Journal of Mathematical Physics, Analysis, Geometry
2019, Vol. 15, No. 2, pp. 239–255
doi: https://doi.org/10.15407/mag15.02.239

Translation-Invariant Gibbs Measures for the Blum–Kapel Model on a Cayley Tree

Nosir Khatamov and Rustam Khakimov

In the paper, translation-invariant Gibbs measures for the Blum–Kapel model on a Cayley tree of order k are considered. An approximate critical temperature T_{cr} is found such that for $T \geq T_{cr}$ there exists a unique translation-invariant Gibbs measure and for $0 < T < T_{cr}$ there are exactly three translation-invariant Gibbs measures. In addition, the problem of (not) extremality for the unique Gibbs measure is studied.

Key words: Cayley tree, configuration, Blum–Kapel model, Gibbs measure, translation-invariant measure, extremality of measure.

Mathematical Subject Classification 2010: 82B26, 60K35.

1. Introduction

The Gibbs measure is a fundamental law determining the probability of a microscopic state of a given physical system and it plays an important role in determining the existence of a phase transition of a physical system, since each Gibbs measure is associated with one phase of the physical system, and if a Gibbs measure is nonunique, then it is said that there is a phase transition. It is well known that the set of all limit Gibbs measures forms a nonempty convex compact subset of the set of all probability measures and each point (i.e., Gibbs measure) of this convex set can be uniquely expanded to its extreme points. Therefore, it is interesting to describe all extreme points of this convex set, i.e., the extreme Gibbs measures (see [4, 13, 16]).

Many papers are devoted to the study of limit Gibbs measures on a Cayley tree for such models of statistical physics as Ising model, Potts model, HC model and SOS model (see, for example, [3, 7–10, 15]). In particular, in [10], the set of translation-invariant Gibbs measures for the ferromagnetic q-state Potts model was fully described and it was proved that the number of translation-invariant measures can be up to $2^q - 1$, and in [9], the extremality problem was studied for these measures. In [15], Gibbs measures for three state HC models were studied on a Cayley tree of order $k \geq 1$ and the nonuniqueness of the translation-invariant Gibbs measure was proved. Moreover, the areas where the measures are (not) extreme are given. In the monograph [14], the results on limit Gibbs measures can be found in more detail.

This paper is devoted to the study of the Blum–Kapel model which has not yet been studied on a Cayley tree. This is a two-dimensional spin system, where spin variables take values from the set: $\Phi = \{-1, 0, +1\}$. It was originally introduced for studying He^3–He^4 phase transition (see [1]). We can consider this model as the system of a particle with a spin. The value $\sigma(x) = 0$ of the spin on the lattice vertex (or on the tree node) x corresponds to the absence of particles (vacancy) and the values $\sigma(x) = +1, -1$, to the presence of a particle with spin $+1, -1$ on the vertex x, respectively (see [1, 5, 17]).

This paper is organized as follows. In Section 2, we present the basic definitions and known facts. In Section 3, we prove a theorem that ensures the condition of consistency of a measure. In Section 4, an approximate critical temperature T_{cr} is found such that for $T \geq T_{cr}$ there exists a unique translation-invariant Gibbs measure and there are exactly three translation-invariant Gibbs measures for the considered model for $0 < T < T_{cr}$. In Section 5, the sets where the existing single measure for $T > 0$ is (not) extremal are given.

2. Preliminary information

A Cayley tree $\Gamma^k = (V, L)$ of order $k \geq 1$ is an infinite tree, i.e., a graph without cycles such that each vertex has precisely $k+1$ edges, where V is the set of vertices of the graph Γ^k, L is the set of its edges. Let i be an incidence function associating each edge $l \in L$ to its endpoints $x, y \in V$. If $i(l) = \{x, y\}$, then x and y are called the nearest neighbors of a vertex and can be written as $\langle x, y\rangle$. The distance $d(x, y), x, y \in V$ on the Cayley tree is defined as

$$d(x,y) = \min\{d \mid \exists\; x = x_0, x_1, \ldots, x_{d-1}, x_d = y \in V\; \langle x_0, x_1\rangle, \ldots, \langle x_{d-1}, x_d\rangle\}.$$

We consider a model in which spin variables take values from the set $\Phi = \{-1, 0, +1\}$. We then define a *configuration* σ on V as a function $x \in V \to \sigma(x) \in \Phi$. The set of all configurations coincides with $\Omega = \Phi^V$. Let $A \subset V$. We denote the space of configurations defined on a set A by Ω_A.

The Hamiltonian of the Blum–Kapel model is given by the formula

$$H(\sigma) = -J \sum_{\langle x,y\rangle, x,y \in V;} \sigma(x)\sigma(y),$$

where $J > 0$.

For a fixed $x^0 \in V$, we write $x < y$ if a path from x^0 to y runs through x.

We denote

$$W_n = \left\{x \in V \mid d(x^0, x) = n\right\}, \quad V_n = \left\{x \in V \mid d(x^0, x) \leq n\right\}.$$

A vertex y is called a *“child”* of a vertex x if $x < y$ and $d(x, y) = 1$.

We let $S(x)$ denote the set of *“children”* of a vertex $x \in V$.

Let $h : x \mapsto h_x = (h_{-1,x}, h_{0,x}, h_{+1,x})$ be a vector-valued function on $x \in V \setminus \{x^0\}$. We consider the probability measure $\mu^{(n)}$ on Ω_{V_n},

$$\mu^{(n)}(\sigma_n) = Z_n^{-1} \exp\left\{-\beta H(\sigma_n) + \sum_{x \in W_n} h_{\sigma(x),x}\right\}. \tag{2.1}$$

Here $\sigma_n \in \Omega_{V_n}$, and Z_n is a normalization factor,

$$Z_n = \sum_{\overline{\sigma}_n \in \Omega_{V_n}} \exp\left\{-\beta H(\overline{\sigma}_n) + \sum_{x\in W_n} h_{\overline{\sigma}(x),x}\right\},$$

where $h_{\overline{\sigma},x} \in R$.

The probability measure $\mu^{(n)}$ is said to be consistent if for all $n \geq 1$ and any $\sigma_{n-1} \in \Omega_{V_{n-1}}$:

$$\sum_{\sigma^{(n)}} \mu^{(n)}(\sigma_{n-1}, \sigma^{(n)}) = \mu^{(n-1)}(\sigma_{n-1}). \tag{2.2}$$

In this case, there is a unique measure μ on Ω_V such that

$$\mu(\{\sigma|_{V_n} = \sigma_n\}) = \mu^{(n)}(\sigma_n)$$

for all $n \geq 1$ and any $\sigma_n \in \Omega_{V_n}$.

3. The system of functional equations

A condition for $h_{i,x}$ ensuring the consistency of the measures $\mu^{(n)}$ is formulated in the next theorem.

