_j(x)$ for all $x \in [a_0, b_0]$, or $a_j(x)$ and $\widetilde{a}_j(x)$ have at most finitely many zeros on $[a_0, b_0]$. Hence there are two cases to be considered.

Case I: If there exists none j such that $a(x) \equiv 0 \equiv \widetilde{a}(x)$ for all $x \in [a_0, b_0]$, then (3.20) deduces that

$$\sum_{j=1}^{N} (\kappa_j^2)^l a_j(x) = \sum_{j=1}^{\widetilde{N}} (\widetilde{\kappa}_j^2)^l \widetilde{a}_j(x). \tag{3.23}$$

Here $a_j(x)$ and $\widetilde{a}_j(x)$ have at most finitely many zeros on $[a_0, b_0]$, which means there exists a common non-zero real constant c and $x_0 \in [a_0, b_0]$ such that

$$a_j(x_0) \neq 0, \quad j = 1, \ldots, N, \quad \text{and} \quad \widetilde{a}_j(x_0) \neq 0, \quad j = 1, \ldots, \widetilde{N}. \tag{3.24}$$

Notice that the Jost solution $f_r(k, x)$ of (2.1) satisfies the reality condition $\overline{f_r(k, x)} = f_r(-\overline{k}, x)$ for Im$k \geq 0$ (see, for example, [7, p. 130]). This gives that for all $k = i\kappa_j$ and $k = i\widetilde{\kappa}_j$, the functions $f_r(k, x)$, $f_l(k, x)$, $\widetilde{f}_r(k, x)$ and $\widetilde{f}_l(k, x)$ are all real-valued. Denote the vector A by $A = (a_1(x_0), \ldots, a_N(x_0))^T \in \mathbb{R}^N$ and the Vandermonde matrix associated with $\{\kappa_j^2\}_{j=1}^N$ by $V_{(N+\widetilde{N})\times N}[\kappa_j^2]_{j=1}^N$. Similar notations can also be introduced for $\{\widetilde{\kappa}_j^2\}_{j=1}^{\widetilde{N}}$ corresponding to the Vandermonde matrix $V_{(N+\widetilde{N})\times N}[\widetilde{\kappa}_j^2]_{j=1}^N$ and the vector $\widetilde{A}$ with $\widetilde{A} = (\widetilde{a}_1(x_0), \ldots, \widetilde{a}_{\widetilde{N}}(x_0))^T \in \mathbb{R}^{\widetilde{N}}$. Then, by (3.23) and $M = N + \widetilde{N}$, we have

$$V_{M\times N}[\kappa_j^2]_{j=1}^N A = V_{M\times \widetilde{N}}[\widetilde{\kappa}_j^2]_{j=1}^{\widetilde{N}} \widetilde{A}. \tag{3.25}$$

Applying Lemma 3.2 to the above equation, we conclude that

$$\kappa_j = \widetilde{\kappa}_j, \quad a_j(x_0) = \widetilde{a}_j(x_0) \quad \text{for } j = 1, \ldots, \widetilde{N}$$

and

$$a_j(x_0) = 0 \quad \text{for } j = \widetilde{N} + 1, \ldots, N. \tag{3.26}$$

Thus the contradiction follows from (3.24) and (3.26), therefore $N = \widetilde{N}$ and further $\kappa_j = \widetilde{\kappa}_j$ for $j = 1, \ldots, N$.

Case II: If there exists some j (for simplicity, we suppose there exists only one, denoted as j_0), such that $a_{j_0}(x) \equiv 0 \equiv \widetilde{a}_{j_0}(x)$ for all $x \in [a_0, b_0]$, then (3.20) deduces that

$$\sum_{j=\overline{1,N}, j\neq j_0} (\kappa_j^2)^l a_j(x) = \sum_{j=\overline{1,\widetilde{N}}, j\neq j_0} (\widetilde{\kappa}_j^2)^l \widetilde{a}_j(x). \tag{3.27}$$

It should be noted that this would happen only if $\kappa_{j_0} = \widetilde{\kappa}_{j_0}$. Here $a_j(x)$ and $\widetilde{a}_j(x)$ have at most finitely many zeros on $[a_0, b_0]$, which means there exists a common non-zero real constant c and $x_0 \in [a_0, b_0]$ such that (3.24) also valid for each j with $j \neq j_0$. Similarly to (3.25), from (3.27) we have that

$$V_{M\times N}[\kappa_j^2]_{j=1,j\neq j_0}^{N} A = V_{M\times \widetilde{N}}[\widetilde{\kappa}_j^2]_{j=1,j\neq j_0}^{\widetilde{N}} \widetilde{A}. \tag{3.28}$$

Applying Lemma 3.2 to the above equation, we will also have a contradiction. Therefore, $N = \widetilde{N}$ and $\kappa_j = \widetilde{\kappa}_j$ for $j = 1, \dots, N$.

