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Automorphisms of Cellular Divisions of 2-Sphere Induced by Functions with Isolated Critical Points

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Let $f : S^2 \to \mathbb{R}$ be a Morse function on the 2-sphere and K be a connected component of some level set of f containing at least one saddle critical point. Then K is a 1-dimensional CW-complex cellularly embedded into S^2, so the complement $S^2 \setminus K$ is a union of open 2-disks $D_1, \dots, D_k$. Let $\mathcal{S}_K(f)$ be the group of isotopic to the identity diffeomorphisms of S^2 leaving invariant K and also each level set $f^{-1}(c)$, $c \in \mathbb{R}$. Then each $h \in \mathcal{S}_K(f)$ induces a certain permutation σ_h of those disks. Denote by $G = \{\sigma_h \mid h \in \mathcal{S}_K(f)\}$ the group of all such permutations. We prove that G is isomorphic to a finite subgroup of $SO(3)$.

Key words: surface, Morse function, diffeomorphisms

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1. Introduction

The studying of groups of automorphisms of discrete structures goes far back in time. One of the first general results obtained by A. Cayley (1854) claims that every finite group G of order n is a subgroup of the permutation group of a set consisting of n elements (see also E. Nummela [34] for extension of this fact to topological groups). C. Jordan [14] (1869) described the structure of groups of automorphisms of finite trees and R. Frucht [13] (1939) showed that every finite group can also be realized as a group of symmetries of certain finite graph.

Given a closed compact surface M endowed with a cellular decomposition Ξ (e.g., with a triangulation), one can consider the group of "combinatorial" automorphisms of M. More precisely, say that a homeomorphism $h : M \to M$ is *cellular* or a Ξ-*homeomorphism* if it maps i-cells to i-cells, and h is Ξ-*trivial* if it preserves every cell with its orientation. Then the group of *combinatorial automorphisms* of Ξ is the group of Ξ-homeomorphisms modulo Ξ-trivial ones. This group is denoted by $\mathrm{Aut}(\Xi)$. It was proved by R. Cori and A. Machi [8] and J. Širáň and M. Škoviera [39] that every finite group is isomorphic with $\mathrm{Aut}(\Xi)$ for some cellular decomposition of some surface which can be taken equally either orientable or non-orientable.

Notice that the 1-skeleton M^1 of Ξ can be regarded as a graph. Suppose that each vertex M^1 has even degree. Then in many cases one can construct a smooth ($\mathcal{C}^\infty$) function $f : M \to \mathbb{R}$ such that M^1 is a critical level containing *all* saddles (i.e., critical points being not local extremes), and the group $\mathrm{Aut}(\Xi)$ can be regarded as the group of *"combinatorial symmetries"* of f.

This point of view was motivated by works of A. Fomenko on classification of Hamiltonian systems, see [10, 11]. The group $\mathrm{Aut}(\Xi)$ is called *the group of symmetries of an "atom" of* f. Such groups for the case when f is a Morse function were studied by A. Fomenko and A. Bolsinov [4], A. Oshemkov and Yu. Brailov [5], Yu. Brailov and E. Kudryavtseva [6], A.A. Kadubovsky and A.V. Klimchuk [15], and A. Fomenko, E. Kudryavtseva and I. Nikonov [25].

In [29], the author gave sufficient conditions for a Ξ-homeomorphism to be Ξ-trivial, and in [30] estimated the number of invariant cells of a Ξ-homeomorphism.

It was proved by A. Fomenko and E. Kudryavtseva [23, 24] that every finite group is the group of combinatorial symmetries of some Morse function f on a compact orientable surface which has a critical level containing all saddles. However, the number of critical points of f, as well as the genus of M, can be arbitrarily large.

In general, if $f : M \to \mathbb{R}$ is an arbitrary smooth function with isolated critical points, then a certain part of its "combinatorial symmetries" is reflected by the so-called *Kronrod–Reeb* graph Δ_f, see, e.g., [2, 3, 18, 19, 26, 33, 35, 38] and Section 2.3. Such a graph is obtained by shrinking each connected component of each level set $f^{-1}(c)$, $c \in \mathbb{R}$, of f into a point.

Let $\mathcal{D}(M)$ be the group of diffeomorphisms of M, and

$$\mathcal{S}(f) = \{h \in \mathcal{D}(M) \mid f(h(x)) = f(x) \text{ for all } x \in M\}$$

be the group of diffeomorphisms h of M which "preserve" f in the sense that h leaves invariant each level set $f^{-1}(c)$, $c \in \mathbb{R}$, of f. Hence it yields a certain permutation of connected components of $f^{-1}(c)$ being the points of Δ_f, and thus induces a certain map $\rho(h) : \Delta_f \to \Delta_f$. It can be shown that $\rho(h)$ is a homeomorphism of Δ_f, and the correspondence $\rho : h \mapsto \rho(h)$ is a *homomorphism* of groups

$$\rho : \mathcal{S}(f) \to \mathcal{H}(\Delta_f),$$

where $\mathcal{H}(\Delta_f)$ is the group of homeomorphisms of Δ_f. One can also verify that the image of $\rho(\mathcal{S}(f))$ is a *finite* group.

Let also $\mathcal{D}_{\mathrm{id}}(M)$ be the identity path component of $\mathcal{D}(M)$, and

$$\mathcal{S}'(f) = \mathcal{S}(f) \cap \mathcal{D}_{\mathrm{id}}(M)$$

be the group of f-preserving diffeomorphisms which are isotopic to the identity via an isotopy consisting of not necessarily f-preserving diffeomorphisms. We will be interested in the group

$$G_f = \rho(\mathcal{S}'(f))$$

of automorphisms of Δ_f induced by elements from $\mathcal{S}'(f)$, see Remark 1.3 for the structure and applications of G_f.

Suppose that the set $\mathrm{Fix}(G_f)$ of common fixed points of all elements of G_f in Δ_f is non-empty. Let $v \in \mathrm{Fix}(G_f)$ be a vertex of Δ_f fixed under G_f, and $\mathrm{Star}(v)$ be a *star* of v, i.e., a small G_f-invariant neighborhood of v. Then each $\gamma \in G_f$ induces a homeomorphism of $\mathrm{Star}(v)$, and we can also define the group

$$G_v^{\mathrm{loc}} = \{\gamma|_{\mathrm{Star}(v)} \mid \gamma \in G_f\}$$

of restrictions of elements of G_f to $\mathrm{Star}(v)$. We will call G_v^{loc} the *local stabilizer* of v.

Remark 1.1. We will give now an equivalent description of the group G_v^{loc}. Let K be the critical component of a level-set of f corresponding to the vertex $v \in \Delta_f$. Since $v \in \mathrm{Fix}(G_f)$, we obtain that $h(K) = K$ for all $h \in \mathcal{S}'(f)$. Let $c = f(K)$ be the value of f on K, and $\varepsilon > 0$ be a small number such that the segment $[c-\varepsilon, c+\varepsilon]$ contains no other critical values of f except for c. Let also N_K be the connected component of $f^{-1}[c-\varepsilon, c+\varepsilon]$ containing K. Notice that the quotient map p induces a bijection between the connected components of ∂N_K and the edges of $\mathrm{Star}(v)$. Moreover, $h(N_K) = N_K$ for all $h \in \mathcal{S}'(f)$, and hence h induces a permutation σ_h of the connected components of ∂N_K. Then G_v^{loc} is the same as the group of permutations of the connected components of ∂N_K induced by h.

In [17,27,28,31,32], the groups G_v^{loc} were calculated for all Morse functions on all orientable surfaces distinct from S^2. In the present paper, we give a complete description of the structure of the group G_v^{loc} to the case when $M = S^2$. For the reader's convenience, we present a general statement about the structure of the group G_v^{loc} for all orientable surfaces.

Theorem 1.2. *Let $f \in \mathcal{C}^\infty(M, \mathbb{R})$ be a Morse function and $v \in \mathrm{Fix}(G_f)$ be some vertex.*

(1) *If $M \neq S^2, T^2$, then $G_v^{\mathrm{loc}} \approx \mathbb{Z}_n$, for some $n \geq 1$,* [31].

(2) *If $M = T^2$, then $G_v^{\mathrm{loc}} \approx \mathbb{Z}_m \times \mathbb{Z}_{mn}$, for some $m, n \geq 1$,* [27,28,32].

(3) *Let $M = S^2$. Then the following statements hold.*

   (a) *For each vertex $v \in \mathrm{Fix}(G_f)$, the group G_v^{loc} is isomorphic to a finite subgroup of $SO(3)$, that is, to one of the following groups, see* [16, pp. 21–23]:
   $$\mathbb{Z}_n, \quad \mathbb{D}_n, \quad \mathbb{A}_4, \quad \mathbb{S}_4, \quad \mathbb{A}_5, \qquad n \geq 1. \tag{1.1}$$

   (b) *If $\mathrm{Fix}(G_f)$ has at least one edge, then for any vertex $v \in \mathrm{Fix}(G_f)$, the group G_v^{loc} is cyclic.*

   (c) *If $\mathrm{Fix}(G_f)$ consists of a unique vertex v and G_v^{loc} is non-trivial and cyclic, then $G_v^{\mathrm{loc}} \cong \mathbb{Z}_2$.*

We need to prove only item (3) of this theorem. In fact, we will establish a more general result including (3) of Theorem 1.2 as a partial case, see Theorem 2.3 and Subsection 2.6.