Theorem 3.1. *Let $k \geq 2$. The sequence of probabilistic measures $\mu^{(n)}(\sigma_n), n = 1, 2, \ldots,$ defined by (2.1) is consistent if and only if the equalities*

$$\begin{cases} z_{+1,x} = \prod\limits_{y\in S(x)} \dfrac{\lambda z_{+1,y} + \frac{1}{\lambda} z_{-1,y} + 1}{z_{+1,y} + z_{-1,y} + 1}, \\ z_{-1,x} = \prod\limits_{y\in S(x)} \dfrac{\frac{1}{\lambda} z_{+1,y} + \lambda z_{-1,y} + 1}{z_{+1,y} + z_{-1,y} + 1}, \end{cases} \tag{3.1}$$

where $\lambda = \exp\{J\beta\}$, $\beta = 1/T$, $z_{i,x} = \exp(h_{i,x} - h_{0,x})$, $i = +1, -1$, hold for any $x \in V$.

Proof. Necessity. By the consistency condition (2.2), we get

$$\frac{Z_{n-1}}{Z_n} \sum_{\omega_n \in \Omega_{W_n}} \prod_{x\in W_{n-1}} \prod_{y\in S(x)} \exp\left(J\beta\sigma_{n-1}(x)\omega_n(y) + h_{\omega_n(y),y}\right)$$
$$= \prod_{x\in W_{n-1}} \exp\left(h_{\sigma_{n-1}(x),x}\right), \tag{3.2}$$

where $\sigma(x) \in \Phi$. Fix $x \in W_{n-1}$ and consider three configurations $\sigma_{n-1} = \overline{\sigma}_{n-1}$, $\sigma_{n-1} = \widetilde{\sigma}_{n-1}$ and $\sigma_{n-1} = \widehat{\sigma}_{n-1}$ on W_{n-1} which coincide on $W_{n-1} \setminus \{x\}$, and rewrite now equality (3.2) for $\overline{\sigma}_{n-1}(x) = -1$, $\widetilde{\sigma}_{n-1}(x) = 0$ and $\widehat{\sigma}_{n-1}(x) = 1$. Thus we obtain

$$\begin{cases} \exp\left(h_{+1,x} - h_{0,x}\right) = \prod\limits_{y\in S(x)} \dfrac{\sum_{\omega_n(y)\in\Phi} \exp\{J\beta\omega_n(y) + h_{\omega_n(y),y}\}}{\sum_{\omega_n(y)\in\Phi} \exp\{h_{\omega_n(y),y}\}}, \\ \exp\left(h_{-1,x} - h_{0,x}\right) = \prod\limits_{y\in S(x)} \dfrac{\sum_{\omega_n(y)\in\Phi} \exp\{-J\beta\omega_n(y) + h_{\omega_n(y),y}\}}{\sum_{\omega_n(y)\in\Phi} \exp\{h_{\omega_n(y),y}\}}. \end{cases}$$

Consequently,

$$\begin{cases} \exp\left(h_{+1,x}-h_{0,x}\right) \\ \quad = \prod\limits_{y\in S(x)} \dfrac{\exp\{J\beta\}\exp\{h_{+1,y}-h_{0,y}\}+\exp\{-J\beta\}\exp\{h_{-1,y}-h_{0,y}\}+1}{\exp\{h_{+1,y}-h_{0,y}\}+\exp\{h_{-1,y}-h_{0,y}\}+1}, \\ \exp\left(h_{-1,x}-h_{0,x}\right) \\ \quad = \prod\limits_{y\in S(x)} \dfrac{\exp\{-J\beta\}\exp\{h_{+1,y}-h_{0,y}\}+\exp\{J\beta\}\exp\{h_{-1,y}-h_{0,y}\}+1}{\exp\{h_{+1,y}-h_{0,y}\}+\exp\{h_{-1,y}-h_{0,y}\}+1}. \end{cases}$$

Hence we can get (3.1).

Sufficiency. Suppose that (3.1) holds. It is equivalent to the representations

$$\prod_{y\in S(x)}\sum_{u\in\{-1,0,+1\}}\exp(J\beta tu+h_{u,y})=a(x)\exp(h_{t,x}),\quad t=-1,0,+1, \tag{3.3}$$

for some function $a(x)>0$, $x\in V$. For l.h.s. of (2.2), we have

$$\sum_{\sigma^{(n)}}\mu^{(n)}(\sigma_{n-1},\sigma^{(n)})=\frac{1}{Z_n}\exp(-\beta H(\sigma_{n-1})) \\ \times\prod_{x\in W_{n-1}}\prod_{y\in S(x)}\sum_{u\in\{-1,0,+1\}}\exp(J\beta\sigma_{n-1}(x)u+h_{u,y}). \tag{3.4}$$

Taking (3.3) into account and denoting

$$A_n(x)=\prod_{x\in W_{n-1}}a(x),$$

for l.h.s. of (3.3) from (3.4), we get

$$\prod_{y\in S(x)}\sum_{u\in\{-1,0,+1\}}\exp(J\beta tu+h_{u,y}) \\ =\frac{A_{n-1}}{Z_n}\exp(-\beta H(\sigma_{n-1}))\prod_{x\in W_{n-1}}\exp(h_{\sigma_{n-1}(x),x}). \tag{3.5}$$

Since $\mu^{(n)}, n\geq 1$ is a probabilistic measure, then the following equation is true:

$$\sum_{\sigma_{n-1}\in\Omega_{V_{n-1}}}\sum_{\omega_n\in\Omega_{W_n}}\mu^{(n)}(\sigma_{n-1},\omega_n)=1.$$

Consequently, from (3.5) we obtain $Z_{n-1}A_{n-1}=Z_n$ and the validity of (2.2). The theorem is proved. □

4. Translation-invariant Gibbs measures

Translation-invariant Gibbs measures correspond to solutions (3.1) with $z_{i,x} = z_i > 0$ for all $x \in V$ and $i = -1, +1$. We introduce the notation $z_{+1} = z_1, z_{-1} = z_2$. Then (3.1) has the form

$$\begin{cases} z_1 = \left(\dfrac{\lambda z_1 + \frac{1}{\lambda} z_2 + 1}{z_1 + z_2 + 1}\right)^k, \\ z_2 = \left(\dfrac{\frac{1}{\lambda} z_1 + \lambda z_2 + 1}{z_1 + z_2 + 1}\right)^k. \end{cases} \tag{4.1}$$

In system (4.1) we subtract the second equation from the first one to have

$$(z_1 - z_2)\left[1 - \frac{(\lambda - \frac{1}{\lambda})((\lambda z_1 + \frac{1}{\lambda} z_2 + 1)^{k-1} + \cdots + (\frac{1}{\lambda} z_1 + \lambda z_2 + 1)^{k-1})}{(z_1 + z_2 + 1)^k}\right] = 0. \tag{4.2}$$

Hence, $z_1 = z_2$ or

$$(z_1 + z_2 + 1)^k = \left(\lambda - \frac{1}{\lambda}\right)\left[\left(\lambda z_1 + \frac{1}{\lambda} z_2 + 1\right)^{k-1} + \cdots + \left(\frac{1}{\lambda} z_1 + \lambda z_2 + 1\right)^{k-1}\right].$$

We consider the case $z_1 = z_2 = z$. Here, from (4.1), we obtain

$$z = \left(\frac{(\lambda + \frac{1}{\lambda})z + 1}{2z + 1}\right)^k. \tag{4.3}$$

For the solutions of the last equation the next proposition holds.

