Journal of Mathematical Physics, Analysis, Geometry
2020, Vol. 16, No. 4, pp. 381–401
doi:

Dissipative Extensions of Linear Relations Generated by Integral Equations with Operator Measures

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In the paper, a minimal relation L_0 generated by an integral equation with operator measures is defined and a description of the adjoint relation L_0^* is given. For this minimal relation, we construct a space of boundary values (a boundary triplet) satisfying the abstract "Green formula" and get a description of maximal dissipative (accumulative) and also self-adjoint extensions of the minimal relation.

Key words: Hilbert space, linear relation, integral equation, dissipative extension, self-adjoint extension, boundary value, operator measure

Mathematical Subject Classification 2010: 47A10, 46G12, 45N05

1. Introduction

In the study of linear operators and relations generated by differential or integral equations with boundary conditions there often arises a problem of finding the boundary conditions that determine an operator or a relation with preassigned properties. A classical example of the solution to this problem is the description of self-adjoint extensions of a symmetric operator generated by an ordinary differential expression. The description was given by M.G. Krein in [17] (see also [18, Chap. 5]).

The method proposed by M. G. Krein essentially uses the finite dimensionality of defect subspaces of the symmetric operator. Therefore it is difficult to apply the results obtained in [17] to operators with infinite defect indices. A significant advance in overcoming these difficulties was made by F. S. Rofe-Beketov [20], who was the first to use linear relations for describing self-adjoint extensions of the minimal operator generated by a differential expression with bounded operator coefficients. The results obtained in [20] were later generalized both to the case of more general (accumulative and dissipative) extensions [14] and to the case of differential expressions with unbounded operator coefficients (see monographs [13] and [21] for detailed bibliography).

In this paper we consider the integral equation

$$y(t) = x_0 - iJ \int_{[a,t)} d\mathbf{p}(s) y(s) - iJ \int_{[a,t)} d\mathbf{m}(s) f(s), \tag{1.1}$$

where y is an unknown function, $a \leqslant t \leqslant b$; J is an operator in a separable Hilbert space H, $J = J^*$, $J^2 = E$ (E is an identical operator); $\mathbf{p}$, $\mathbf{m}$ are the operator-valued measures defined on Borel sets $\Delta \subset [a,b]$ that take values in the set of linear bounded operators acting in H; $x_0 \in H$, $f \in L_2(H, d\mathbf{m}; a, b)$. We assume that the measures $\mathbf{p}$, $\mathbf{m}$ have bounded variations, $\mathbf{p}$ is self-adjoint and $\mathbf{m}$ is non-negative.

We define a minimal relation L_0 generated by equation (1.1) and give a description of the adjoint relation L_0^*. For this minimal relation, we construct a space of boundary values (boundary triplet) satisfying the abstract "Green formula" (see [4,5,16]) and get a description of maximal dissipative (accumulative) and also self-adjoint extensions of the minimal relation.

If the measures $\mathbf{p}$, $\mathbf{m}$ are absolutely continuous (i.e., $\mathbf{p}(\Delta) = \int_\Delta p(t)\,dt$, $\mathbf{m}(\Delta) = \int_\Delta m(t)\,dt$ for all Borel sets $\Delta \subset [a,b]$, where the functions $\|p(t)\|$, $\|m(t)\|$ belong to $L_1(a,b)$), then integral equation (1.1) is transformed into a differential equation with a non-negative weight operator function. Linear relations and operators generated by such differential equations were considered in many works (see [6,7,19], further detailed bibliography can be found, for example, in [3,15]).

The study of integral equation (1.1) differs essentially from the study of differential equations by the presence of the following features:

i) a representation of a solution of equation (1.1) using an evolutional family of operators is possible if the measures $\mathbf{p}$, $\mathbf{m}$ do not have common single-point atoms (see [8]);

ii) the Lagrange formula contains summands that are related to single-point atoms of the measures $\mathbf{p}$, $\mathbf{m}$ (see [9]).

Note that this paper partially corrects the errors made in [10].

Under tighter assumptions imposed on the measures $\mathbf{p}$, $\mathbf{m}$, a description of self-adjoint or maximal dissipative (accumulative) extension of L_0 is given in the papers: [9] (where $\mathbf{m}$ is the usual Lebesgue measure on $[a,b]$ and the measure $\mathbf{p}$ has a finite number of single-point atoms); [11] (where $\mathbf{m}$ is the usual Lebesgue measure on $[a,b]$ and the set of single-point atoms of the measure $\mathbf{p}$ can be arranged as an increasing sequence converging to b); [12] (where $\mathbf{m}$ is a non-negative continuous measure and the measure $\mathbf{p}$ is the same as in [11]). In [9,11], L_0, L_0^* are operators.

2. Preliminary assertions

Let H be a separable Hilbert space with a scalar product $(\cdot,\cdot)$ and a norm $\|\cdot\|$. We consider a function $\Delta \to \mathbf{P}(\Delta)$ defined on Borel sets $\Delta \subset [a,b]$ that takes values in the set of linear bounded operators acting in H. The function $\mathbf{P}$ is called an operator measure on $[a,b]$ (see, for example, [2, Chap. 5]) if it is zero on the empty set and the equality

$$\mathbf{P}\left(\bigcup_{n=1}^{\infty} \Delta_n\right) = \sum_{n=1}^{\infty} \mathbf{P}(\Delta_n)$$

holds for disjoint Borel sets Δ_n, where the series converges weakly. Further, we extend any measure $\mathbf{P}$ on $[a,b]$ to a segment $[a,b_0]$ $(b_0>b)$ letting $\mathbf{P}(\Delta)=0$ for each Borel set $\Delta\subset\ (b,b_0]$.