Remark 1.3. Notice that $\mathcal{S}(f)$ can be regarded as the *stabilizer* of f with respect to the natural right action $\phi : \mathcal{C}^{\infty}(M,\mathbb{R}) \times \mathcal{D}(M) \to \mathcal{C}^{\infty}(M,\mathbb{R})$ of $\mathcal{D}(M)$ on the space $\mathcal{C}^{\infty}(M,\mathbb{R})$ of smooth functions on M defined by $\phi(f,h) = f \circ h$. Then the group G_f plays a key role in determining the homotopy type of the orbit $\mathcal{O}(f) = \{f \circ h \mid h \in \mathcal{D}(M)\}$ of f with respect to the above action, see E. Kudryavtseva [20–22], S. Maksymenko [29,31].

A function $f \in \mathcal{C}^{\infty}(M,\mathbb{R})$ is *Morse* if the following conditions are fulfilled:

- f takes constant values on the connected components of the boundary ∂M;
- each critical point f is nondegenerate and is contained in $\mathrm{Int} M$.

We will denote by $\mathcal{M}(M,\mathbb{R})$ the set of all Morse functions on M.

Let $\mathcal{P}$ be the minimal set of isomorphism classes of groups satisfying the following conditions:

- a unit group $\{1\} \in \mathcal{P}$;
- if $A, B \in \mathcal{P}$ and $n \in \mathbb{N}$, then $A \times B$, $A \wr \mathbb{Z}_n \in \mathcal{P}$,

where $A \wr \mathbb{Z}_n$ is the *wreath product* of the groups A and $\mathbb{Z}_n$ which can be defined as the *direct product of the sets* $\underbrace{A \times \cdots \times A}_{n} \times \mathbb{Z}_n$ with the following multiplication:

$$(a_0, a_1, \dots, a_{n-1}, k)\,(b_0, b_1, \dots, b_{n-1}, l) = (a_0 b_k,\, a_1 b_{k+1},\, \dots,\, a_{n-1} b_{k-1},\, k+l),$$

where all indices are taken modulo n.

In [17], the authors described the structure of the set

$$G(M,\mathbb{R}) = \{G_f \mid f \in \mathcal{M}(M,\mathbb{R})\} \tag{1.2}$$

of groups G_f for all Morse functions on all orientable surfaces distinct from 2-sphere and 2-torus. It was proved that $G(M,\mathbb{R}) = \mathcal{P}$. The structure of the groups $G(T^2,\mathbb{R})$ and $G(S^2,\mathbb{R})$ will be studied in forthcoming papers.

2. Main result

2.1. Isolated critical points of smooth functions on the plane. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a smooth function such that $0 \in \mathbb{R}^2$ is an isolated critical point of f. Then there exists an open neighborhood U of 0 in $\mathbb{R}^2$ and a topological embedding (homeomorphism onto its image) $h : U \to \mathbb{R}^2$ such that $h(0) = 0$, and the composition $f \circ h : U \to \mathbb{R}$ is given by one of the following formulas (see [9,36]):

$$(f \circ h)(x,y) = \begin{cases} \pm\left(x^2 + y^2\right) & \text{if } 0 \in \mathbb{R}^2 \text{ is a local extreme of } f \\ \mathrm{Im}\left((x+iy)^k\right) & \text{for some } k \geq 1 \text{ otherwise} \end{cases}.$$

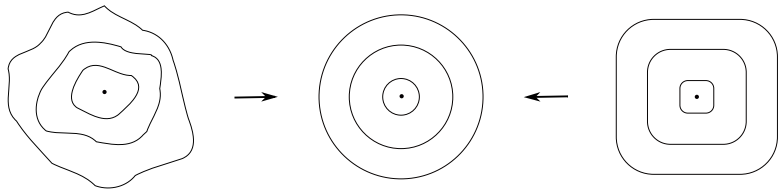


Fig. 2.1: Local extreme.

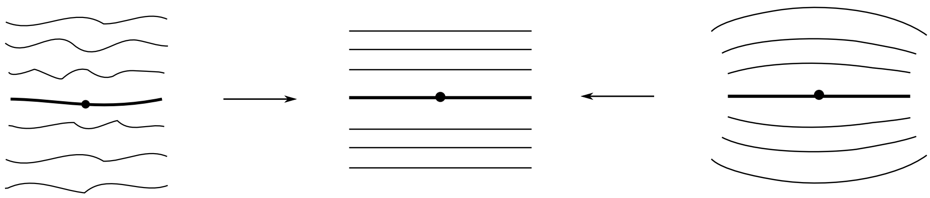


Fig. 2.2: Saddle point of order 1.

The structures of level sets of f near isolated critical points are shown in Figures 2.1, 2.2, 2.3.

In particular, if 0 is a local extreme, then the level sets of f are concentric simple closed curves wrapping around 0.

Otherwise, 0 is called a *saddle*, and the critical level set of f near 0 consists of $2k$ arcs

$$\alpha_0, \alpha_1, \ldots, \alpha_{2k-1}$$

starting from 0. They split a neighborhood of 0 into $2k$-sectors $\widehat{\alpha_i\alpha_{i+1}}$ such that the values of f in the consecutive sectors $\widehat{\alpha_{i-1}\alpha_i}$ and $\widehat{\alpha_i\alpha_{i+1}}$ are of opposite signs, see Figure 2.4. In particular, the number k does not depend on h, and will be called the *order* of the critical point.

2.2. Functions with isolated critical points on compact surfaces. From now on, M will be a compact two-dimensional manifold. Let $f: M \to \mathbb{R}$ be a smooth ($\mathcal{C}^\infty$) function, Σ_f be the set of all critical points of f, and K be a connected component of some level set $f^{-1}(c)$, $c \in \mathbb{R}$, of f. Then K is called *regular* if it contains no critical points of f, and *critical* otherwise.

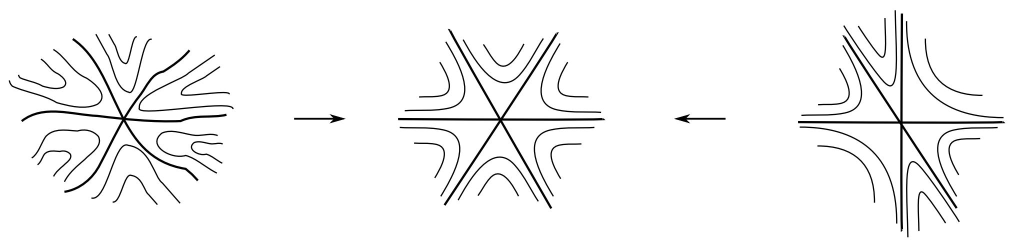
Definition 2.1. We will say that f belongs to *class* $\mathcal{Z}$ if

1) f takes constant values on the connected components of the boundary ∂M;

2) each critical point of f is isolated and is contained in $\mathrm{Int}M$.

In particular, every Morse function belongs to $\mathcal{Z}$.

Suppose $f \in \mathcal{Z}$. Then every connected component K of some level set of f has the following structure.



$f \circ h(x,y) = Im((x+iy)^3) = 3x^2y - y^3$ $\qquad$ $f(x,y) = xy(x+y)$

Fig. 2.3: Saddle point of order 3.

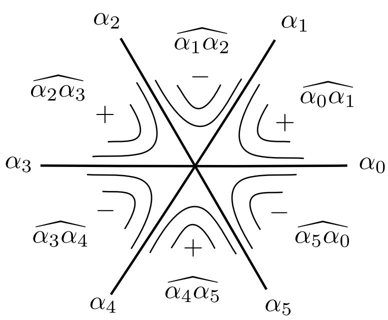


Fig. 2.4: Arcs and sectors near the saddle point, $k = 3$.

A) Suppose K is regular, so it is a closed connected 1-submanifold of $\mathrm{Int}M$. Therefore K is diffeomorphic with the circle S^1. Moreover, there exists an open neighborhood U of K, $\varepsilon > 0$, and a diffeomorphism $\phi : S^1 \times (-\varepsilon, \varepsilon) \to U$ such that $\phi(S^1 \times 0) = K$ and $f \circ \phi(z,t) = t + c$ for all $(z,t) \in S^1 \times (-\varepsilon, \varepsilon)$.

B) If K is critical, then it follows from 2) that K is homeomorphic to a finite 1-dimensional CW-complex ("topological graph"). Moreover, let N be a connected component of $f^{-1}[c - \varepsilon, c + \varepsilon]$ containing K, where $\varepsilon > 0$ is so small that $N \cap \partial M = \varnothing$ and $N \cap \Sigma_f = K \cap \Sigma_f$. We will call N an *f-regular* neighborhood of K, or an *atom* in the sense of A. Fomenko, see, e.g., [4].