Step 2. We show $m_j^- = \widetilde{m}_j^-$ for $j = 1, \dots, N$ by virtue of $L(k) = \widetilde{L}(k)$ for $k \in \mathbb{R}$, $V(x) = \widetilde{V}(x)$ for $x \in [a, b]$, $x_j' = \widetilde{x}_j' \in [a, b]$ with $2 \leq j \leq N$, and $\int_{-\infty}^{a} V(t)\mathrm{d}t = \int_{-\infty}^{a} \widetilde{V}(t)\mathrm{d}t$. The case for $m_j^+ = \widetilde{m}_j^+$ can be treated in a similar way.

Once $N = \widetilde{N}$ and $\kappa_j = \widetilde{\kappa}_j$ for $j = 1, \dots, N$, it follows from (3.6) that

$$\sum_{j=1}^{N} (\kappa_j^2)^l (m_j^- - \widetilde{m}_j^-)(f_r \widetilde{f}_r)'(i\kappa_j, x) = 0 \tag{3.29}$$

for $x \in [a_0, b_0]$ and $l = 0, 1, \dots, N-1$, which implies that

$$(m_j^- - \widetilde{m}_j^-)(f_r \widetilde{f}_r)'(i\kappa_j, x) = 0, \quad x \in [a_0, b_0],\ j = 1, \dots, N. \tag{3.30}$$

In terms of $V(x) = \widetilde{V}(x)$ for a.e. $x \in [a, b]$, the functions $f_r(i\kappa_j, x)$ and $\widetilde{f}_r(i\kappa_j, x)$ are both solutions of the equation $-y''(k, x) + V(x)y(k, x) = -\kappa_j^2 y(k, x)$ for $x \in [a, b]$, further the condition $x_j' = \widetilde{x}_j' \in [a, b]$ with $j = 2, \dots, N$ deduces that the functions $f_r(i\kappa_j, x)$ and $\widetilde{f}_r(i\kappa_j, x)$ satisfy the same initial condition $y(i\kappa_j, x_j') = 0$. Hence, $f_r(i\kappa_j, x)$ and $\widetilde{f}_r(i\kappa_j, x)$ are linearly dependent for $x \in [a_0, b_0]$, i.e., there exists a non-zero real constant c_j such that

$$\widetilde{f}_r(i\kappa_j, x) = c_j f_r(i\kappa_j, x), \quad \text{for } x \in [a_0, b_0],\ j = 2, \dots, N.$$

Since $f_r(i\kappa_j, x)$ is the eigenfunction of equation (2.1), which is not constant and has at most one zero on $[a_0, b_0]$, $f_r'(i\kappa_j, x)$ has at most finitely many zeros on $[a_0, b_0]$ by virtue of Rolle mean value theorem [19]. It gives that $(f_r \widetilde{f}_r)'(i\kappa_j, x) = 2c_j(f_r f_r')(i\kappa_j, x)$ has only a finite number of zeros on $[a_0, b_0]$, which means there exists a common $x_0' \in [a_0, b_0]$ such that

$$(f_r \widetilde{f}_r)'(i\kappa_j, x_0') \neq 0 \quad \text{for } j = 2, \dots, N,$$

which together with (3.30) gives that

$$m_j^- = \widetilde{m}_j^- \quad \text{for } j = 2, \dots, N. \tag{3.31}$$

On the other hand, by means of $L(k) = \widetilde{L}(k)$ for $k \in \mathbb{R}$, $\int_{-\infty}^{a} V(t)\mathrm{d}t = \int_{-\infty}^{a} \widetilde{V}(t)\mathrm{d}t$ and $V(x) = \widetilde{V}(x)$ for $x \in [a, b]$, we have from (3.15) and $N = \widetilde{N}$, $\kappa_j = \widetilde{\kappa}_j$ for $j =$