By $\mathbf{V}_\Delta(\mathbf{P})$, we denote

$$\mathbf{V}_\Delta(\mathbf{P})=\rho_{\mathbf{P}}(\Delta)=\sup\sum_n\|\mathbf{P}(\Delta_n)\|,$$

where the supremum is taken over finite sums of disjoint Borel sets $\Delta_n\subset\Delta$. The number $\mathbf{V}_\Delta(\mathbf{P})$ is called the variation of the measure $\mathbf{P}$ on the Borel set Δ. Suppose that the measure $\mathbf{P}$ has the bounded variation on $[a,b]$. Then for $\rho_{\mathbf{P}}$-almost all $\xi\in[a,b]$ there exists an operator function $\xi\to\Psi_{\mathbf{P}}(\xi)$ such that $\Psi_{\mathbf{P}}$ possesses the values in the set of linear bounded operators acting in H, $\|\Psi_{\mathbf{P}}(\xi)\|=1$, and the equality

$$\mathbf{P}(\Delta)=\int_\Delta\Psi_{\mathbf{P}}(s)\,d\rho_{\mathbf{P}}\tag{2.1}$$

holds for each Borel set $\Delta\subset[a,b]$. The function $\Psi_{\mathbf{P}}$ is uniquely determined up to values on a set of zero $\rho_{\mathbf{P}}$-measure. Integral (2.1) converges in the sense of usual operator norm ([2, Chap. 5]).

Further, $\int_{t_0}^t$ stands for $\int_{[t_0t)}$ if $t_0<t$, for $-\int_{[t,t_0)}$ if $t_0>t$, and for 0 if $t_0=t$. A function h is integrable with respect to the measure $\mathbf{P}$ on a set Δ if there exists the Bochner integral

$$\int_\Delta\Psi_{\mathbf{P}}(t)h(t)\,d\rho_{\mathbf{P}}=\int_\Delta(d\mathbf{P})\,h(t).$$

Then the function

$$(y(t)=\int_{t_0}^t(d\mathbf{P})\,h(s)$$

is continuous from the left.

By $\mathcal{S}_{\mathbf{P}}$, denote a set of single-point atoms of the measure $\mathbf{P}$ (i.e., a set $t\in[a,b]$ such that $\mathbf{P}(\{t\})\neq0$). The set $\mathcal{S}_{\mathbf{P}}$ is at most countable. The measure $\mathbf{P}$ is continuous if $\mathcal{S}_{\mathbf{P}}=\varnothing$, it is self-adjoint if $(\mathbf{P}(\Delta))^*=\mathbf{P}(\Delta)$ for each Borel set $\Delta\subset[a,b]$, it is non-negative if $(\mathbf{P}(\Delta)x,x)\geqslant0$ for all Borel sets $\Delta\subset[a,b]$ and for all elements $x\in H$.

In Lemma 2.1 below, $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{q}$ are operator measures having bounded variations and taking values in the set of linear bounded operators acting in H. Suppose that the measure $\mathbf{q}$ is self-adjoint and assume that these measures are extended on the segment $[a,b_0]\supset[a,b_0)\supset[a,b]$ in the manner described above.

Lemma 2.1 ([9]). *Let f, g be functions integrable on $[a,b_0]$ with respect to the measure $\mathbf{q}$ and $y_0,z_0\in H$. Then the functions*

$$y(t)=y_0-iJ\int_{t_0}^t d\mathbf{p}_1(s)y(s)-iJ\int_{t_0}^t d\mathbf{q}(s)f(s),$$
$$z(t)=z_0-iJ\int_{t_0}^t d\mathbf{p}_2(s)z(s)-iJ\int_{t_0}^t d\mathbf{q}(s)g(s),\quad a\leqslant t_0<b_0,\ t_0\leqslant t\leqslant b_0,$$

satisfy the following formula (analogous to the Lagrange one):

$$\begin{aligned}\int_{c_1}^{c_2} (d\mathbf{q}(t)f(t), z(t)) - \int_{c_1}^{c_2} (y(t), d\mathbf{q}(t)g(t))\\ = (iJy(c_2), z(c_2)) - (iJy(c_1), z(c_1))\\ + \int_{c_1}^{c_2} (y(t), d\mathbf{p}_2(t)z(t)) - \int_{c_1}^{c_2} (d\mathbf{p}_1(t)y(t), z(t))\\ - \sum_{t\in\mathcal{S}_{\mathbf{p}_1}\cap\mathcal{S}_{\mathbf{p}_2}\cap[c_1,c_2)} (iJ\mathbf{p}_1(\{t\})y(t), \mathbf{p}_2(\{t\})z(t))\\ - \sum_{t\in\mathcal{S}_{\mathbf{q}}\cap\mathcal{S}_{\mathbf{p}_2}\cap[c_1,c_2)} (iJ\mathbf{q}(\{t\})f(t), \mathbf{p}_2(\{t\})z(t))\\ - \sum_{t\in\mathcal{S}_{\mathbf{p}_1}\cap\mathcal{S}_{\mathbf{q}}\cap[c_1,c_2)} (iJ\mathbf{p}_1(\{t\})y(t), \mathbf{q}(\{t\})g(t))\\ - \sum_{t\in\mathcal{S}_{\mathbf{q}}\cap[c_1,c_2)} (iJ\mathbf{q}(\{t\})f(t), \mathbf{q}(\{t\})g(t)), \quad t_0 \leqslant c_1 < c_2 \leqslant b_0. \quad (2.2)\end{aligned}$$

Further we will assume that the measures $\mathbf{p}$, $\mathbf{m}$ have bounded variations, $\mathbf{p}$ is self-adjoint and $\mathbf{m}$ is non-negative. We consider the equation

$$y(t) = x_0 - iJ\int_a^t d\mathbf{p}(s)y(s) - iJ\int_a^t d\mathbf{m}(s)\, f(s), \quad (2.3)$$

where $x_0 \in H$, f is integrable with respect to the measure $\mathbf{m}$ on $[a, b]$, $a \leqslant t \leqslant b_0$.