Proposition 4.1. *If $\overline{z}$ is the solution of equation (4.3), then*

$$1 \leq \overline{z} < \left(\frac{\lambda + \frac{1}{\lambda}}{2}\right)^k$$

and $\overline{z} = 1$ for $\lambda = 1$.

The *proof of Proposition* 4.1 is obtained directly from equation (4.3).

Proposition 4.2. *For $k \geq 2$ and for any values $\lambda > 0$ equation (4.3) has a unique positive solution.*

Proof. The proof will be carried out in three steps.
Step 1. Denoting $\sqrt[k]{z} = x$, we rewrite equation (4.3) in the form

$$\varphi(x) = 2x^{k+1} - ax^k + x - 1 = 0, \tag{4.4}$$

where $a = \lambda + \frac{1}{\lambda} \geq 2$. Then the inequality from Proposition 1 has the form $1 \leq x < \frac{a}{2}$.

If $a = 2$ (i.e., $\lambda = 1$), then equation (4.4) (equation (4.3)) has a unique solution $x = 1$ ($z = 1$). Therefore we consider the case $a > 2$ ($\lambda \neq 1$).

By Proposition 4.1, it is clear that $1 \leq x < \frac{a}{2}$. Notice that $\varphi(1) = 2 - a < 0$ and $\varphi(\frac{a}{2}) = 1 > 0$, i.e., equation (4.4) has at least one positive solution for $1 \leq x < \frac{a}{2}$. Moreover, since there are three sign changes in the polynomial $\varphi(x) = 2x^{k+1} - ax^k + x - 1$, it follows from the known Descartes theorem on the number of positive roots of a polynomial [12, Corollary 1, p. 39] that equation (4.4) has at most three positive solutions.

Step 2. In the second step of the proof, we use the Jacobi method for estimating the number of roots of a polynomial between α and β [12, Remark, p. 39]. To do this, we make a substitution

$$y = \frac{x-1}{\frac{a}{2} - x} \quad \text{i.e., } x = \frac{1 + \frac{a}{2}y}{1+y},$$

and consider the polynomial

$$\begin{aligned}(1+y)^{k+1}\varphi\left(\frac{1+\frac{a}{2}y}{1+y}\right) &= (a-2)\left[\frac{y}{2}(y+1)^k - \left(\frac{a}{2}y+1\right)^k\right] \\ &= (a-2)\left[\frac{1}{2}y^{k+1} + \left(\frac{1}{2}C_k^1 - \frac{a^k}{2^k}\right)y^k + \left(\frac{1}{2}C_k^2 - C_k^1\frac{a^{k-1}}{2^{k-1}}\right)y^{k-1} + \dots \right.\\ &\quad \left. + \left(\frac{1}{2} - C_k^{k-1}\frac{a}{2}\right)y - 1\right] \\ &= (a-2)\left(\frac{1}{2}y^{k+1} + b_0y^k + b_1y^{k-1} + \dots + b_{k-1}y + b_k\right) = (a-2)\psi(y).\end{aligned}$$

Here

$$b_i = \frac{1}{2}C_k^{i+1} - C_k^i\left(\frac{a}{2}\right)^{k-i}, \quad i = 0, 1, 2, \dots, k-1, \ b_k = -1.$$

By the Jacobi method, the number of positive roots of the polynomial $\psi(y)$ is the number of positive roots of the polynomial $\varphi(x)$ for $\left[1, \frac{a}{2}\right)$.

We note that if $b_i < 0$ for all $i = 1, 2, \dots, k-1$ ($b_k = -1 < 0$), then independently of the sign of b_0, by the Descartes theorem, the polynomial $\psi(y)$ has the unique positive solution. Thus we consider the case $i \neq 0$.

If $b_i > 0$, then

$$a < 2\sqrt[k-i]{\frac{k-i}{2(i+1)}} = t_1$$

and $i < \frac{k-2}{3} = i_0$, $i \in \{1, 2, \dots, k-1\}$. Indeed, after solving the inequality $b_i > 0$ for a, the inequality $a < t_1$ is obtained directly. On the other hand, the inequality $b_i > 0$ is equivalent to the inequality

$$\frac{1}{2}C_k^{i+1} > C_k^i\left(\frac{a}{2}\right)^{k-i}.$$

From this inequality, we get

$$\frac{k-i}{2(i+1)} > \left(\frac{a}{2}\right)^{k-i},$$

here the right side is greater than one. Hence we have

$$i < \frac{k-2}{3} = i_0.$$

Consequently, $b_i < 0$ for any $i \geq i_0$.

Step 3. In this step we prove that if $b_i > 0$ for $0 \neq i < i_0$, then b_{i-1} is also positive. We suppose $b_i > 0$ but $b_{i-1} < 0$. If $b_i > 0$, then it is already known that

$$a < 2 \sqrt[k-i]{\frac{k-i}{2(i+1)}} = t_1.$$

From $b_{i-1} < 0$, we have

$$a > 2 \sqrt[k-i+1]{\frac{k-i+1}{2i}} = t_2.$$

We prove that $t_1 < t_2$. Indeed, $t_1 < t_2$ is equivalent to the inequality

$$\left(\frac{k-i}{2(i+1)}\right)^{k-i+1} < \left(\frac{k-i+1}{2i}\right)^{k-i}.$$

Denoting $k - i = n, 1 \leq n < k$ (since $i \neq 0$ here $n \neq k$), we rewrite the last inequality

$$\left(\frac{n}{2(k-n+1)}\right)^{n+1} < \left(\frac{n+1}{2(k-n)}\right)^{n}. \tag{4.5}$$

Using mathematical induction, we prove inequality (4.5). For $n = 1$, we obtain the inequality $4k^2 - k + 1 > 0$ which is true for any k. We suppose that (4.5) holds for n. We prove the inequality

$$\left(\frac{n+1}{2(k-n)}\right)^{n+2} < \left(\frac{n+2}{2(k-n-1)}\right)^{n+1}.$$