   Let also V be a connected component of $N \setminus K$. Then there exists a continuous map $\phi : S^1 \times [-1,1] \to N$ with the following properties:

   - the set $F := \phi^{-1}(\Sigma_f)$ is finite and is contained in $S^1 \times 1$;
   - the restriction of ϕ to $(S^1 \times [-1,1]) \setminus F$ is an embedding;
   - $\phi\big(S^1 \times [-1,1)\big) = V$;
   - ϕ homeomorphically maps each connected component J of $(S^1 \times 1) \setminus F$ onto some edge of K.

   Thus, saying informally, N can be obtained from K by gluing to it cylinders $S^1 \times [-1,1]$ via maps $\psi : S^1 \times 1 \to K$ which are homeomorphisms except some finite subsets, see Figure 2.5.

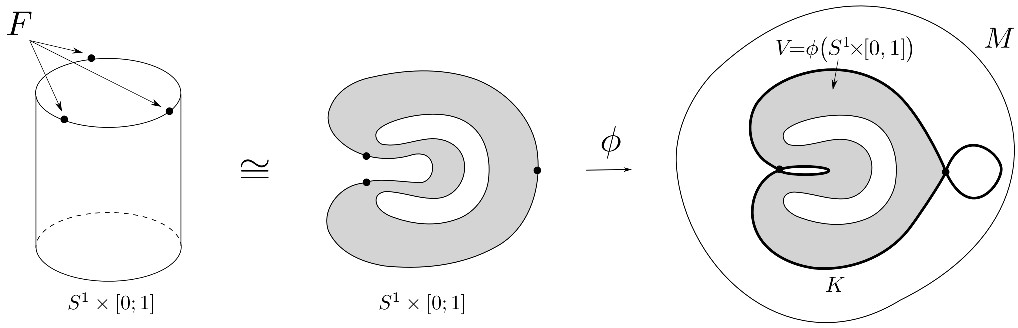


Fig. 2.5: The gluing of cylinders to a critical component of a level set.

2.3. Kronrod–Reeb graph For each *continuous* function $f \in \mathcal{C}(M, \mathbb{R})$ we will denote by Δ_f the partition of M into connected components of the level set of f. Let also $p : M \to \Delta_f$ be the canonical quotient map associating to each point $x \in M$ the connected component of the level set $f^{-1}(f(x))$ containing x. Endow Δ_f with the *factor* topology with respect to the mapping p. Thus a subset $A \subset \Delta_f$ is open if and only if its inverse image $p^{-1}(A)$ is open in M. Then f induces a unique continuous function $\hat{f} : \Delta_f \to \mathbb{R}$ such that $f = \hat{f} \circ p$.

It follows from A) and B) above that for $f \in \mathcal{Z}$ the space Δ_f has a structure of a one-dimensional CW-complex: the vertices of Δ_f correspond to the critical components of the level-sets of f, while the points of edges correspond to the regular ones. The space Δ_f is often called the *Kronrod–Reeb* graph, or the *Lyapunov* graph, or simply the *graph* of f, [1, 12, 18, 37].

2.4. The action of the stabilizers of f on Δ_f. Notice that for each subgroup $\mathcal{G}$ of the group $\mathcal{H}(M)$ of homeomorphisms of M one can define a natural action $\phi : \mathcal{C}(M, \mathbb{R}) \times \mathcal{G} \to \mathcal{C}(M, \mathbb{R})$ of $\mathcal{G}$ on the space $\mathcal{C}(M, \mathbb{R})$ of the continuous functions on M defined by

$$\phi(f, h) = f \circ h : M \to \mathbb{R}.$$

Given $f \in \mathcal{C}(M, \mathbb{R})$, we will denote by

$$\mathcal{S}^{\mathcal{G}}(f) = \{h \in \mathcal{G} \mid f \circ h = f\}$$

its stabilizer with respect to the above action. Notice that the relation $f \circ h = f$ means that $h(f^{-1}(c)) = f^{-1}(c)$ for all $c \in \mathbb{R}$, that is, h leaves invariant each level-set $f^{-1}(c)$ of f. Hence it interchanges the connected components of $f^{-1}(c)$ and therefore induces a map $\rho(h) : \Delta_f \to \Delta_f$ making the following diagram commutative, see Figure 2.6:

$$\begin{tikzcd} M \arrow[r, "p"] \arrow[d, "h"'] & \Delta_f \arrow[r, "\hat{f}"] \arrow[d, "\rho(h)"] & \mathbb{R} \arrow[d, equal] \\ M \arrow[r, "p"] & \Delta_f \arrow[r, "\hat{f}"] & \mathbb{R} \end{tikzcd} \tag{2.1}$$

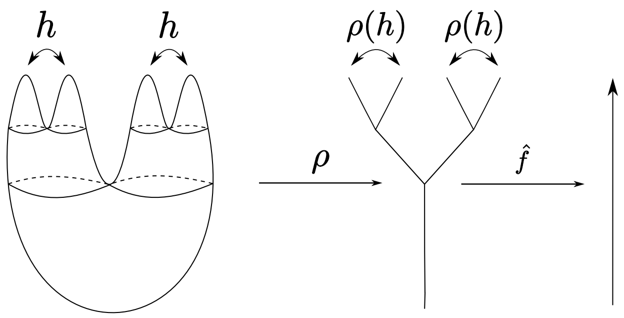


Fig. 2.6: The action of $\mathcal{S}^{\mathcal{G}}(f)$ on Δ_f.

Denote by $\mathcal{H}(\Delta_f)$ the group of homeomorphisms of Δ_f. Then the following Lemma 2.2 implies that $\rho(h)$ is a *homeomorphism* of Δ_f, and one easily checks that the correspondence

$$\rho : \mathcal{S}^{\mathcal{G}}(f) \to \mathcal{H}(\Delta_f) \tag{2.2}$$

is a *homomorphism of groups*. In other words, $\mathcal{S}^{\mathcal{G}}(f)$ acts on Δ_f.

Lemma 2.2. *Suppose we have the following commutative diagram:*

$$\begin{tikzcd} M \arrow[r, "p"] \arrow[d, "h"'] & Y \arrow[d, "g"] \\ M \arrow[r, "p"] & Y \end{tikzcd}$$

in which M and Y are topological spaces, h is continuous, and p is a surjective factor map. Then g is continuous as well. Moreover, if h is a homeomorphism, then so is g.

Proof. We should show that $g^{-1}(U)$ is open in Y for each open $U \subset Y$. Since h and p are continuous, we see that

$$p^{-1}\big(g^{-1}(U)\big) = h^{-1}\big(p^{-1}(U)\big)$$

is open in M. But p is a factor map and thus the openness of the inverse image $p^{-1}\big(g^{-1}(U)\big)$ implies that $g^{-1}(U)$ is open in Y. $\square$

Assume now that $f \in \mathcal{Z}$ and denote

$$G_f = \rho(\mathcal{S}^{\mathcal{G}}(f)),$$

so G_f is the group of all homeomorphisms of Δ_f induced by some homeomorphism $\mathcal{G}$ preserving f, i.e., belonging to $\mathcal{S}^{\mathcal{G}}(f)$.

One easily checks that if $h(e) = e$ for some edge e of Δ_f and $h \in \mathcal{S}^{\mathcal{G}}(f)$, then $h|_e = \mathrm{id}_e$. This implies that G_f is a finite subgroup of $\mathcal{H}(\Delta_f)$ and can be regarded as a group of certain *automorphisms* of a “graph” Δ_f.

2.5. Functions on the 2-sphere. Suppose now $M = S^2$ and $f \in \mathcal{Z}$. Then Δ_f is always a tree. We claim that the set $\mathrm{Fix}(G_f)$ of common fixed points of G_f is a non-empty subtree of Δ_f.

Indeed, it is well known that the group of automorphisms of a finite tree always has

- either a common fixed point or
- an invariant edge e such that some automorphisms of Δ_f change the orientation of e.

However, as mentioned above, the elements of G_f do not change the orientation of the edges, whence G_f must have fixed points. Moreover, if v, w are two fixed points of G_f, then there exists a unique path π in Δ_f connecting them, whence this path is G_f-invariant, and hence it must consist of fixed points of G_f as well. Hence $\mathrm{Fix}(G_f)$ is a non-empty subtree of Δ_f.

For a vertex $v \in \mathrm{Fix}(G_f)$, let $\mathrm{Star}(v)$ be the *star* of v in Δ_f that is the union of v and all edges incident to v. Then $\mathrm{Star}(v)$ is invariant with respect to G_f, whence we can define the group

$$G_v^{\mathrm{loc}} = \{\phi|_{\mathrm{Star}(v)} \mid \phi \in G_f\}$$

consisting of the restrictions of automorphisms of G_f to $\mathrm{Star}(v)$. Notice that G_f can be regarded as a subgroup of the permutation group of edges of $\mathrm{Star}(v)$. We will call G_v^{loc} the *local stabilizer* of v, see Remark 1.1.