We construct a continuous measure $\mathbf{p}_0$ from the measure $\mathbf{p}$ in the following way. We set $\mathbf{p}_0(\{t_k\}) = 0$ for $t_k \in \mathcal{S}_{\mathbf{p}}$ and we set $\mathbf{p}_0(\Delta) = \mathbf{p}(\Delta)$ for all Borel sets such that $\Delta \cap \mathcal{S}_{\mathbf{p}} = \varnothing$. Similarly, we construct a continuous measure $\mathbf{m}_0$ from the measure $\mathbf{m}$. The measures $\mathbf{p}_0$, $\mathbf{m}_0$ are self-adjoint and the measure $\mathbf{m}_0$ is non-negative. We replace $\mathbf{p}$ by $\mathbf{p}_0$ and $\mathbf{m}$ by $\mathbf{m}_0$ in (2.3). Then we obtain the equation

$$y(t) = x_0 - iJ\int_a^t d\mathbf{p}_0(s)y(s) - iJ\int_a^t d\mathbf{m}_0(s)f(s). \quad (2.4)$$

Equations (2.3) and (2.4) have unique solutions (see [8]).

By W, denote the operator solution of the equation

$$W(t)x_0 = x_0 - iJ\int_a^t d\mathbf{p}_0(s)W(s)x_0, \quad (2.5)$$

where $x_0 \in H$. Using Lemma 2.1, we get

$$W^*(t)JW(t) = J \quad (2.6)$$

by the standard method (see [11]). The functions $t \to W(t)$ and $t \to W^{-1}(t) = JW^*(t)J$ are continuous with respect to the uniform operator topology. Consequently, there exist constants $\varepsilon_1 > 0$, $\varepsilon_2 > 0$ such that the inequality

$$\varepsilon_1 \|x\|^2 \leqslant \|W(t)x\|^2 \leqslant \varepsilon_2 \|x\|^2 \quad (2.7)$$

holds for all $x \in H$, $t \in [a, b_0]$. The following Lemma 2.2 is established in [12] for the case of a continuous measure $\mathbf{m}$.

Lemma 2.2. *The function y is a solution of the equation*

$$y(t) = x_0 - iJ \int_a^t d\mathbf{p}_0(s)y(s)x - iJ \int_a^t d\mathbf{m}(s)f(s), \quad x_0 \in H, \quad a \leqslant t \leqslant b_0, \tag{2.8}$$

if and only if y has the form

$$y(t) = W(t)x_0 - W(t)iJ \int_a^t W^*(\xi)d\mathbf{m}(\xi)f(\xi). \tag{2.9}$$

Proof. Equation (2.8) has a unique solution (see [8]). It is enough to prove that if we substitute the function from the right-hand side of (2.9) instead of y in equation (2.8), then we get the identity. With this substitution, the right-hand side of (2.8) takes the form

$$\begin{aligned} x_0 - iJ \int_a^t d\mathbf{p}_0(s) \Big(W(s)x_0 - W(s)iJ \int_a^s W^*(\xi)d\mathbf{m}(\xi)\, f(\xi) \Big) - iJ \int_a^t d\mathbf{m}(s)f(s) \\ = x_0 - iJ \int_a^t d\mathbf{p}_0(s)\, W(s)x_0 \\ - J \int_a^t d\mathbf{p}_0(s)\, W(s)J \int_a^s W^*(\xi)\, d\mathbf{m}(\xi)\, f(\xi) - iJ \int_a^t d\mathbf{m}(s)\, f(s). \end{aligned} \tag{2.10}$$

We change the limits of integration in the third term of the right-hand side of (2.10). Then the third term takes the form

$$\begin{aligned} J \int_a^t d\mathbf{p}_0(s)\, W(s)J \int_a^s W^*(\xi)d\mathbf{m}(\xi)\, f(\xi) \\ = J \int_{[a,t)} \Big(\int_{(\xi,t)} d\mathbf{p}_0(s)\, W(s) \Big) JW^*(\xi)\, d\mathbf{m}(\xi)\, f(\xi) \\ = J \int_{[a,t)} \Big(\int_{[\xi,t)} d\mathbf{p}_0(s)\, W(s) \Big) JW^*(\xi)\, d\mathbf{m}(\xi)\, f(\xi) \\ - J \int_{[a,t)} \Big(\int_{\{\xi\}} d\mathbf{p}_0(s)\, W(s) \Big) JW^*(\xi)\, d\mathbf{m}(\xi)\, f(\xi). \end{aligned} \tag{2.11}$$

The last term in (2.11) is equal to zero since the measure $\mathbf{p}_0$ is continuous. Using (2.5), we continue equality (2.10):

$$W(t)x_0 - \int_a^t J \Big(\int_\xi^t d\mathbf{p}_0(s)\, W(s) \Big) JW^*(\xi)\, d\mathbf{m}(\xi)\, f(\xi) - iJ \int_a^t d\mathbf{m}(s)\, f(s). \tag{2.12}$$

It follows from (2.5) that (2.12) is equal to

$$W(t)x_0 - \int_a^t i((W(t) - E) - (W(\xi) - E))JW^*(\xi)d\mathbf{m}(\xi)\, f(\xi) - iJ \int_a^t d\mathbf{m}(s)\, f(s)$$

$$= W(t)x_0 - i\int_a^t W(t)JW^*(\xi)\,d\mathbf{m}(\xi)\,f(\xi)$$
$$+ i\int_a^t W(\xi)JW^*(\xi)\,d\mathbf{m}(\xi)\,f(\xi) - iJ\int_a^t d\mathbf{m}(s)\,f(s).$$