We transform and estimate the left-hand side of the last inequality

$$\begin{aligned}
\left(\frac{n+1}{2(k-n)}\right)^{n+2} &= \left(\frac{n+1}{2(k-n)}\right)^{n+2} \left(\frac{n}{2(k-n+1)}\right)^{n+1} \left(\frac{2(k-n+1)}{n}\right)^{n+1} \\
&= \left(\frac{n}{2(k-n+1)}\right)^{n+1} \left(\frac{n+1}{2(k-n)}\right)^{n+2} \left(\frac{2(k-n+1)}{n}\right)^{n+1} \\
&< \left(\frac{n+1}{2(k-n)}\right)^{n} \left(\frac{n+1}{2(k-n)}\right)^{n+2} \left(\frac{2(k-n+1)}{n}\right)^{n+1} \\
&= \left(\frac{n+1}{2(k-n)}\right)^{n} \left(\frac{n+1}{2(k-n)}\right)^{n+2} \left(\frac{2(k-n+1)}{n}\right)^{n+1} \\
&\qquad \times \left(\frac{n+2}{2(k-n-1)}\right)^{n+1} \left(\frac{2(k-n-1)}{n+2}\right)^{n+1} \\
&= \left(\frac{n+2}{2(k-n-1)}\right)^{n+1} \left(\frac{n+1}{2(k-n)}\right)^{2n+2} \left(\frac{2(k-n+1)2(k-n-1)}{n(n+2)}\right)^{n+1}.
\end{aligned}$$

Consequently, it is necessary to prove the inequality

$$\left(\frac{n+2}{2(k-n-1)}\right)^{n+1}\left(\frac{n+1}{2(k-n)}\right)^{2n+2}\left(\frac{2(k-n+1)2(k-n-1)}{n(n+2)}\right)^{n+1} < \left(\frac{n+2}{2(k-n-1)}\right)^{n+1},$$

which is equivalent to the inequality

$$\left(\frac{n+1}{2(k-n)}\right)^{2n+2} < \left(\frac{n(n+2)}{4((k-n)^2-1)}\right)^{n+1}.$$

From the last inequality we obtain $i < \frac{k+1}{2}$. Since $i < \frac{k-2}{3}$ and $\frac{k-2}{3} < \frac{k+1}{2}$, the inequality $i < \frac{k+1}{2}$ holds. Hence equation (4.3) has the unique solution for any values $\lambda > 0$ and $k \geq 2$. The proposition is proved. $\square$

For the case $z_1 = z_2 = z$, by Proposition 4.2, we get that system (4.1) has the unique solution (z^*, z^*) for $\lambda > 0$ and $k \geq 2$.

The following theorem holds.

Theorem 4.3. *Let $k = 2$. Then for the Blum–Kapel model there is $\lambda_{cr} \approx 2.1132163$ such that there exists one translation-invariant Gibbs measure μ_0 for $0 < \lambda \leq \lambda_{cr}$ and there are exactly three translation-invariant Gibbs measures μ_0, μ_1, μ_2 for $\lambda > \lambda_{cr}$.*

Proof. For the case $k = 2$, from (4.2), we get

$$(z_1 - z_2)\cdot\left[1 - \frac{(\lambda-\frac{1}{\lambda})((\lambda+\frac{1}{\lambda})(z_1+z_2)+2)}{(z_1+z_2+1)^2}\right] = 0.$$

In the case of $z_1 = z_2$, it is already known that there is a unique solution for any $\lambda > 0$.

Let $z_1 \neq z_2$. Then

$$(z_1+z_2+1)^2 = \left(\lambda - \frac{1}{\lambda}\right)\left[\left(\lambda+\frac{1}{\lambda}\right)(z_1+z_2)+2\right].$$

This equation is equivalent to the equation for $(z_1 + z_2)$:

$$(z_1+z_2)^2 - \left(\lambda^2 - \frac{1}{\lambda^2} - 2\right)(z_1+z_2) + 1 - 2\left(\lambda - \frac{1}{\lambda}\right) = 0,$$

whose solutions have the form

$$(z_1+z_2)_{1,2} = \frac{\lambda^4 - 2\lambda^2 - 1 \pm \sqrt{D}}{2\lambda^2} = \varphi_{1,2}(\lambda),$$

where

$$D = (\lambda+1)(\lambda-1)^2(\lambda^5+\lambda^4-2\lambda^3+6\lambda^2+\lambda+1) \geq 0$$

for any $\lambda > 0$.

It is not difficult to show that

$$\varphi_1(\lambda) = \frac{\lambda^4 - 2\lambda^2 - 1 - \sqrt{D}}{2\lambda^2} < 0$$

for any $\lambda > 0$ and

$$\varphi_2(\lambda) = \frac{\lambda^4 - 2\lambda^2 - 1 + \sqrt{D}}{2\lambda^2} > 0$$

for $\lambda > \frac{1+\sqrt{17}}{4} \approx 1.28078$.

Thus $z_1 + z_2 = \varphi_2(\lambda)$. From the system of equations (4.1), we obtain

$$\begin{aligned}&(z_1+z_2)(z_1+z_2+1)^2\\ &= \left(\lambda^2 + \frac{1}{\lambda^2}\right)(z_1+z_2)^2 + 2\left(\lambda + \frac{1}{\lambda}\right)(z_1+z_2) + 2\left(2 - \left(\lambda^2 + \frac{1}{\lambda^2}\right)\right) z_1 z_2 + 2.\end{aligned}$$

In respect that $z_1 + z_2 = \varphi_2(\lambda)$, we have the quadratic equation for z_1:

$$\begin{aligned}&2\left(2 - \left(\lambda^2 + \frac{1}{\lambda^2}\right)\right) z_1^2 - 2\left(2 - \left(\lambda^2 + \frac{1}{\lambda^2}\right)\right)\varphi_2(\lambda) z_1\\ &- \left[\left(\lambda^2 + \frac{1}{\lambda^2}\right)\varphi_2^2(\lambda) + 2\left(\lambda + \frac{1}{\lambda}\right)\varphi_2(\lambda) - \varphi_2(\lambda)(\varphi_2(\lambda)+1)^2 + 2\right] = 0. \quad (4.6)\end{aligned}$$

The discriminant of this quadratic equation is

$$\begin{aligned}D_1 = {}& 2^2\left(2 - \left(\lambda^2 + \frac{1}{\lambda^2}\right)\right)^2 \varphi_2^2(\lambda) + 8\left(2 - \left(\lambda^2 + \frac{1}{\lambda^2}\right)\right)\\ &\times\left[\left(\lambda^2 + \frac{1}{\lambda^2}\right)\varphi_2^2(\lambda) + 2\left(\lambda + \frac{1}{\lambda}\right)\varphi_2(\lambda) - \varphi_2(\lambda)(\varphi_2(\lambda)+1)^2 + 2\right] > 0\end{aligned}$$

for $\lambda > \lambda_{cr} \approx 2.1132163$. Then equation (4.6) has two positive solutions for $\lambda > \lambda_{cr}$:

$$z_1^{(1)}(\lambda) = \frac{1}{2}\varphi_2(\lambda) + \frac{\sqrt{D_1}}{4\left(\lambda - \frac{1}{\lambda}\right)^2}, \quad z_1^{(2)}(\lambda) = \frac{1}{2}\varphi_2(\lambda) - \frac{\sqrt{D_1}}{4\left(\lambda - \frac{1}{\lambda}\right)^2}.$$

Cumbersome calculations show that

$$\begin{aligned}&\lim_{\lambda\to+\infty} z_1^{(1)}(\lambda) = +\infty, \quad \lim_{\lambda\to+\infty} z_1^{(2)}(\lambda) = 0,\\ &\lim_{\lambda\to\lambda_{cr}} z_1^{(1)}(\lambda) = \lim_{\lambda\to\lambda_{cr}} z_1^{(2)}(\lambda) = \frac{1}{2}\varphi_2(\lambda_{cr}) \approx 1.487\end{aligned}$$

and $z_1^{(1)} > 0$, $z_1^{(2)} > 0$ (see Fig. 4.1).