In particular, we have an epimorphism

$$r_v : G_f \to G_v^{\mathrm{loc}}, \qquad\qquad r_v(\phi) = \phi|_{\mathrm{Star}(v)},$$

and the composition

$$\rho_v\colon\ \mathcal{S}^{\mathcal{G}}(f) \xrightarrow{\rho} G_f \xrightarrow{r_v} G_v^{\mathrm{loc}}. \tag{2.3}$$

Our aim is to prove the following statement.

Theorem 2.3. *Let $f \in \mathcal{C}\left(S^2, \mathbb{R}\right)$ be a function from class $\mathcal{Z}$ on 2-sphere S^2. Let also $\mathcal{G} \subset \mathcal{H}^+(S^2)$ be a subgroup of the group of orientation preserving homeomorphisms of S^2, $G_f = \rho(\mathcal{S}^{\mathcal{G}}(f))$, and $v \in \mathrm{Fix}(G_f)$ be a common fixed vertex of the group G_f. Then the following statements hold.*

(a) *G_v^{loc} is isomorphic to a finite subgroup of $\mathcal{H}^+(S^2)$. Therefore, by the Brouwer–Kerekjarto theorem [7], G_v^{loc} is isomorphic to a finite subgroup of $SO(3)$ and thus to one of the following groups:*

$$\mathbb{Z}_n, \quad \mathbb{D}_n, \quad \mathbb{A}_4, \quad \mathbb{S}_4, \quad \mathbb{A}_5, \quad n \geq 1, \tag{2.4}$$

see, e.g., [16, pp. 21–23].

(b) *If $\mathrm{Fix}(G_f)$ has at least one edge, then for any vertex $v \in \mathrm{Fix}(G_f)$, the group G_v^{loc} is cyclic.*

(c) *Suppose* $\mathrm{Fix}(G_f)$ *consists of a unique vertex* v *and* $G_v^{\mathrm{loc}} \cong \mathbb{Z}_k$ *for some* $k \geq 2$. *Let also* X^1 *be the critical component of a level set of* f *corresponding to* v. *Then there are two critical saddle points* $z_1, z_2 \in X^1$ *of orders* k_1 *and* k_2, *respectively, such that* k *divides* $GCD(k_1, k_2)$, *and every* $h \in \mathcal{S}^{\mathcal{G}}(f)$ *fixes* z_1 *and* z_2.

Theorem 2.3 will be deduced in Sections 4 and 5 from Theorems 3.1 and 3.2 below about cellular homeomorphisms of CW-complexes.

2.6. Proof of Theorem 1.2. We need only to establish statement (3) of Theorem 1.2. Notice that each Morse function $f : S^2 \to \mathbb{R}$ has isolated critical points, whence Theorem 2.3 is applicable for f. In particular, statements (a) and (b) of Theorem 1.2 are the same as the corresponding statements in Theorem 2.3.

(c) Suppose $\mathrm{Fix}(G_f)$ consists of a unique point and $G_v^{\mathrm{loc}} \cong \mathbb{Z}_k$ for some $k \geq 2$. We need to show that $k = 2$. Indeed, by (c) of Theorem 2.3, k must divide the order of some saddle critical point of f. But each *non-degenerate* saddle has order 2, whence $k = 2$. □

3. Cellular homeomorphisms of CW-complexes

In what follows D^k, $k \geqslant 1$, is the closed k-disk of radius one in $\mathbb{R}^k$ with center at the origin.

Let X be a CW-complex. Then, by X^k, $k \geqslant 0$, we will denote its k-skeleton and by $j_k : X^k \hookrightarrow X$, the canonical embedding. A *cell* always means an *open* cell. Also 0- and 1-cells will often be called *vertices* and *edges* respectively.

A homeomorphism $h : X \to X$ is *cellular* if $h(X^k) = X^k$ for all $k \geq 0$, i.e., for every cell e of X, its image $h(e)$ is also a cell of X. We will denote by $\mathcal{H}^{\mathrm{cw}}(X)$ the group of all cellular homeomorphisms of X.

Note that for each $k \geq 0$ there is a *restriction to* X^k homomorphism:

$$\rho_k : \mathcal{H}^{\mathrm{cw}}(X) \to \mathcal{H}^{\mathrm{cw}}(X^k), \qquad\qquad \rho_k(h) = h|_{X^k}.$$

Let also

$$\mathcal{H}^{\mathrm{cw}}(X^k, j_k) := \rho_k\big(\mathcal{H}^{\mathrm{cw}}(X)\big)$$

be the image of ρ_k. Evidently, it consists of the cellular homeomorphisms h of X^k, which can be extended to some cellular homeomorphism of X. In particular, we have an *epimorphism*

$$\rho_k : \mathcal{H}^{\mathrm{cw}}(X) \to \mathcal{H}^{\mathrm{cw}}\big(X^k, j_k\big).$$

Finally, let $\mathcal{H}_0^{\mathrm{cw}}(X) \subset \mathcal{H}^{\mathrm{cw}}(X)$ be the subgroup consisting of homeomorphisms which leave invariant each cell e of X and preserve its orientation if $\dim e \geq 1$.

Theorem 3.1. *Let X^1 be a 1-dimensional CW-complex, and d be a metric on X^1 such that the length of each edge equals 1. Let also $\mathrm{Isom}^{\mathrm{cw}}(X^1) \subset \mathcal{H}^{\mathrm{cw}}(X^1)$ be the subgroup consisting of all cellular isometries of X^1. Then the following statements hold.*

(1) *There exists a homomorphism*

$$q : \mathcal{H}^{\mathrm{cw}}(X^1) \to \mathrm{Isom}^{\mathrm{cw}}(X^1),$$

which is a retraction onto $\mathrm{Isom}^{\mathrm{cw}}(X^1)$, that is, $q(h) = h$ for each $h \in \mathrm{Isom}^{\mathrm{cw}}(X)$.

(2) $\ker(q) = \mathcal{H}_0^{\mathrm{cw}}(X^1)$, *so we get the following exact sequence:*

$$1 \longrightarrow \mathcal{H}_0^{\mathrm{cw}}(X^1) \longrightarrow \mathcal{H}^{\mathrm{cw}}(X^1) \underset{\eta}{\overset{q}{\rightleftarrows}} \mathrm{Isom}^{\mathrm{cw}}(X^1) \longrightarrow 1 \tag{3.1}$$

in which the natural inclusion $\eta : \mathrm{Isom}^{\mathrm{cw}}(X^1) \subset \mathcal{H}^{\mathrm{cw}}(X^1)$ is a section of q.

Thus, we have a splitting of $\mathcal{H}^{\mathrm{cw}}(X^1)$ into a semidirect product of its subgroups:

$$\mathcal{H}^{\mathrm{cw}}(X^1) = \mathcal{H}_0^{\mathrm{cw}}(X^1) \rtimes \mathrm{Isom}^{\mathrm{cw}}(X^1).$$

In particular, for each $h \in \mathcal{H}_0^{\mathrm{cw}}(X^1)$, its image $\{q(h)\}$ is the only element of the intersection of its adjacent class $h\mathcal{H}_0^{\mathrm{cw}}(X^1)$ with $\mathrm{Isom}^{\mathrm{cw}}(X^1)$:

$$\{q(h)\} \;=\; h\mathcal{H}_0^{\mathrm{cw}}(X^1) \;\cap\; \mathrm{Isom}^{\mathrm{cw}}(X^1). \tag{3.2}$$

Proof. (1) Let $\{e_i\}_{i\in\Lambda}$ be all 1-cells of X^1. For each $i \in \Lambda$, let

$$\phi_i : [-1, 1] \to X^1$$

be the characteristic map of e_i, so the restriction $\phi_i|_{\{-1,1\}} : \{-1, 1\} \to X^0$ of ϕ_i to the boundary $\partial[-1, 1] = \{-1, 1\}$ is the *gluing map* of the cell e_i, and

$$\phi_i|_{(-1,1)} : (-1, 1) \to X^1 \setminus X^0$$

is an embedding.

By assumption, the length of each 1-cell in the metric d equals 1 and thus, without loss of generality, we may assume that the restriction

$$\phi_i|_{(-1,1)} : (-1, 1) \to e_i$$

is an isometry.

Let $h \in \mathcal{H}^{\mathrm{cw}}(X^1)$. We need to construct an isometry $q(h) : X^1 \to X^1$ so that the correspondence $q(h) \mapsto h$ will satisfy the assertion of the theorem.

We will define $q(h)$ in such a way that:

(a) $q(h)(v) = h(v)$ for each vertex $v \in X^0$;

(b) $q(h)(e) = h(e)$ for each edge e of X^1;

(c) if $h(e) = e$, then h preserves the orientation of e if and only if $q(h)|_e = \mathrm{id}_e$.