Taking into account (2.6), we continue the last equality

$$W(t)x_0 - iW(t)J\int_a^t W^*(\xi)\,d\mathbf{m}(\xi)\,f(\xi)$$
$$+ iJ\int_a^t d\mathbf{m}(\xi)\,f(\xi) - iJ\int_a^t d\mathbf{m}(s)\,f(s) = y(t).$$

The lemma is proved. □

3. Linear relations generated by the integral equation

Let $\mathbf{B}$ be a Hilbert space. A linear relation T is understood as a linear manifold $T \subset \mathbf{B}\times\mathbf{B}$. The terminology of the linear relations can be found, for example, in [1, 13]. In what follows we make use of the following notations: $\{\cdot,\cdot\}$ is an ordered pair, $\mathcal{D}(T)$ is the domain of T, $\mathcal{R}(T)$ is the range of T, $\ker T$ is the set of elements $x \in \mathbf{B}$ such that $\{x, 0\} \in T \subset \mathbf{B}\times\mathbf{B}$. A relation T^* is called adjoint for T if T^* consists of all pairs $\{y_1, y_2\}$ such that the equality $(x_2, y_1) = (x_1, y_2)$ holds for all pairs $\{x_1, x_2\} \in T$. A linear relation T is called dissipative (accumulative, symmetric) if for any $\{x, x'\} \in T$ we have $\operatorname{Im}(x', x) \geqslant 0$ (respectively, $\operatorname{Im}(x', x) \leqslant 0$, or $\operatorname{Im}(x', x) = 0$). A dissipative (accumulative, symmetric) relation T is called maximal dissipative (accumulative, symmetric) if it has no dissipative (accumulative, symmetric) extensions $T_1 \supset T$ such that $T_1 \neq T$. A symmetric relation is called self-adjoint if it is maximal dissipative and maximal accumulative at the same time. As it is known, a relation T is symmetric if and only if $T \subset T^*$ and it is self-adjoint if and only if $T = T^*$. As linear operators are treated as linear relations, the notation $\{x_1, x_2\} \in T$ is also used for the operator T. Since all considered relations are linear, we will often omit the word "linear".

Let $\mathbf{m}$ be a non-negative operator-valued measure defined on Borel sets $\Delta \subset [a, b]$ that takes values in the set of linear bounded operators acting in the space H. The measure $\mathbf{m}$ is assumed to have a bounded variation on $[a, b]$. We introduce the quasi-scalar product

$$(x, y)_{\mathbf{m}} = \int_a^{b_0} ((d\mathbf{m})x(t), y(t))$$

on a set of step-like functions with values in H defined on the segment $[a, b_0]$. Identifying with zero functions y obeying $(y, y)_{\mathbf{m}} = 0$ and making the completion, we arrive at the Hilbert space denoted by $L_2(H, d\mathbf{m}; a, b) = \mathfrak{H}$. The elements of $\mathfrak{H}$ are the classes of functions identified with respect to the norm $\|y\|_{\mathbf{m}} = (y, y)_{\mathbf{m}}^{1/2}$. In order not to complicate the terminology, the class of functions with a representative y is indicated by the same symbol and we write $y \in \mathfrak{H}$. The equalities of the functions in $\mathfrak{H}$ are understood as the equalities for associated equivalence classes.

Let us define a minimal relation L_0 in the following way. The relation L_0 consists of pairs $\{\widetilde{y}, \widetilde{f}\} \in \mathfrak{H} \times \mathfrak{H}$ satisfying the condition: for each pair $\{\widetilde{y}, \widetilde{f}\}$ there exists a pair $\{y, f\}$ such that the pairs $\{\widetilde{y}, \widetilde{f}\}$, $\{y, f\}$ are identical in $\mathfrak{H} \times \mathfrak{H}$ and $\{y, f\}$ satisfies equation (2.3) and the equalities

$$y(a) = y(b_0) = y(\alpha) = 0, \quad \alpha \in \mathcal{S}_{\mathbf{p}}; \qquad \mathbf{m}(\{\beta\})f(\beta) = 0, \quad \beta \in \mathcal{S}_{\mathbf{m}}. \tag{3.1}$$

In general, the relation L_0 is not an operator since the function y may happen to be identified with zero in $\mathfrak{H}$, while f is non-zero. It follows from Lemma 2.1 that the relation L_0 is symmetric. Further, without loss of generality it can be assumed that if a pair $\{y, f\} \in L_0$, then equalities (2.3) and (3.1) hold for this pair.

Lemma 3.1. *Equalities (2.3), (2.4), and (2.8) hold simultaneously for any pair $\{y, f\} \in L_0$.*

Proof. We denote $\overline{\mathbf{p}} = \mathbf{p} - \mathbf{p}_0$, $\overline{\mathbf{m}} = \mathbf{m} - \mathbf{m}_0$. Then $\overline{\mathbf{p}}(\{t_k\}) = \mathbf{p}(\{t_k\})$ for all $t_k \in \mathcal{S}_{\mathbf{p}}$ and $\overline{\mathbf{p}}(\Delta) = 0$ for all Borel sets Δ such that $\Delta \cap \mathcal{S}_{\mathbf{p}} = \varnothing$. Similar equalities hold for the measure $\overline{\mathbf{m}}$. Using (2.3), we get

$$\begin{aligned} y(t) = x_0 - iJ \int_a^t d\mathbf{p}_0(s)\, y(s) - iJ \int_a^t d\overline{\mathbf{p}}(s)\, y(s) \\ - iJ \int_a^t d\overline{\mathbf{m}}(s)\, f(s) - iJ \int_a^t d\mathbf{m}_0(s)\, f(s). \end{aligned}$$