In addition, from the notation $z_1 + z_2 = \varphi_2(\lambda)$ we have $z_2^{(1)} = z_1^{(2)}$, $z_1^{(1)} = z_2^{(2)}$, i.e., the solutions of (4.1) are symmetric: (z_1, z_2) and (z_2, z_1).

It is known from Proposition 4.2 that the system of equations (4.1) has the unique positive solution for $k \geq 2$, $\lambda > 0$ and $z_1 = z_2 = z^*$. In particular, we can

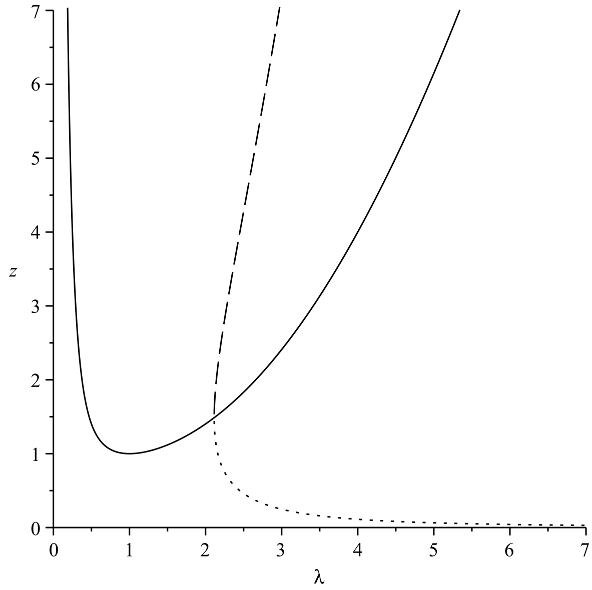

Fig. 4.1: Graph of the functions $z^*(\lambda)$ (continuous curve), $z_1(\lambda)$ (shaded curve), $z_2(\lambda)$ (pointwise curve).

find an explicit form of this solution for $k = 2$. For this, we consider equation (4.3) for $k = 2$:

$$z = \left(\frac{(\lambda + \frac{1}{\lambda})z + 1}{2z + 1} \right)^2, \tag{4.7}$$

which is equivalent to the equation

$$g(z) = 4z^3 + (4 - a^2)z^2 + (1 - 2a)z - 1 = 0.$$

Using the Cardano formula, we find the solution of the last equation:

$$z^* = \frac{1}{12\lambda^2} \left(\sqrt[3]{A + 6\lambda^4 \sqrt{\frac{3B}{\lambda}}} + \frac{C}{\sqrt[3]{A + 6\lambda^4 \sqrt{\frac{3B}{\lambda}}}} + (\lambda^2 - 1)^2 \right), \tag{4.8}$$

where

$$\begin{aligned} A &= \lambda^{12} - 6\lambda^{10} + 36\lambda^9 - 3\lambda^8 - 36\lambda^7 + 232\lambda^6 - 36\lambda^5 - 3\lambda^4 + 36\lambda^3 - 6\lambda^2 + 1, \\ B &= 4\lambda^{10} - 17\lambda^9 + 4\lambda^8 + 188\lambda^7 - 616\lambda^6 + 874\lambda^5 - 616\lambda^4 \\ &\qquad + 188\lambda^3 + 4\lambda^2 - 17\lambda + 4, \\ C &= \lambda^8 - 4\lambda^6 + 24\lambda^5 - 6\lambda^4 + 24\lambda^3 - 4\lambda^2 + 1. \end{aligned}$$

Thus, for $0 < \lambda \leq \lambda_{cr}$, there is a unique translation-invariant Gibbs measure μ_0, corresponding to the unique solution (z^*, z^*) of the system of equations (4.1), and for $\lambda > \lambda_{cr}$ there are three translation-invariant Gibbs measures μ_0, μ_1, μ_2, corresponding to the solutions (z^*, z^*), (z_1, z_2) and (z_2, z_1), respectively. The theorem is proved. □

Remark 4.4. Since $\lambda = \exp(\frac{J}{T})$, where $T > 0$ is the temperature, then $T_{cr} = \frac{J}{\ln \lambda_{cr}}$ and, by Theorem 4.3, for the Blum–Kapel model, there is a unique translation-invariant Gibbs measure μ_0 for $T \geq T_{cr}$, and there are exactly three translation-invariant Gibbs measures μ_0, μ_1, μ_2 for $0 < T < T_{cr}$.

5. Extremality of measure μ_0

In this section, we study the extremality of the measure μ_0 corresponding to the solution (z^*, z^*). To check the extremality of the Gibbs measure, we apply the arguments of a reconstruction on trees from [2] and the methods from [6,11]. We consider Markov chain with states $\{-1, 0, 1\}$ and transition probabilities matrix $\mathbb{P} = (P_{ij})$,

$$P_{\sigma(x)\sigma(y)} = \frac{\exp\{-J\beta\sigma(x)\sigma(y) + h_{\sigma(y)}\}}{\sum_{\sigma(y)\in\{-1,0,+1\}} \exp\{-J\beta\sigma(x)\sigma(y) + h_{\sigma(y)}\}}.$$

Hence, using $z'_{i,x} = \frac{z_{i,x}}{z_{0,x}}$, $i = 1, 2$, we get

$$\begin{aligned}
P_{-1,-1} &= \frac{\lambda^2 z'_1}{\lambda^2 z'_1 + \lambda + z'_2}, & P_{-1,0} &= \frac{\lambda}{\lambda^2 z'_1 + \lambda + z'_2}, & P_{-1,+1} &= \frac{z'_2}{\lambda^2 z'_1 + \lambda + z'_2},\\
P_{0,-1} &= \frac{z'_1}{z'_1 + 1 + z'_2}, & P_{0,0} &= \frac{1}{z'_1 + 1 + z'_2}, & P_{0,+1} &= \frac{z'_2}{z'_1 + 1 + z'_2},\\
P_{+1,-1} &= \frac{z'_1}{z'_1 + \lambda + \lambda^2 z'_2}, & P_{+1,0} &= \frac{\lambda}{z'_1 + \lambda + \lambda^2 z'_2}, & P_{+1,+1} &= \frac{\lambda^2 z'_2}{z'_1 + \lambda + \lambda^2 z'_2}.
\end{aligned}$$