According to (a), we must put $q(h)(v) = h(v)$ for all $v \in X^0$, so it remains to extend $q(h)$ to all of X^1.

Let e_i be an edge of X^1 and $e_j = h(e_i)$ be its image under h. Then there is a unique homeomorphism $\alpha_i : [-1,1] \to [-1,1]$ such that the following diagram is commutative:

$$\begin{tikzcd}
(-1,1) \arrow[r, "\alpha_i"] \arrow[d, "\phi_i"'] & (-1,1) \arrow[d, "\phi_j"] \\
e_i \arrow[r, "h"] & e_j
\end{tikzcd} \tag{3.3}$$

that is, $\alpha_i|_{(-1,1)} = \phi_j \circ h \circ {\phi_i}^{-1}$.

Due to (b), define $q(h)|_{e_i} : e_i \to h(e_i) = e_j$ by

$$q(h)|_{e_i} = \begin{cases} \phi_j \circ \mathrm{id}_{(-1,1)} \circ {\phi_i}^{-1} & \text{if } \alpha_i \text{ preserves the orientation} \\ \phi_j \circ (-\mathrm{id}_{(-1,1)}) \circ {\phi_i}^{-1} & \text{if } \alpha_i \text{ reverses the orientation} \end{cases},$$

where $\mathrm{id}_{(-1,1)}$ is the identity map of $(-1,1)$, and $-\mathrm{id}_{(-1,1)}(t) = -t$ for all $t \in (-1,1)$. In other words, we get the following commutative diagram:

$$\begin{tikzcd}
(-1,1) \arrow[r, "\pm\mathrm{id}_{(-1,1)}"] \arrow[d, "\phi_i"'] & (-1,1) \arrow[d, "\phi_j"'] \\
e_i \arrow[r, "q(h)"] & e_j
\end{tikzcd} \tag{3.4}$$

Then (c) also holds. Moreover, since $\pm\mathrm{id}_{(-1,1)}$ and $\phi_i|_{(-1,1)}$ for all i are isometries, we see that $q(h)|_{e_i}$ is an isometry as well. Thus, $q(\mathcal{H}^{\mathrm{cw}}(X^1)) \subset \mathrm{Isom}^{\mathrm{cw}}(X^1)$.

Note that if $h \in \mathrm{Isom}^{\mathrm{cw}}(X^1)$, then for each cell e_i the homeomorphism

$$\alpha_i = \phi_j \circ h \circ {\phi_i}^{-1}$$

is an isometry of the segment $[-1,1]$. Hence $\alpha|_{(-1,1)} = \pm\mathrm{id}_{(-1,1)}$ and therefore

$$q(h)|_{e_i} = \phi_j \circ \alpha_i \circ {\phi_i}^{-1} = h|_{e_i}.$$

In other words, $q(h) = h$ for each $h \in \mathrm{Isom}^{\mathrm{cw}}(X^1)$. Thus q is a retraction $\mathcal{H}^{\mathrm{cw}}(X^1)$ by $\mathrm{Isom}^{\mathrm{cw}}(X^1)$.

Verification that q is a homomorphism we leave for the reader.

(2) The identity $\ker q = \mathcal{H}^{\mathrm{cw}}_0(X^1)$ is a direct consequence of (a)–(c). All other statements are easy and we also leave them for the reader. □

Theorem 3.2. *Let X^{k+1}, $k \geq 0$, be a $(k+1)$-dimensional CW-complex. Suppose that for each $(k+1)$-cell of e, its gluing map $\psi_e : S^k \to X^k$ has the following property: there exists a (possibly empty) finite subset $F_e \subset S^k$ such that:*

(a) ${\psi_e}^{-1}(X^0) = F_e$;

(b) $\psi_e|_{S^k\setminus F_e} : S^k \setminus F_e \to X^k \setminus X^0$ *is an embedding, i.e., a homeomorphism on its image.*

Then the following statements hold true.

1) *There exists a homomorphism* $s : \mathcal{H}^{\mathrm{cw}}(X^k, j_k) \to \mathcal{H}^{\mathrm{cw}}(X^{k+1})$ *which is a section of* $\rho_k : \mathcal{H}^{\mathrm{cw}}(X^{k+1}) \to \mathcal{H}^{\mathrm{cw}}(X^k, j_k)$, *i.e.,* $\rho_k(s(h)) = s(h)|_{X^k} = h$, *for all* $h \in \mathcal{H}^{\mathrm{cw}}(X^k, j_k)$.

2) *Suppose* $k = 1$, *so* $\dim X^2 = 2$. *Then*

(i) $\mathcal{H}^{\mathrm{cw}}_0(X^1) \subset \mathcal{H}^{\mathrm{cw}}(X^1, j_1)$.

(ii) *Fix a metric* d *on* X^1 *in which each* 1*-cell has the length* 1*, and let*

$$q : \mathcal{H}^{\mathrm{cw}}(X^1) \to \mathrm{Isom}^{\mathrm{cw}}(X^1)$$

be the homomorphism constructed in Theorem 3.1. Let also

$$\mathrm{Isom}^{\mathrm{cw}}(X^1, j_1) = \mathcal{H}^{\mathrm{cw}}(X^1, j_1) \cap \mathrm{Isom}^{\mathrm{cw}}(X^1). \tag{3.5}$$

Then

$$q(\mathcal{H}^{\mathrm{cw}}(X^1, j_1)) = \mathrm{Isom}^{\mathrm{cw}}(X^1, j_1). \tag{3.6}$$

In other words, if a cellular homeomorphism $h : X^1 \to X^1$ *extends to a cellular homeomorphism of* X^2*, then the corresponding isometry* $q(h)$ *also extends to a cellular homeomorphism of* X^2.

In particular, we have the following commutative diagram in which the first line coincides with (3.1):

$$\begin{tikzcd}
1 \arrow[r] & \mathcal{H}^{\mathrm{cw}}_0(X^1) \arrow[r] & \mathcal{H}^{\mathrm{cw}}(X^1) \arrow[r, bend left, "q"] & \mathrm{Isom}^{\mathrm{cw}}(X^1) \arrow[l, bend left, "\eta"] \arrow[r] & 1 \\
1 \arrow[r] & \mathcal{H}^{\mathrm{cw}}_0(X^1) \arrow[r] \arrow[u, equal] & \mathcal{H}^{\mathrm{cw}}(X^1, j_1) \arrow[r, bend left, "q"] \arrow[u, hook] & \mathrm{Isom}^{\mathrm{cw}}(X^1, j_1) \arrow[l, bend left, "\eta"] \arrow[r] \arrow[u, hook] & 1
\end{tikzcd} \tag{3.7}$$

Proof. 1) Let $h \in \mathcal{H}^{\mathrm{cw}}(X^k, j_k) = \rho_k(\mathcal{H}^{\mathrm{cw}}(X^{k+1}))$. Thus,

$$\rho(\hat{h}) = \hat{h}|_{X^k} = h$$

for some $\hat{h} \in \mathcal{H}^{\mathrm{cw}}(X^{k+1})$. In other words, h can be extended to some cellular homeomorphism $\hat{h}$ of X^{k+1}, though this extension is not unique.

We need to construct extensions $\hat{h} = s(h)$ of all $h \in \mathcal{H}^{\mathrm{cw}}(X^k, j_k)$ so that the correspondence of $s : h \to \hat{h}$ will be a homomorphism of groups. In fact, one needs to define $s(h)$ only for each $(k + 1)$-cell.

Let $\psi_e : D^{k+1} \to X^{k+1}$ be the characteristic mapping of a $(k+1)$-cell e. Then, by (a) and (b), there exists a finite subset $F_e \subset \partial D^{k+1} = S^k$ such that the gluing map

$$\psi_e|_{S^k \setminus F_e} : \; S^k \setminus F_e \;\to\; X^k \setminus X^0$$

is an embedding. One easily checks that then the restriction

$$\psi_e|_{D^{k+1} \setminus F_e} : \; D^{k+1} \setminus F_e \;\to\; X^k \setminus X^0$$

will also be an embedding.