Now the desired statement follows from (3.1). The lemma is proved. □

Corollary 3.2. *If $y \in \mathcal{D}(L_0)$, then y is continuous and $y(b) = 0$.*

Lemma 3.3. *A pair $\{\widetilde{y}, \widetilde{f}\} \in \mathfrak{H} \times \mathfrak{H}$ belongs to the relation L_0 if and only if there exists a pair $\{y, f\}$ such that the pairs $\{\widetilde{y}, \widetilde{f}\}$, $\{y, f\}$ are identical in $\mathfrak{H} \times \mathfrak{H}$ and the equalities*

$$y(t) = -W(t)\, iJ \int_a^t W^*(s)\, d\mathbf{m}_0(s)\, f(s), \tag{3.2}$$

$$y(\alpha) = W(\alpha)\, iJ \int_a^\alpha W^*(s)\, d\mathbf{m}_0(s)\, f(s) = 0, \tag{3.3}$$

$$\mathbf{m}(\{\beta\})f(\beta) = 0 \tag{3.4}$$

hold, where $\alpha \in \mathcal{S}_{\mathbf{p}} \cup \{b_0\}$, $\beta \in \mathcal{S}_{\mathbf{m}}$.

Proof. It follows from Lemmas 2.2 and 3.1 that equalities (3.2)–(3.4) hold together with equalities (2.3) and (3.1). By the definition of the relation L_0, a pair $\{y, f\} \in L_0$ if and only if (2.3) and (3.1) hold. The lemma is proved. □

Lemma 3.4. *The relation L_0 is closed.*

Proof. Suppose $\{y_n, f_n\} \in L_0$. Using (3.2)–(3.4), we obtain

$$y_n(t) = -W(t)\, iJ \int_a^t W^*(s)\, d\mathbf{m}_0(s)\, f_n(s), \tag{3.5}$$

$$y_n(\alpha) = W(\alpha)\, iJ \int_a^\alpha W^*(s)\, d\mathbf{m}_0(s)\, f_n(s) = 0, \quad \mathbf{m}(\{\beta\}) f_n(\beta) = 0, \tag{3.6}$$

where $\alpha \in \mathcal{S}_\mathbf{p} \cup \{b_0\}$, $\beta \in \mathcal{S}_\mathbf{m}$. Suppose that the sequences $\{y_n\}$, $\{f_n\}$ converge in $\mathfrak{H}$ to y, f, respectively. We note that if a sequence converges in $\mathfrak{H} = L_2(H, d\mathbf{m}; a, b)$, then this sequence converges in $L_2(H, d\mathbf{m}_0; a, b)$. Moreover,

$$\|f_n - f\|_\mathfrak{H}^2 \geqslant (\mathbf{m}(\{\beta\})(f_n(\beta) - f(\beta)), f_n(\beta) - f(\beta)) = (\mathbf{m}(\{\beta\}) f(\beta), f(\beta)),$$

where $\beta \in \mathcal{S}_\mathbf{m}$. Passing to the limit as $n \to \infty$ in (3.5) and (3.6), we obtain equalities (3.2)–(3.4). It follows from Lemma 3.3 that the pair $\{y, f\} \in L_0$. The lemma is proved. □

Corollary 3.5. *The function $f \in \mathfrak{H}$ belongs to the range $\mathcal{R}(L_0)$ if and only if f satisfies the conditions*

$$\int_a^\alpha W^*(s)\, d\mathbf{m}_0(s)\, f(s) = 0, \quad \mathbf{m}(\{\beta\}) f(\beta) = 0, \tag{3.7}$$

where $\alpha \in \mathcal{S}_\mathbf{p} \cup \{b_0\}$, $\beta \in \mathcal{S}_\mathbf{m}$.

Remark 3.6. The first equality in (3.7) is equivalent to the following:

$$\int_{\alpha_1}^{\alpha_2} W^*(s)\, d\mathbf{m}_0(s)\, f(s) = 0, \quad \alpha_1, \alpha_2 \in \mathcal{S}_\mathbf{p} \cup \{a\} \cup \{b_0\}. \tag{3.8}$$

Remark 3.7. It follows from Lemma 3.1, Corollary 3.2, and equality (3.4) that we can replace $\mathbf{m}_0$ by $\mathbf{m}$ and b_0 by b in (3.2), (3.3), (3.7), and (3.8).

By $\overline{\mathcal{S}}_\mathbf{p}$, denote the closure of the set $\mathcal{S}_\mathbf{p}$.

Lemma 3.8. *Suppose $\{y, f\} \in L_0$. Then $y(t) = 0$ for all $t \in \overline{\mathcal{S}}_\mathbf{p}$ and $f(t) = 0$ for $\mathbf{m}$-almost all $t \in \overline{\mathcal{S}}_\mathbf{p} \cup \{a, b\}$.*

Proof. It follows from Corollary 3.2 that the functions $y \in \mathcal{D}(L_0)$ are continuous. Taking into account (3.1), we obtain $y(t) = 0$ for $t \in \overline{\mathcal{S}}_\mathbf{p}$. Using Corollary 3.5 and Remark 3.7, we get

$$\int_a^\alpha (d\mathbf{m}_0(s) f(s), W(s)x) = 0, \quad \mathbf{m}(\{\beta\}) f(\beta) = 0$$

for all $x \in H$ and for all $\alpha \in \overline{\mathcal{S}}_\mathbf{p} \cup \{b\}$, $\beta \in \mathcal{S}_\mathbf{m}$. Hence equality (2.1) implies

$$\int_a^\alpha (\Psi_{\mathbf{m}_0}(s) f(s), W(s)x)\, d\rho_{\mathbf{m}_0}(s) = 0, \quad \mathbf{m}(\{\beta\}) f(\beta) = 0. \tag{3.9}$$