$1, \ldots, N$ that

$$\sum_{j=1}^{N} (\widetilde{m}_j^- - m_j^-)(f_r \widetilde{f}_r)(i\kappa_j, x) = 0, \quad x \in [a, b], \tag{3.32}$$

which together with (3.31) deduces that

$$(\widetilde{m}_1^- - m_1^-)(f_r \widetilde{f}_r)(i\kappa_1, x) = 0,$$

this means

$$\widetilde{m}_1^- = m_1^-. \tag{3.33}$$

In all, we have $L(k) = \widetilde{L}(k)$ for $k \in \mathbb{R}, N = \widetilde{N}$ and $\kappa_j = \widetilde{\kappa}_j, m_j^- = \widetilde{m}_j^-$ for $j = 1, \ldots, N$. Thus, by Marchenko's uniqueness theorem [15], we get $V = \widetilde{V}$ a.e. on $\mathbb{R}$. The proof is completed. □

Basing on the proof of Theorem 2.1, if all the bound-state energies and the reflection coefficient $L(k)$ for $k \in \mathbb{R}$ are given, then the knowledge of the potential on a finite interval will give N algebraic equations associated with the unspecified norming constants (see (3.29) and (3.30)). We have used the nodal points $x_j' \in [a, b]$ with $j = 2, \ldots, N$ and the value $\int_{-\infty}^{a} V(t)\mathrm{d}t$ to determine uniquely the norming constants m_j^- with $j = 1, \ldots, N$. Thus the proofs of Theorems 2.2 and 2.4 follow that of **Step 2** of Theorem 2.1, and we will give the sketch.

In virtue of (3.29) and (3.32), we give the proof of Theorem 2.2.

Proof of Theorem 2.2. For the sake of simplicity, we shall consider the uniqueness only for the left reflection coefficient $L(k)$, the case for $R(k)$ can be treated in a similar way. According to the hypothesis of Theorem 2.2, for two operators H and $\widetilde{H}$, since $L(k) = \widetilde{L}(k)$ for $k \in \mathbb{R}$ and $V(x) = C = \widetilde{V}(x)$ for a.e. $x \in [a_0, b_0]$, it follows from Lemma 3.3 that (3.6) holds, by virtue of the fact $\{\kappa_j\}_{j=1}^N = \{\widetilde{\kappa}_j\}_{j=1}^N$, we have (3.29) and further (3.30) is valid. Then we derive from the condition $x_j' = \widetilde{x}_j' \in [a, b]$ with $j \notin \{l_s\}_{s=1}^n \cup \{1\}$ that

$$m_j^- = \widetilde{m}_j^- \quad \text{for } j \notin \{l_s\}_{s=1}^n \cup \{1\}.$$

We further get $\widetilde{m}_1^- = m_1^-$ for the same reason of (3.33). Thus, the proof is completed. □

In virtue of (3.32), we give the proof of Theorem 2.4.

Proof of Theorem 2.4. According to the hypothesis of Theorem 2.4, for two operators H and $\widetilde{H}$, since $L(k) = \widetilde{L}(k)$ for $k \in \mathbb{R}$ and $V(x) = \widetilde{V}(x)$ for $x \in [a, b]$, $\int_{-\infty}^{a} V(t)\mathrm{d}t = \int_{-\infty}^{a} \widetilde{V}(t)\mathrm{d}t$, and $\{\kappa_j\}_{j=1}^N = \{\widetilde{\kappa}_j\}_{j=1}^N$, we have that (3.32) holds. It follows from the condition $\{m_j^-\}_{j=1, j\neq j_0}^N = \{\widetilde{m}_j^-\}_{j=1, j\neq j_0}^N$ that

$$(\widetilde{m}_{j_0}^- - m_{j_0}^-)(f_r \widetilde{f}_r)(i\kappa_{j_0}, x) = 0, \quad x \in [a, b],$$

which means $m_{j_0}^- = \widetilde{m}_{j_0}^-$, and thus the proof is completed. □

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Обернені задачі розсіювання з потенціалом, заданим на внутрішньому підінтервалі

Yongxia Guo and Guangsheng Wei

Розглянуто обернену задачу для одновимірного оператора Шредінгера на прямій у випадку, коли потенціал є дійсно значним, інтегрованим та має скінчений перший момент. Показано, що цей потенціал на прямій однозначно визначений змішаними даними розсіювання, які містять матрицю розсіювання, заданий на скінченому інтервалі потенціал та одну вузлову точку на заданому інтервалі для кожної власної функції.

Ключові слова: рівняння Шредінгера, обернена задача розсіювання, відновлення потенціалу за частковими даними.