Let $e' = \hat{h}(e)$ be the image of e. Then the homeomorphism

$$\alpha_e = \psi_{e'}^{-1} \circ \hat{h} \circ \psi_e : \; D^{k+1} \setminus F_e \;\to\; D^{k+1} \setminus F_{e'}$$

makes the following diagram commutative:

$$\begin{tikzcd}
D^{k+1} \setminus F_e \arrow[r, "\psi_e"] \arrow[d, "\alpha_e"'] & X^{k+1} \arrow[d, "\hat{h}"] & X^k \arrow[l, hook', "j_k"'] \arrow[d, "h"] \\
D^{k+1} \setminus F_{e'} \arrow[r, "\psi_{e'}"] & X^{k+1} & X^k \arrow[l, hook', "j_k"']
\end{tikzcd} \tag{3.8}$$

Since F_e and $F_{e'}$ are finite subsets of ∂D^{k+1}, they must consist of the same number of points. Therefore α_e extends to a unique homeomorphism $\alpha_e : D^{k+1} \to D^{k+1}$ making the following diagram commutative:

$$\begin{tikzcd}
\partial D^{k+1} \arrow[r, "\psi_e"] \arrow[d, "\alpha|_{D^{k+1}}"'] & X^k \arrow[d, "h"] \\
\partial D^{k+1} \arrow[r, "\psi_{e'}"] & X^k
\end{tikzcd} \tag{3.9}$$

Define another homeomorphism $\beta(h) : D^{k+1} \to D^{k+1}$ by

$$\beta_e(h)(x) = \begin{cases} 0, & x = 0 \\ \alpha(x/|x|)|x|, & x \neq 0 \end{cases}. \tag{3.10}$$

Then $\beta(h) = \alpha_e$ on ∂D^{k+1}. Extend h to a homeomorphism $s(h) : X^{k+1} \to X^{k+1}$ by

$$s(h) = \begin{cases} h(x), & x \in X^k \\ \psi_{e'} \circ \beta_e(h) \circ \psi_e^{-1}, & x \in e \subset X^{k+1} \setminus X^k \end{cases}. \tag{3.11}$$

Then $s(h)$ is a cellular homeomorphism, i.e., it belongs to $\mathcal{H}^{\mathrm{cw}}(X^{k+1})$, and for each $(k+1)$-cell e it makes the following diagram commutative:

$$\begin{tikzcd}
D^{k+1} \arrow[r, "\psi_e"] \arrow[d, "\beta_e(h)"'] & X^{k+1} \arrow[d, "s(h)"] & X^k \arrow[l, hook', "j_k"'] \arrow[d, "h"] \\
D^{k+1} \arrow[r, "\psi_{e'}"] & X^{k+1} & X^k \arrow[l, hook', "j_k"']
\end{tikzcd} \tag{3.12}$$

It is easy to verify that the correspondence $h \to s(h)$ is a homomorphism of groups

$$s : \mathcal{H}^{\mathrm{cw}}(X^k, j_k) \to \mathcal{H}^{\mathrm{cw}}(X^k),$$

being also a section of ρ_k, i.e., $s(h)|_{X^k} = h$ for all $h \in \mathcal{H}^{\mathrm{cw}}(X^k, j_k)$.

2) Now consider the case $\dim X^2 = 2$.

(i) We need to show that $\mathcal{H}^{\mathrm{cw}}_0(X^1) \subset \mathcal{H}^{\mathrm{cw}}(X^1, j_1)$, i.e., that every homeomorphism $h \in \mathcal{H}^{\mathrm{cw}}_0(X^1)$ can be extended to some homeomorphism

$$\gamma(h) : X^2 \to X^2.$$

It suffices to show that for each 2-cell e there exists a homeomorphism β_e which makes the following diagram commutative:

$$\begin{tikzcd}
D^2 \arrow[r, "\psi_e"] \arrow[d, "\beta_e"'] & X^2 \arrow[d, dashed, "\gamma(h)"] & X^1 \arrow[l, hook', "j_1"'] \arrow[d, "h"] \\
D^2 \arrow[r, "\psi_e"] & X^2 & X^1 \arrow[l, hook', "j_1"']
\end{tikzcd} \tag{3.13}$$

where $\psi_e : D^2 \to X^1$ is the characteristic mapping of the cell e. Then $\gamma(h)$ will be uniquely determined by its restrictions on each 2-cell e by the formula

$$\gamma(h)|_e = \psi_e^{-1} \circ \beta_e \circ \psi_e.$$

To construct β_e, recall that there is a finite subset of $F_e \subset \partial D^2 = S^1$ such that the corresponding gluing map

$$\psi_e|_{S^1 \setminus F_e} : S^1 \setminus F_e \to X^1 \setminus X^0$$

is an embedding. Therefore, there is a homeomorphism $\alpha_e : S^1 \setminus F_e \to S^1 \setminus F_e$ which makes the following diagram commutative:

$$\begin{tikzcd}
S^1 \setminus F_e \arrow[r, hook] \arrow[d, "\alpha_e"] & S^1 \arrow[r, hook] & D^2 \arrow[r, "\psi_e"] & X^2 & X^1 \arrow[l, hook', "j_1"'] \arrow[d, "h"] \\
S^1 \setminus F_e \arrow[r, hook] & S^1 \arrow[r, hook] & D^2 \arrow[r, "\psi_e"] & X^2 & X^1 \arrow[l, hook', "j_1"']
\end{tikzcd}$$

Notice that for each connected component K of the set $S^1 \setminus F_e$ its image $\psi_e(K)$ is an edge in X^2. As $h \in \mathcal{H}^{\mathrm{cw}}_0(X^1)$, it follows that h leaves invariant that edge and preserves its orientation. In addition, h fixes each vertex of X^1, and therefore α_e extends to a homeomorphism

$$\alpha_e : S^1 \to S^1,$$

which is fixed on F_e and makes the following diagram commutative:

$$\begin{tikzcd}
S^1 \setminus F_e \arrow[r, hook] \arrow[d, "\alpha_e"] & S^1 \arrow[r, hook] \arrow[d, "\alpha_e"] & D^2 \arrow[r, "\psi_e"] & X^2 & X^1 \arrow[l, hook', "j_1"'] \arrow[d, "h"] \\
S^1 \setminus F_e \arrow[r, hook] & S^1 \arrow[r, hook] & D^2 \arrow[r, "\psi_e"] & X^2 & X^1 \arrow[l, hook', "j_1"']
\end{tikzcd} . \tag{3.14}$$

Using Alexander's trick, extend α_e to some homeomorphism $\beta_e : D^2 \to D^2$. Then β_e is a desired homeomorphism which makes the diagram (3.13) commutative.

(ii) We need to verify that $q(\mathcal{H}^{\mathrm{cw}}(X^1, j_1)) = \mathrm{Isom}^{\mathrm{cw}}(X^1, j_1)$, see (3.6). Since

$$\mathcal{H}_0^{\mathrm{cw}}(X^1) \subset \mathcal{H}^{\mathrm{cw}}(X^1, j_1),$$

it follows that for each $h \in \mathcal{H}^{\mathrm{cw}}(X^1, j_1)$ we have

$$h\mathcal{H}_0^{\mathrm{cw}}(X^1) \subset \mathcal{H}^{\mathrm{cw}}(X^1, j_1). \tag{3.15}$$

Hence,

$$\begin{aligned}\{q(h)\} = h\mathcal{H}_0^{\mathrm{cw}}(X^1) \cap \mathrm{Isom}^{\mathrm{cw}}(X^1)& \\ \subset \mathcal{H}^{\mathrm{cw}}(X^1, j_1) \cap \mathrm{Isom}^{\mathrm{cw}}(X^1) &= \mathrm{Isom}^{\mathrm{cw}}(X^1, j_1).\end{aligned}$$

Here we conclude that the first and last equalities hold according to (3.2) and (3.5), respectively, and the subset relation holds according to (3.15). In other words,

$$q(\mathcal{H}^{\mathrm{cw}}(X^1, j_1)) \subset \mathrm{Isom}^{\mathrm{cw}}(X^1, j_1).$$

To check the inverse inclusion recall that the homomorphism q from Theorem 3.1 is a retraction onto $\mathrm{Isom}^{\mathrm{cw}}(X^1)$. Therefore, for each

$$h \in \mathrm{Isom}^{\mathrm{cw}}(X^1, j_1) \subset \mathcal{H}^{\mathrm{cw}}(X^1, j_1),$$

$q(h) = h$, whence

$$\mathrm{Isom}^{\mathrm{cw}}(X^1, j_1) = q(\mathrm{Isom}^{\mathrm{cw}}(X^1, j_1)) \subset q(\mathcal{H}^{\mathrm{cw}}(X^1, j_1)).$$

Theorem 3.2 is proved. □

4. Proof of (a) of Theorem 2.3

Proof of (a) of Theorem 2.3. Let X^1 be the critical component of a level set of f corresponding to the vertex v, X^0 be the set of critical points of f belonging to X^1, and $j_1 : X^1 \hookrightarrow S^2$ be the canonical embedding.

Then S^2 has a structure of a 2-dimensional CW-complex in which X^0, X^1, and $X^2 = S^2$ are 0-, 1 and 2-skeletons, respectively. We will denote by Ξ the corresponding CW-partition of S^2.

Fix a metric d such that the length of each edge equals 1 and denote by $\mathrm{Isom}^{\mathrm{cw}}(X^1)$ the group of isometries of X^1. Similarly to Theorem 3.2, consider the group

$$\mathrm{Isom}^{\mathrm{cw}}(X^1, j_1) = \mathrm{Isom}^{\mathrm{cw}}(X^1) \cap \mathcal{H}^{\mathrm{cw}}(X^1, j_1)$$

of isometries of X^1 which can be extended to some homeomorphisms of S^2 with respect to this embedding j_1. Then $\mathrm{Isom}^{\mathrm{cw}}(X^1, j_1)$ is a finite subgroup of $\mathcal{H}^{\mathrm{cw}}(X^1, j_1)$.