We denote

$$\varphi_x(t) = (\Psi_{\mathbf{m}_0}(t)f(t), W(t)x), \quad \Phi_x(t) = \int_a^t \varphi_x(s)\, d\rho_{\mathbf{m}_0}(s).$$

The function Φ_x is continuous. Hence, it follows from (3.9) that $\Phi_x(t) = 0$ for all $t \in \overline{\mathcal{S}}_{\mathbf{p}} \cup \{a,b\}$. Therefore, $\varphi_x(t) = 0$ for $\rho_{\mathbf{m}_0}$-almost all $t \in \overline{\mathcal{S}}_{\mathbf{p}} \cup \{a,b\}$.

Let $\{x_n\}$ be a countable everywhere dense set in H and let $\mathcal{X}_n$ be a set $t \in \overline{\mathcal{S}}_{\mathbf{p}}$ such that $\varphi_{x_n}(t) = 0$. Then $\varrho_{\mathbf{m}_0}(\mathcal{X}_n) = \varrho_{\mathbf{m}_0}(\overline{\mathcal{S}}_{\mathbf{p}})$. We denote $\mathcal{X} = \cap_n \mathcal{X}_n$. Then $\varrho_{\mathbf{m}_0}(\mathcal{X}) = \varrho_{\mathbf{m}_0}(\overline{\mathcal{S}}_{\mathbf{p}})$ and $\varphi_{x_n}(t) = 0$ for all n. If a sequence $\{z_n\}$ converges to z in H, then the sequence $\{W(t)z_n\}$ converges to $W(t)z$ for fixed t. Therefore, $\varphi_x(t) = 0$ for all $x \in H$ and for all $t \in \mathcal{X}$. The operator $W(t)$ has a bounded inverse for all t. It follows that $\Psi_{\mathbf{m}_0}(t)f(t) = 0$ for all $t \in \mathcal{X}$. Consequently, $\Psi_{\mathbf{m}_0}(t)f(t) = 0$ for $\rho_{\mathbf{m}_0}$-almost all $t \in \overline{\mathcal{S}}_{\mathbf{p}} \cup \{a,b\}$. It follows from (2.1) that

$$\int_a^b (d\mathbf{m}_0(t)f(t), f(t)) = \int_a^b (\Psi_{\mathbf{m}_0}(t)f(t), f(t))\, d\rho_{\mathbf{m}_0}(t) = 0.$$

Hence, using (3.4), we obtain $f(t) = 0$ for $\mathbf{m}$-almost all $t \in \overline{\mathcal{S}}_{\mathbf{p}} \cup \{a,b\}$. The lemma is proved. □

By $\mathfrak{H}_0$ (by $\mathfrak{H}_1$), denote a subspace of functions that vanish on $(a,b) \setminus \overline{\mathcal{S}}_{\mathbf{p}}$ (on $\overline{\mathcal{S}}_{\mathbf{p}} \cup \{a,b\}$, respectively) with respect to the norm in $\mathfrak{H}$. The subspaces $\mathfrak{H}_0$, $\mathfrak{H}_1$ are orthogonal and $\mathfrak{H} = \mathfrak{H}_0 \oplus \mathfrak{H}_1$. We note that $\mathfrak{H}_0 = \{0\}$ if and only if $\mathbf{m}(\overline{\mathcal{S}}_{\mathbf{p}} \cup \{a,b\}) = 0$.

We denote $L_{10} = L_0 \cap (\mathfrak{H}_1 \times \mathfrak{H}_1)$. Then $\mathcal{D}(L_{10}) \subset \mathfrak{H}_1$, $\mathcal{R}(L_{10}) \subset \mathfrak{H}_1$. It follows from Lemma 3.8 that

$$L_0^* = (\mathfrak{H}_0 \times \mathfrak{H}_0) \oplus L_{10}^*, \tag{3.10}$$

i.e., the relation L_0^* consists of all pairs $\{y, f\} \in \mathfrak{H}$ of the form

$$\{y, f\} = \{u, v\} + \{z, g\} = \{u + z, v + g\},$$

where $u, v \in \mathfrak{H}_0$, $\{z, g\} \in L_{10}^*$.

The set $\mathcal{T}_{\mathbf{p}} = (a,b) \setminus \overline{\mathcal{S}}_{\mathbf{p}}$ is open and it is the union of at most a countable number of disjoint open intervals, i.e., $\mathcal{T}_{\mathbf{p}} = \bigcup_{k=1}^{\Bbbk_1} \mathcal{J}_k$, $\mathcal{J}_k \cap \mathcal{J}_j = \varnothing$ for $k \neq j$, where $\Bbbk_1$ is a natural number (equal to the number of intervals if this number is finite) or the symbol ∞ (if the number of intervals is infinite). By $\mathbb{J}$, denote the set of these intervals $\mathcal{J}_k$. Note that the boundaries α_k, β_k of any interval $\mathcal{J}_k = (\alpha_k, \beta_k) \in \mathbb{J}$ belong to $\overline{\mathcal{S}}_{\mathbf{p}} \cup \{a,b\}$.