Consequently, (we set $z'_i = z_i$ in what follows):

$$\mathbb{P} = \begin{pmatrix} \frac{\lambda^2 z_1}{\lambda^2 z_1+\lambda+z_2} & \frac{\lambda}{\lambda^2 z_1+\lambda+z_2} & \frac{z_2}{\lambda^2 z_1+\lambda+z_2}\\ \frac{z_1}{z_1+1+z_2} & \frac{1}{z_1+1+z_2} & \frac{z_2}{z_1+1+z_2}\\ \frac{z_1}{z_1+\lambda+\lambda^2 z_2} & \frac{\lambda}{z_1+\lambda+\lambda^2 z_2} & \frac{\lambda^2 z_2}{z_1+\lambda+\lambda^2 z_2} \end{pmatrix}.$$

For the considered solution $\mathbb{P}$, the matrix has the form $(z_1 = z_2 = z)$:

$$\mathbb{P} = \begin{pmatrix} \frac{\lambda^2 z}{\lambda^2 z+\lambda+z} & \frac{\lambda}{\lambda^2 z+\lambda+z} & \frac{z}{\lambda^2 z+\lambda+z}\\ \frac{z}{2z+1} & \frac{1}{2z+1} & \frac{z}{2z+1}\\ \frac{z}{z+\lambda+\lambda^2 z} & \frac{\lambda}{z+\lambda+\lambda^2 z} & \frac{\lambda^2 z}{z+\lambda+\lambda^2 z} \end{pmatrix}.$$

5.1. Conditions for non-extremality of measure μ_0. It is known that a sufficient condition (i.e., the Kesten–Stigum condition) for non-extremality of a Gibbs measure μ corresponding to the matrix $\mathbb{P}$ is that $k\lambda_2^2 > 1$, where λ_2 is the second largest (in absolute value) eigenvalue of $\mathbb{P}$ (see [6]).

We shall find the conditions of non-extremality of the measure corresponding to the unique solution $(z^*, z^*)(z^* = z)$. It is clear that the eigenvalues of this matrix are

$$s_1 = \frac{(\lambda-1)^2 z}{((\lambda^2+1)z+\lambda)(2\lambda+1)}, \quad s_2 = \frac{(\lambda^2-1)z}{(\lambda^2+1)z+\lambda}, \quad s_3 = 1,$$

where z is the solution (4.7). We find $\max\{|s_1|, |s_2|\}$:

$$|s_1| - |s_2| = \frac{(\lambda-1)^2 z}{((\lambda^2+1)z+\lambda)(2\lambda+1)} - \frac{|\lambda-1|(\lambda+1)z}{(\lambda^2+1)z+\lambda}.$$

Let $\lambda > 1$, then

$$|s_1| - |s_2| = \frac{2(1-\lambda)(\lambda^2+\lambda+1)z}{((\lambda^2+1)z+\lambda)(2\lambda+1)} < 0.$$

For $\lambda < 1$,

$$|s_1| - |s_2| = \frac{2\lambda(\lambda-1)(\lambda+2)z}{((\lambda^2+1)z+\lambda)(2\lambda+1)} < 0.$$

Then for any $\lambda > 0$, we have

$$\max\{|s_1|, |s_2|\} = | s_2 | .$$

Consequently, $s_1 <| s_2 |< s_3 = 1$.

Now we check the Kesten–Stigum condition for non-extremality of the measure μ_0: $2{s_2}^2 > 1$, i.e.,

$$2{s_2}^2 - 1 = 2\left(\frac{(\lambda^2-1)z}{(\lambda^2+1)z+\lambda}\right)^2 - 1 > 0,$$

where z has the form (4.8). Using Maple, one can see that the last inequality holds for $\lambda \in (0, \lambda_1) \cup (\lambda_2, +\infty)$, where $\lambda_1 \approx 0.336135$ and $\lambda_2 \approx 2.975$, i.e., the measure μ_0 is non-extremal under this condition (see Fig. 5.1).

Thus, the following theorem holds.

Theorem 5.1. *Let $k = 2, \lambda \in (0, \lambda_1) \cup (\lambda_2, +\infty)$, where $\lambda_1 \approx 0.336135$ and $\lambda_2 \approx 2.975$. Then, for the Blum–Kapel model, the measure μ_0 is non-extremal.*

Remark 5.2. We note that $T = \frac{J}{\ln \lambda}$, where $T > 0$ is the temperature, and since $T_1 = \frac{J}{\ln \lambda_1} < 0$, then in the case $k = 2$ the measure μ_0 is non-extremal for $T \in (0, T_2)$.

5.2. Conditions for extremality of the measure μ_0. If from a Cayley tree Γ^k we remove an arbitrary edge $\langle x^0, x^1 \rangle = l \in L$, then it is divided into two components $\Gamma^k_{x^0}$ and $\Gamma^k_{x^1}$, each called semi-infinite Cayley tree or Cayley subtree.

Let us first give some necessary definitions from [11]. We consider the finite complete subtrees $\mathcal{T}$ that are the initial points of Cayley tree $\Gamma^k_{x^0}$. The boundary $\partial\mathcal{T}$ of the subtree $\mathcal{T}$ consists of the neighbors which are on $\Gamma^k_{x^0} \setminus \mathcal{T}$. We identify

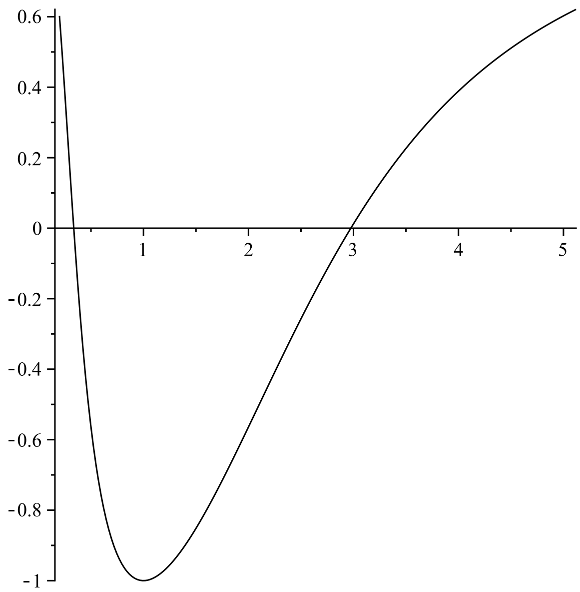

Fig. 5.1: Graph of the function $2s_2^2 - 1$.

the subgraphs of $\mathcal{T}$ with their vertex sets and write $E(A)$ for the edges within either a subset A or ∂A.