As f has only finitely many critical points, and they are isolated, one can assume that the gluing map $\psi_e : \partial D^1 \to X^0$ of each 2-cell e is a homeomorphism outside some finite subset $F_e \subset \partial D^1$.

Thus, this cellular partition of S^2 satisfies the conditions of Theorem 3.2. Therefore, the group $\mathcal{H}^{\mathrm{cw}}(X^1, j_1)$, and thus its finite subgroup $\mathrm{Isom}^{\mathrm{cw}}(X^1, j_1)$, is isomorphic to some subgroups of $\mathcal{H}(S^2)$.

Hence, to prove the theorem it suffices to show that G_v^{loc} is isomorphic to a subgroup of $\mathrm{Isom}^{\mathrm{cw}}(X^1, j)$.

Recall that there is an epimorphism (2.3):

$$\rho_v : \mathcal{S}'(f) \xrightarrow{\ \rho\ } G_f \xrightarrow{\ r_v\ } G_v^{\mathrm{loc}}.$$

On the other hand, we have the homomorphism

$$\sigma : \mathcal{S}^{\mathcal{G}}(f) \ \subset\ \mathcal{H}^{\mathrm{cw}}(S^2) \xrightarrow{\ \omega\ } \mathcal{H}^{\mathrm{cw}}(X^1, j_1) \xrightarrow{\ q\ } \mathrm{Isom}^{\mathrm{cw}}(X^1, j_1),$$

where $\omega(h) = h|_{X^1}$ is the restriction homomorphism on X^1, and q is the homomorphism defined in Theorem 3.1. We will show that

$$\ker(\rho_v) = \ker(\sigma). \tag{4.1}$$

As ρ_v is an epimorphism, there will exist a unique *monomorphism*

$$\mu : G_v^{\mathrm{loc}} \hookrightarrow \mathrm{Isom}^{\mathrm{cw}}(X^1, j),$$

which makes the following diagram commutative:

$$\begin{tikzcd} & \mathcal{S}'(f) \arrow[dl, "\rho_v"'] \arrow[dr, "\sigma"] & \\ G_v^{\mathrm{loc}} \arrow[rr, hook, "\mu"'] & & \mathrm{Isom}^{\mathrm{cw}}(X^1, j_1) \end{tikzcd} \tag{4.2}$$

In other words, we will get that G_v^{loc} is isomorphic to some subgroup of $\mathrm{Isom}^{\mathrm{cw}}(X^1, j)$ and hence of $\mathcal{H}(S^2)$.

Consider the following conditions on $h \in \mathcal{S}'(f)$:

(a1)	$h \in \ker(\sigma)$;	(b1)	$h \in \ker(\rho_v)$;
(a2)	h fixes each vertex of X^1, and also leaves invariant every edge X^1 and preserves the orientation.	(b2)	h leaves each 2-cell S^2 invariant.

Then it follows from the definitions of ρ_v and σ that (a1) is equivalent to (a2), and (b1) is equivalent to (b2).

Moreover, (a2) implies (b2), because each 2-cell is uniquely determined by the edges to which it is glued and therefore $\ker(\sigma) \subset \ker(\rho_v)$.

Conversely, (b2) implies (a2) due to [29, Theorem 7.1]. Hence $\ker(\rho_v) \subset \ker(\sigma)$ as well, and so $\ker(\rho_v) = \ker(\sigma)$. Thus G_v^{loc} is isomorphic to a certain subgroup of $\mathcal{H}(S^2)$. $\square$

5. Proof of (b) and (c) of Theorem 2.3

Now let

$$\xi : G_v^{\mathrm{loc}} \xrightarrow{\mu} \mathrm{Isom}^{\mathrm{cw}}(X^1, j_1) \subset \mathcal{H}^{\mathrm{cw}}(X^1, j_1) \xrightarrow{s} \mathcal{H}^{\mathrm{cw}}(S^2) \subset \mathcal{H}(S^2)$$

be the embedding constructed in (a) of Theorem 2.3. We will first discuss its properties. To simplify notations for each $\gamma \in G_v^{\mathrm{loc}}$, denote its image in $\mathcal{H}(S^2)$ by $\widehat{\gamma}$, that is, $\widehat{\gamma} := \xi(\gamma)$.

Lemma 5.1.

(1) *Each* $\widehat{\gamma} \in \xi(G_v^{\mathrm{loc}}) \subset \mathcal{H}(S^2)$ *is a cellular homeomorphism of* Ξ*. Also,* $\widehat{\gamma} = \mathrm{id}_{S^2}$ *if and only if* $\widehat{\gamma}$ *leaves invariant every* 2*-cell of* Ξ*.*

(2) *If* $h \in \mathcal{S}^{\mathcal{G}}(f)$ *and* $\gamma = \rho_v(h) \in G_v^{\mathrm{loc}}$*, then* $h(e) = \widehat{\gamma}(e)$ *for each cell* $e \in \Xi$*.*

(3) *In each cell* $e \in \Xi$*, we can choose a point* $z_e \in e$ *such that their collection* $\{z_e\}_{e\in\Xi}$ *is* $\xi(G_v^{\mathrm{loc}})$*-invariant. In particular, if* $\widehat{\gamma}(e) = e$ *for some* $\gamma \in G_v^{\mathrm{loc}}$*, then* $\widehat{\gamma}(z_e) = z_e$*.*

(4) *Let* e *be a cell in* Ξ *and* A_e *be the subgroup in* $\xi(G_v^{\mathrm{loc}})$*, which leaves* e *invariant.*

   (a) *Suppose that* $\dim e = 0$ *which means that* $z_e = e$ *is the critical point* f *belonging to the critical level* X^1*. Let also* $\alpha_0, \ldots, \alpha_{2k-1}$ *be cyclically ordered arcs on* X^1 *starting from the point* z_e *for some* $k \geq 1$ *and having the same length* $d < 0.5$*, see Figure 2.4. Then* A_e *is a cyclic group freely acting on the set of arcs* $\alpha_0, \alpha_2, \ldots, \alpha_{2k-2}$ *with even indices. In particular, its order divides* k*.*

   (b) *If* $\dim e = 1$*, that is,* e *is an edge* X^1*, then* $A_e = \{\mathrm{id}_{S^2}\}$ *is a unit group.*

   (c) *Suppose* $\dim e = 2$*. Let also* $\psi_e : D^2 \to X^1$ *be the characteristic mapping* 2*-cell* e *and* $F \subset S^1 = \partial D^2$ *be that finite subset such that the corresponding gluing map* $\psi_e|_{S^1}$ *satisfies the conditions of Theorem 3.2. Denote by* n *the number of points in* F*, which coincides with the number of arcs in* $S^1 \setminus F$ *as well as with the number of connected components of* $\overline{e} \setminus e$*. Then* A_e *is a cyclic group freely acting on that set of arcs. In particular, its order divides* n*.*

(5) *If* $\widehat{\gamma} \in \xi(G_v^{\mathrm{loc}})$ *is not the identity mapping, then* $\widehat{\gamma}$ *has exactly* 2 *invariant cells* $e_1, e_2 \in \Xi$*, which may have different dimensions. In particular,* $\widehat{\gamma}$ *has two fixed points* z_{e_1} *and* z_{e_2}*.*

Proof. (1) By the construction, $\xi(G_v^{\mathrm{loc}})$ consists of cellular homeomorphisms of S^2. Also notice that there exists a canonical bijection between 2-cells of Ξ and edges of the star $\mathrm{Star}(v)$ of v such that for each $\gamma \in G_v^{\mathrm{loc}}$ the following conditions are equivalent:

(i) $\widehat{\gamma} \in \mathcal{H}(S^2)$ leaves invariant every 2-cell of Ξ,

(ii) γ leaves invariant each edge of $\mathrm{Star}(v)$,

(iii) γ is the identity homeomorphism of $\mathrm{Star}(v)$,

(iv) $\widehat{\gamma} = \xi(\gamma) = \xi(\mathrm{id}_{\mathrm{Star}(v)}) = \mathrm{id}_{S^2}$.

(2) This statement also follows from the construction of the embedding ξ.

(3) In each e we should choose a point $z_e \in e$ such that if $\widehat{\gamma}(e) = e$ for some $\widehat{\gamma} \in \xi(G_v^{\mathrm{loc}})$, then $\widehat{\gamma}(z_e) = z_e$.

a) If $\dim e = 0$, i.e., e is a critical point of f belonging to X^1, then we must put $z_e = e$.

b) Suppose $\dim e = 1$, i.e., e is an edge X^1. Recall that we have chosen a metric on X^1 in which each edge (in particular, e) has the length 1. Let z_e be the middle point of e, so it splits e into two arcs each of length $\frac{1}{2}$. Hence, if $\widehat{\gamma}(e) = e$, then $\widehat{\gamma}|_e$ is either the identity map or the unique reversing orientation isometry. In both cases $\widehat{\gamma}(z_e) = z_e$.

c) Let $\dim e = 2$ and $\psi_e : D^2 \to X^1$ be the characteristic mapping of the cell e constructed in 2) of Theorem 3.2. Put $z_e = \psi_e(0)$, where $0 \in D^2 \subset \mathbb{R}^2$ is the origin. Then it follows from formulas (3.10) and (3.11) that $\widehat{\gamma}(z_e) = z_e$ whenever $\widehat{\gamma}(e) = e$.