Further, let χ_A denote the characteristic function of a set A. We denote

$$w_k(t) = \chi_{[\alpha_k, \beta_k)} W(t) W^{-1}(\alpha_k), \tag{3.11}$$

where $(\alpha_k, \beta_k) = \mathcal{J}_k \in \mathbb{J}$. Using (2.6), we get

$$w_k^*(t) J w_k(t) = J, \quad \alpha_k \leqslant t < \beta_k. \tag{3.12}$$

Lemma 3.9. *Let $g \in \mathfrak{H}$ and let the function G_k be given by the equality*

$$G_k(t) = -w_k(t)\, iJ \int_{\alpha_k}^{t} w_k^*(s)\, d\mathbf{m}(s)\, g(s),$$

where $(\alpha_k, \beta_k) = \mathcal{J}_k \in \mathbb{J}$. Then the pair $\{G_k, g\} \in L_{10}^*$ if g vanishes outside $[\alpha_k, \beta_k)$.

Proof. Equalities (2.6) and (3.11) imply

$$G_k(t) = -\chi_{[\alpha_k,\beta_k)} W(t)\, iJ \int_{\alpha_k}^{t} W^*(s)\, d\mathbf{m}(s)\, g(s).$$

It follows from Lemma 2.2 that the function G_k is a solution of equation (2.8) on the segment $[\alpha_k, \gamma]$, $\gamma < \beta_k$ (for $a = \alpha_k$, $y = G_k$, $f = g$, $x_0 = 0$).

Suppose a pair $\{y, f\} \in L_0$. According to Lemma 3.1, the pair $\{y, f\}$ satisfies equation (2.8) for $x_0 = 0$. Therefore we can apply formula (2.2) to the functions y, f, G_k, g for $c_1 = \alpha_k$, $c_2 = \gamma$, $\mathbf{q} = \mathbf{m}$, $\mathbf{p}_1 = \mathbf{p}_2 = \mathbf{p}_0$. Since the measure $\mathbf{p}_0$ is continuous, self-adjoint and (3.4) holds, we obtain

$$\int_{\alpha_k}^{\gamma} (g(s), d\mathbf{m}(s)\, y(s)) = \int_{\alpha_k}^{\gamma} (G_k(s), d\mathbf{m}(s)\, f(s)) + (iJG_k(\gamma), y(\gamma)). \tag{3.13}$$

The function y is continuous from the left and $y(\beta_k) = 0$. Hence, passing to the limit as $\gamma \to \beta_k - 0$ in (3.13), we obtain

$$\int_{\alpha_k}^{\beta_k} (g(s), d\mathbf{m}(s)\, y(s)) = \int_{\alpha_k}^{\beta_k} (G_k(s), d\mathbf{m}(s)\, f(s)).$$

This implies the desired statement. The lemma is proved. □

Let $\mathbb{M}$ be a set consisting of intervals $\mathcal{J} \in \mathbb{J}$ and single-point sets $\{\tau\}$, where $\tau \in \mathcal{S}_\mathbf{m} \setminus \overline{\mathcal{S}}_\mathbf{p}$. The set $\mathbb{M}$ is at most countable. We arrange the elements of $\mathbb{M}$ in the form of a finite or infinite sequence and denote these elements by $\mathcal{E}_k$, where k is any natural number if the number of elements in $\mathbb{M}$ is infinite, and $1 \leqslant k \leqslant \Bbbk$ if the number of elements in $\mathbb{M}$ is finite and equal to $\Bbbk$.

We will assign an operator function v_k to each element $\mathcal{E}_k \in \mathbb{M}$ in the following way. If $\mathcal{E}_k$ is the interval, $\mathcal{E}_k = \mathcal{J}_k = (\alpha_k, \beta_k) \in \mathbb{J}$, then

$$v_k(t) = \chi_{[\alpha_k,\beta_k)\setminus\mathcal{S}_\mathbf{m}} w_k(t). \tag{3.14}$$

If $\mathcal{E}_k$ is a single-point set, $\mathcal{E}_k = \{\tau_k\}$, $\tau_k \in \mathcal{S}_\mathbf{m} \setminus (\overline{\mathcal{S}}_\mathbf{p} \cup \{a, b\})$, and $\tau_k \in \mathcal{J}_n = (\alpha_n, \beta_n)$, then

$$v_k(t) = \chi_{\{\tau_k\}} w_n(\tau_k). \tag{3.15}$$

It follows from the definition of functions v_k that for each element $x_1, x_2 \in H$ the functions $v_k(\cdot)x_1$, $v_j(\cdot)x_2$ are orthogonal in $\mathfrak{H}$ for $k \neq j$. Moreover, $v_k(\cdot)x \in \mathfrak{H}_1$ for all $x \in H$ and for all k.

Lemma 3.10. *The linear span of functions $t \to v_k(t)\xi$, $\xi \in H$, is dense in* $\ker L_{10}^*$. *Here $k \in \mathbb{N}$ if $\Bbbk = \infty$, and $1 \leqslant k \leqslant \Bbbk$ if $\Bbbk$ is finite.*

Proof. It follows from Corollary 3.5, Remark 3.7, and (3.10) that the range $\mathcal{R}(L_{10})$ consists of all functions $f \in \mathfrak{H}$ orthogonal to functions of the form $v_k(\cdot)\xi$, where $\xi \in H$. The equality $\ker(L_{10}^*) \oplus \mathcal{R}(L_{10}) = \mathfrak{H}_1$ implies the desired assertion. The lemma is proved. □

Let $Q_{k,0}$ be a set $x \in H$ such that the functions $t \to v_k(t)x$ are identical with zero in $\mathfrak{H}$. We put $Q_k = H \ominus Q_{k,0}$. On the linear space Q_k, we introduce a norm $\|\cdot\|_-$ by the equality

$$\|\xi_k\|_- = \|v_k(\cdot)\xi_k\|_{\mathfrak{H}}\,, \quad \xi_k \in Q_k. \tag{3.16}$$