In [11], the key ingredients are the two quantities κ and γ. Both are the properties of the collection of Gibbs measures $\{\mu_{\mathcal{T}}^{\tau}\}$, where the boundary condition τ is fixed and $\mathcal{T}$ ranges over all initial finite complete subtrees of $\Gamma_{x^0}^k$. For a given subtree $\mathcal{T}$ of $\Gamma_{x^0}^k$ and a vertex $x \in \mathcal{T}$, we write $\mathcal{T}_x$ for the (maximal) subtree of $\mathcal{T}$ rooted at x. When x is not the root of $\mathcal{T}$, let $\mu_{\mathcal{T}_x}^{s}$ denote the (finite-volume) Gibbs measure in which the parent of x has its spin fixed to s and the configuration on the bottom boundary of $\mathcal{T}_x$ (i.e., on $\partial\mathcal{T}_x \setminus \{\text{parent of } x\}$) is specified by τ.

For two measures μ_1 and μ_2 on Ω, $\|\mu_1 - \mu_2\|_x$ denotes the variation distance between the projections of μ_1 and μ_2 onto the spin at x, i.e.,

$$\|\mu_1 - \mu_2\|_x = \frac{1}{2} \sum_{i \in \{-1,0,+1\}} |\mu_1(\sigma(x) = i) - \mu_2(\sigma(x) = i)|.$$

Let $\eta^{x,s}$ be the configuration η with the spin at x set to s.

Following ([11]), define

$$\kappa \equiv \kappa(\mu) = \sup_{x \in \Gamma^k} \max_{x,s,s'} \left\| \mu_{\mathcal{T}_x}^{s} - \mu_{\mathcal{T}_x}^{s'} \right\|_x,$$

$$\gamma \equiv \gamma(\mu) = \sup_{A \subset \Gamma^k} \max \left\| \mu_A^{\eta^{y,s}} - \mu_A^{\eta^{y,s'}} \right\|_x,$$

where the maximum is taken over all boundary conditions η, all sites $y \in \partial A$, all neighbors $x \in A$ of y and all spins $s, s' \in \{-1, 0, +1\}$.

It is known that a sufficient condition for extremality of the translation-invariant Gibbs measure is $k\kappa\gamma < 1$ (see [11], Theorem 9.3).

Note that κ has the particularly simple form

$$k = \frac{1}{2} \max \sum_{l \in \{-1,0,+1\}} \|P_{il} - P_{jl}\|.$$

Hence, it is clear that $\mid P_{il} - P_{jl} \mid = 0$ for $i = j$. Using the methods from [11], we compute (for $i \neq j$):

$$\sum_{l \in \{-1,0,+1\}} \|P_{il} - P_{jl}\|$$
$$= \begin{cases} \frac{((\lambda+1)(2z+1)+|\lambda-1|)|\lambda-1|z}{(\lambda^2 z+z+\lambda)(2z+1)}, & i=-1,\ j=0 \text{ or } i=0,\ j=-1 \\ \frac{2|\lambda^2-1|z}{\lambda^2 z+z+\lambda}, & i=-1,\ j=+1 \text{ or } i=+1,\ j=-1 \\ \frac{((\lambda+1)(2z+1)+|\lambda-1|)|\lambda-1|z}{(\lambda^2 z+z+\lambda)(2z+1)}, & i=0,\ j=+1 \text{ or } i=+1,\ j=0 \end{cases}.$$

We note that

$$\kappa = \frac{|\lambda^2 - 1|z}{\lambda^2 z + z + \lambda}.$$

Now, in the same way as in ([11], p.15), we can find the estimate for γ in the following form:

$$\gamma = \max\left\{ \left\| \mu_A^{\eta^{y,-1}} - \mu_A^{\eta^{y,0}} \right\|_x, \left\| \mu_A^{\eta^{y,-1}} - \mu_A^{\eta^{y,+1}} \right\|_x, \left\| \mu_A^{\eta^{y,0}} - \mu_A^{\eta^{y,+1}} \right\|_x \right\},$$

where

$$\begin{aligned}
\left\| \mu_A^{\eta^{y,-1}} - \mu_A^{\eta^{y,0}} \right\|_x &= \frac{1}{2} \sum_{s \in \{-1,0,+1\}} \left| \mu_A^{\eta^{y,-1}}(\sigma(x)=s) - \mu_A^{\eta^{y,0}}(\sigma(x)=s) \right| \\
&= \frac{1}{2} \left(|P_{-1,-1} - P_{0,-1}| + |P_{-1,0} - P_{0,0}| + |P_{-1,+1} - P_{0,+1}| \right) \\
&= \frac{1}{2} \frac{((\lambda+1)(2z+1) + |\lambda-1|)|\lambda-1|z}{(\lambda^2 z + z + \lambda)(2z+1)} \le \frac{|\lambda^2-1|z}{\lambda^2 z + z + \lambda}, \\
\left\| \mu_A^{\eta^{y,-1}} - \mu_A^{\eta^{y,+1}} \right\|_x &= \frac{1}{2} \sum_{l \in \{-1,0,+1\}} |P_{-1,l} - P_{+1,l}| = \frac{|\lambda^2-1|z}{\lambda^2 z + z + \lambda}, \\
\left\| \mu_A^{\eta^{y,0}} - \mu_A^{\eta^{y,+1}} \right\|_x &= \frac{1}{2} \sum_{l \in \{-1,0,+1\}} |P_{0,l} - P_{+1,l}| \\
&= \frac{1}{2} \frac{((\lambda+1)(2z+1) + |\lambda-1|)|\lambda-1|z}{(\lambda^2 z + z + \lambda)(2z+1)} \le \frac{|\lambda^2-1|z}{\lambda^2 z + z + \lambda}.
\end{aligned}$$

Consequently,

$$\gamma \le \frac{|\lambda^2 - 1|z}{\lambda^2 z + z + \lambda}.$$

We check the condition $2\kappa\gamma < 1$ for μ_0 which is equivalent to the inequality

$$(\lambda^4 - 6\lambda^2 + 1)z^2 - 2\lambda(\lambda^2 + 1)z - \lambda^2 < 0,$$

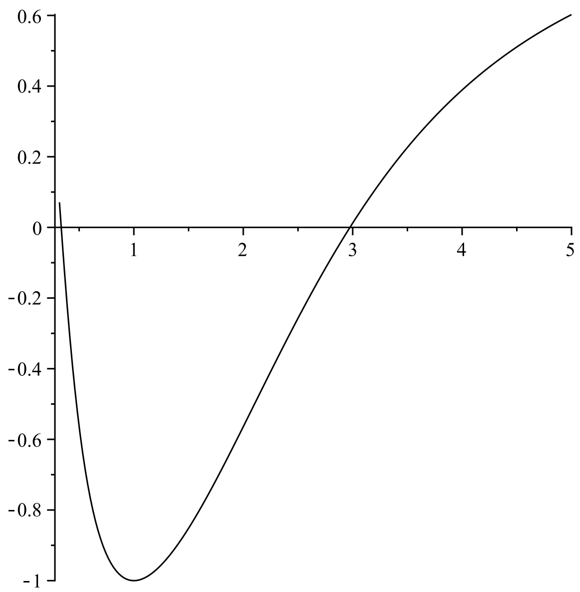

Fig. 5.2: Graph of the function $2\kappa\gamma - 1$.

where z is defined by (4.8). Using computer analysis, we obtain that the last inequality holds for $\lambda_1 < \lambda < \lambda_2$, where $\lambda_1 \approx 0.336135$ and $\lambda_2 \approx 2.975$ (see Fig. 5.2).