(4) Let e be a cell in Ξ. We need to compute the subgroup $A_e \subset \xi(G_v^{\mathrm{loc}})$ of homeomorphisms that leave e invariant.

Let $\widehat{\gamma} \in A_e \in \xi(G_v^{\mathrm{loc}})$, so there exists $h \in \mathcal{S}^{\mathcal{G}}(f)$ such that $\gamma = \rho_v(h) \in G_v^{\mathrm{loc}}$. In particular, e is an h-invariant cell of $\widehat{\gamma}$ and $z_e \in e$ is the corresponding fixed point of $\widehat{\gamma}$.

(b) Suppose $\dim e = 1$. We will show that h leaves each 2-cell invariant. Then, due to (1), we will get that $\widehat{\gamma} = \mathrm{id}_{S^2}$, which will prove that $A_e = \{\mathrm{id}_{S^2}\}$.

First, we claim that h *preserves orientation of* e. Indeed, since e is an edge, i.e., a part of the critical component of some level set of f, it follows that e belongs to the closure of precisely two 2-cells $\alpha, \beta \in \Xi$, and $f(a) < f(e) < f(b)$ for all $a \in \alpha$ and $b \in \beta$. But h preserves f, whence $h(\alpha) = \alpha$ and $h(\beta) = \beta$. Moreover, since $h \in \mathcal{S}^{\mathcal{G}}(f) \subset \mathcal{H}^+(S^2)$ also preserves orientation of S^2, it must preserve orientations of open subsets α and β of S^2, and therefore h also preserves orientation of e.

Hence h fixes each vertex $x \in \overline{e} \setminus e$ of the edge e being a saddle critical point of f. Let k_x be the order of x, see Figure 2.4. Then there are $2k_x$ arcs in X^1 starting from x which are cyclically ordered and h preserves their cyclic order. Therefore h leaves invariant all the edges of X^1 incident to x.

This implies that the closure B of the set of h-invariant edges is open in X^1. Then it follows from the connectedness of X^1 that $B = X^1$, i.e., all edges of X^1 are invariant with respect to h. Therefore h leaves invariant each 2-cells, whence $\gamma = \mathrm{id}_{S^2}$.

(a) Suppose $\dim e = 0$. Since h preserves the cyclic order of the arcs $\alpha_0, \dots, \alpha_{2k-1}$, it follows that $h(\alpha_i) = \alpha_{i+\eta}$ for some $\eta \in \{0, \dots, 2k-1\}$. Not loosing generality, one can assume that f equals to 0 on X^1. Then on the consecutive sectors $\widehat{\alpha_{i-1}\alpha_i}$ and $\widehat{\alpha_i\alpha_{i+1}}$ the function f takes the values of different signs. As h preserves the values of f, it follows that η must be even, so $h(\alpha_i) = \alpha_{i+2\tau}$ for some $\tau \in \{0, \dots, k-1\}$. Therefore the set of k arcs with even numbers is also invariant with respect to γ.

In other words, we get an action of the group A_e on the set of arcs with even numbers by cyclic shifts, which can be viewed as a homomorphism $q : A_e \to \mathbb{Z}_k$. Note that if $\gamma(\alpha_i) = \alpha_i$ for some $i \in \{0, 2, \dots, 2k-1\}$, then, according to the previous statement, (b) $\gamma = \mathrm{id}_{S^2}$. This means that the action of the group A_e is free, and thus q is a monomorphism. Therefore, A_e is a subgroup of $\mathbb{Z}_k$, whence it is cyclic and its order divides k.

(c) Suppose $\dim e = 2$ and denote by n the number of points in F. Then, according to the diagram 3.13, γ cyclically shifts n edges $\delta_0, \dots, \delta_{n-1}$ of X^1 along which e is glued to X^1. Therefore, similarly to the previous paragraph (a), we get a free action of A_e on the set of those edges by cyclic shifts, which implies that A_e is also cyclic and its order divides n.

(5) According to [30, Corollary 5.4], a cellular homeomorphism $\widehat{\gamma}$ of a closed orientable surface M with a cell partition Ξ

- either leaves every cell invariant and preserves its orientation,
- or the number of invariant cells is equal to the Lefschetz number $L(\widehat{\gamma})$ of $\widehat{\gamma}$.

In our case, γ is a preserving orientation homeomorphism of S^2. Therefore $\widehat{\gamma}$ is isotopic to the identity and we get $L(\widehat{\gamma}) = \chi(S^2) = 2$. Thus, if $\widehat{\gamma}$ is not the identity, then it has precisely two invariant cells. □

Proof of (b) *and* (c) *of Theorem* 2.3. Now we can prove (b) and (c) of Theorem 2.3.

(b) Suppose $\mathrm{Fix}(G_f)$ has a fixed edge. Let v be a vertex in $\mathrm{Fix}(G_f)$ and Ξ be the cellular partition of S^2 constructed in (a) of Theorem 2.3. As $\mathrm{Fix}(G_f)$ is a tree, it follows that v belongs to some edge $\delta \subset \mathrm{Fix}(G_f)$ corresponding to a 2-cell e of Ξ. This cell is therefore invariant with respect to the action of $\xi(G_v^{\mathrm{loc}})$ on S^2, that is, $G_v^{\mathrm{loc}} = A_e$. But then, by (4)(c) of Lemma 5.1, $G_v^{\mathrm{loc}} = A_e$ is a cyclic group.

(c) Suppose $\mathrm{Fix}(G_f)$ consists of a unique vertex v and $G_v^{\mathrm{loc}} \cong \mathbb{Z}_k$ for some $k \geq 2$. As $\xi(G_v^{\mathrm{loc}})$ is conjugated to a finite (cyclic) subgroup of $SO(3)$, all elements of $\xi(G_v^{\mathrm{loc}})$ have exactly two common fixed points, which we will denote by a and b.

On the other hand, according to (5) of Lemma 5.1, every nontrivial element $\widehat{\gamma} \in \xi(G_v^{\mathrm{loc}})$ has exactly two invariant cells e_1 and e_2, and, respectively, two fixed points z_{e_1} and z_{e_2}. Hence, one can assume that $a = z_{e_1}$ and $b = z_{e_2}$. In particular, $G_v^{\mathrm{loc}} = A_{e_1} = A_{e_2}$. Therefore, it suffices to consider the following three cases.

a) If $\dim e_1 = 2$, then $\mathrm{Fix}(G_f)$ must have a fixed edge that corresponds to the 2-cell e, which contradicts to the assumption that $\mathrm{Fix}(G_f) = \{v\}$ consists of a unique vertex.

b) If $\dim e_1 = 1$, then, according to (4)(b) of Lemma 5.1, $G_v^{\mathrm{loc}} = A_{e_1}$ is a trivial group, so $\mathrm{Fix}(G_f) = \Delta_f \neq \{v\}$, which again contradicts to the assumption.

c) Thus the remained situation is when both e_1 and e_2 are vertices of X^1 being therefore saddle critical points of f. Let k_i, $i = 1, 2$, be the order of e_i. Then $G_v^{\mathrm{loc}} = A_{e_i}$ is isomorphic to a subgroup of $\mathbb{Z}_{k_i}$ for both $i = 1, 2$. Hence k divides both k_1 and k_2, and therefore $GCD(k_1, k_2)$. □

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Автоморфізми кліткових ділень 2-сфери індуковані функціями із ізольованими критичними точками

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Нехай $f : S^2 \to \mathbb{R}$ – функція Морса на 2-сфері і K – компонента зв'язності деякої множини рівня функції f, що містить хоча б одну сідлову критичну точку. Тоді K – це 1-вимірний CW-комплекс, клітково вкладений в S^2, так що доповнення $S^2 \setminus K$ є об'єднанням відкритих 2-дисків $D_1, \dots, D_k$. Нехай $\mathcal{S}_K(f)$ група дифеоморфізмів S^2, які ізотопні до тотожного відображення і залишають інваріантними множину K і кожну множину рівня $f^{-1}(c)$, $c \in \mathbb{R}$. Тоді кожен $h \in \mathcal{S}_K(f)$ індукує певну перестановку σ_h вказаних вище дисків. Позначимо через $G = \{\sigma_h \mid h \in \mathcal{S}_K(f)\}$ групу всіх таких перестановок. Ми доведемо, що G ізоморфна скінченній підгрупі в $SO(3)$.

Ключові слова: поверхня, функція Морса, дифеоморфізми