Thus the following theorem is true.

Theorem 5.3. *Let $k = 2$. Then for the Blum–Kapel model the measure μ_0 is extremal for $\lambda_1 < \lambda < \lambda_2$.*

Remark 5.4. Since $T_1 < 0$, then it follows from Remark 5.2 and Theorem 5.3 that in the case $k = 2$ the measure μ_0 is extremal for $T > T_2$.

Remark 5.5. To check (not) the extremality of measures μ_1, μ_2 is very difficult even with the help of computer analysis. Therefore this problem remains open.

Since the set of all limit Gibbs measures forms a nonempty convex compact subset of the set of all probability measures ([4, 13, 16]), then the following theorem is true.

Theorem 5.6. *If $k = 2$ and $\lambda_{cr} < \lambda < \lambda_2$ (i.e., for $0 < T < T_{cr}$ and $T > T_2$), then there are at least two extremal Gibbs measures for the Blum–Kapel model.*

Proof. By Theorem 4.3, it is known that if $0 < \lambda \leq \lambda_{cr}$, then there is the unique translation-invariant Gibbs measure μ_0. By Theorem 5.3, if $\lambda_1 < \lambda < \lambda_2$, then the measure μ_0 is extremal. For $\lambda > \lambda_{cr}$, we have the measure μ_0 and at least two new measures μ_1, μ_2 mentioned in Theorem 4.3. If we assume that all the new measures are not extremal in $(\lambda_{cr}, \lambda_2)$, then only one known extremal measure μ_0 remains. But in this case, the non-extremal measures can not be decomposed only into the unique measure μ_0. Consequently, for $\lambda_{cr} < \lambda < \lambda_2$, at least one of the new measures must be extremal. The theorem is proved. $\square$

Acknowledgments. The authors are very grateful to Professor U.A. Rozikov for his useful advice.

References

[1] E.N. Cirillo and E. Olivieri, *Metastabilty and nucleation for the Blume–Capel model. Different mechanisms of transition,* J. Statist. Phys. **83** (1996), No. 3–4, 473–554.

[2] M. Formentin and C. Külske, *A symmetric entropy bound on the non-reconstruction regime of Markov chains on Galton–Watson trees,* Electron. Commun. Probab. **14** (2009), 587–596.

[3] N.N. Ganikhodzhaev and U.A. Rozikov, *Description of periodic extreme Gibbs measures for some lattice models on the Cayley tree,* Teoret. Mat. Fiz. **111**(1) (1997), 109–117 (Russian); Engl. transl.: Theoret. and Math. Phys. **111** (1997), 480–486.

[4] H.-O. Georgii, *Gibbs Measures and Phase Transitions,* De Gruyter Studies in Mathematics, **9**, Walter de Gruyter & Co., Berlin, 1988.

[5] O. Hryniv and R. Kotecký, *Surfase tension and the Orustein–Zernike behavior for the 2D Blume–Capel model,* J. Statist. Phys. **106** (2002), 431–476.

[6] H. Kesten and B.P. Stigum, *Additional limit theorem for indecomposable multidimensional Galton–Watson processes,* Ann. Math. Statist. **37** (1966), 1463–1481.

[7] N.M. Khatamov, *New classes of ground states for the Potts model with random competing interactions on a Cayley tree,* Teoret. Mat. Fiz. **180** (2014), 86–93 (Russian); Engl. transl.: Theoret. and Math. Phys. **180** (2014), 827–834.

[8] N.M. Khatamov, *Nonuniqueness of a Gibbs measure for the Ising ball model,* Teoret. Mat. Fiz. **180** (2014), 318–328 (Russian); Engl. transl.: Theoret. and Math. Phys. **180** (2014), 1030–1039.

[9] C. Külske and U.A. Rozikov, *Fuzzy transformations and extremality of Gibbs measures for the Potts model on a Cayley tree,* Random Structures and Algorithms **50** (2017), 636–678.

[10] C. Külske, U.A. Rozikov, and R.M. Khakimov, *Description of all translation-invariant splitting Gibbs measures for the Potts model on a Cayley tree,* J. Stat. Phys. **156** (2014), 189–200.

[11] F. Martinelli, A. Sinclair, and D. Weitz, *Fast mixing for independent sets, coloring and other models on trees,* Random Structures and Algoritms **31** (2007), 134–172.

[12] V.V. Prasolov, *Polynomials,* Algorithms and Computation in Mathematics, **11**, Springer-Verlag, Berlin, 2004.

[13] C.J. Preston, *Gibbs States on Countable Sets,* Cambridge Tracts Math., No. 68, Cambridge Univ. Press, London-New York, 1974.

[14] U.A. Rozikov, *Gibbs Measures on Cayley Trees,* World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2013.

[15] U.A. Rozikov and R.M. Khakimov, *Gibbs measures for the fertile three-state hard core models on a Cayley tree,* Queueing Syst. **81**(1) (2015), 49–69.

[16] Ya.G. Sinai, *Theory of Phase Transitions: Rigorous Results,* Nauka, Moscow (1980) (Russian); Engl. transl.: International Series in Natural Philosophy, **108**, Pergamon Press, Oxford-Elmsford, N.Y., 1982.

[17] P.E. Theodorakis and N.J. Fytas, *Monte Carlo study of the triangular Blume–Capel model under bond randomness*, Phys. Rev. E **86** (2012), 011140.

Received July 26, 2017, revised September 14, 2018.

Nosir Khatamov,
Namangan State University, 316 Uychi Str., 160119, Namangan, Uzbekistan,
E-mail: nxatamov@mail.ru

Rustam Khakimov,
Namangan State University, 316 Uychi Str., 160119, Namangan, Uzbekistan,
E-mail: rustam-7102@rambler.ru

Трансляційно-інваріантні міри Гіббса для моделі Блюма–Капеля на дереві Кейлі

Nosir Khatamov and Rustam Khakimov

У даній роботі розглянуто трансляційно-інваріантні міри Гіббса для моделі Блюма–Капеля на дереві Кейлі порядку k. Знайдено таку приблизну критичну температуру T_{cr}, що для $T \geq T_{cr}$ існує єдина трансляційно-інваріантна міра Гіббса, а для $0 < T < T_{cr}$ є рівно три трансляційно-інваріантні міри Гіббса. Крім того, вивчено проблему (не)екстремальності для унікальної міри Гіббса.

Ключові слова: дерево Кейлі, конфігурація, модель Блюма–Капеля, міра Гіббса, трансляційно-інваріантна міра, екстремальність міри.