We note that if v_k has the form (3.14), then

$$\|\xi_k\|_- = \left(\int_{[\alpha_k,\beta_k)\setminus\mathcal{S}_{\mathbf{m}}} (d\mathbf{m}(s)\, w_k(s)\xi_k, w_k(s)\xi_k)\right)^{1/2}, \quad \xi_k \in Q_k.$$

If v_k has the form (3.15), then

$$\|\xi_k\|_- = (\mathbf{m}(\{\tau_k\})w_n(\tau_k)\xi_k, w_n(\tau_k)\xi_k)^{1/2}, \quad \xi_k \in Q_k.$$

By Q_k^-, denote the completion of Q_k with respect to the norm (3.16). The norm (3.16) is generated by the scalar product $(\xi_k, \eta_k)_- = (v_k(\cdot)\xi_k, v_k(\cdot)\eta_k)_{\mathfrak{H}}$, where $\xi_k, \eta_k \in Q_k$. From the formula (2.1), in which the measure $\mathbf{P}$ is replaced by $\mathbf{m}$, it follows that

$$\|\xi_k\|_- \leqslant \gamma\,\|\xi_k\|\,, \quad \xi_k \in Q_k, \tag{3.17}$$

where $\gamma > 0$ is independent of $\xi_k \in Q_k$.

It follows from (3.17) that the space Q_k^- can be treated as a space with a negative norm with respect to Q_k [2, Chap. 1] and [13, Chap. 2]. By Q_k^+, we denote the associated space with a positive norm. The definition of spaces with positive and negative norms implies that $Q_k^+ \subset Q_k$. By $(\cdot,\cdot)_+$ and $\|\cdot\|_+$, we denote the scalar product and the norm in Q_k^+, respectively.

Suppose that a sequence $\{x_{kn}\}$, $x_{kn} \in Q_k$, converges in the space Q_k^- to $x_0 \in Q_k^-$ as $n \to \infty$. Then a sequence $\{v_k(\cdot)x_{kn}\}$ is fundamental in $\mathfrak{H}$. Therefore this sequence converges to some element $x_0 \in \mathfrak{H}$. We denote this element by $v_k(\cdot)x_0$.

Let $\widetilde{Q}_n^- = Q_1^- \times \cdots \times Q_n^-$ $(\widetilde{Q}_n^+ = Q_1^+ \times \cdots \times Q_n^+)$ be the Cartesian product of the first n sets Q_k^- (Q_k^+, respectively) and let $V_n(t) = (v_1(t), \ldots, v_n(t))$ be the operator one-row matrix. It is convenient to treat elements from $\widetilde{Q}_n^-$ as one-column matrices, and to assume that $V_n(t)\widetilde{\xi}_n = \sum_{k=1}^n v_k(t)\xi_k$, where we denote $\widetilde{\xi}_n = \operatorname{col}(\xi_1, \ldots, \xi_n) \in \widetilde{Q}_n^-$, $\xi_k \in Q_k^-$.

Let $\ker_k$ be a linear space of functions $t \to v_k(t)\xi_k$, $\xi_k \in Q_k^-$. By (3.16), it follows that $\ker_k$ is closed in $\mathfrak{H}$. The spaces $\ker_k$ and $\ker_j$ are orthogonal for $k \neq j$. We denote $\mathcal{K}_n = \ker_1 \oplus \cdots \oplus \ker_n$. Obviously, $\mathcal{K}_n \subset \mathcal{K}_m$ for $n < m$.

Lemma 3.11. *The set $\bigcup_n \mathcal{K}_n$ is dense in* $\ker L_{10}^*$.

Proof. The required statement follows immediately from Lemma 3.10. □

By $\mathcal{V}_n$, denote the operator $\widetilde{\xi}_n \to V_n(\cdot)\widetilde{\xi}_n$ ($\widetilde{\xi}_n \in \widetilde{Q}_n^-$). The operator $\mathcal{V}_n$ maps continuously and one-to-one $\widetilde{Q}_n^-$ onto $\mathcal{K}_n \subset \mathfrak{H}_1 \subset \mathfrak{H}$. Hence the adjoint operator $\mathcal{V}_n^*$ maps $\mathfrak{H}$ onto $\widetilde{Q}_n^+$ continuously. We find the form of the operator $\mathcal{V}_n^*$. For all $\widetilde{\xi}_n \in \widetilde{Q}_n = Q_1 \times ...Q_n$, $f \in \mathfrak{H}$, we have

$$(f, \mathcal{V}_n \widetilde{\xi}_n)_{\mathfrak{H}} = \int_a^{b_0} (d\mathbf{m}(s)\, f(s), V_n(s)\widetilde{\xi}_n) = \int_a^{b_0} (V_n^*(s)\, d\mathbf{m}(s) f(s), \widetilde{\xi}_n) = (\mathcal{V}_n^* f, \widetilde{\xi}_n).$$

Since $\widetilde{Q}_n$ is dense in $\widetilde{Q}_n^-$, we obtain

$$\mathcal{V}_n^* f = \int_a^{b_0} V_n^*(s)\, d\mathbf{m}(s)\, f(s). \tag{3.18}$$

So we proved the following statement:

Lemma 3.12. *The operator $\mathcal{V}_n$ maps continuously and one-to-one $\widetilde{Q}_n^-$ onto $\mathcal{K}_n$. The adjoint operator $\mathcal{V}_n^*$ maps continuously $\mathfrak{H}$ onto $\widetilde{Q}_n^+$ and acts by the formula* (3.18)*. Moreover, $\mathcal{V}_n^*$ maps one-to-one $\mathcal{K}_n$ onto $\widetilde{Q}_n^+$.*

Let $\mathcal{Q}_-$, $\mathcal{Q}_+$, $\mathcal{Q}$ be linear spaces of sequences $\widetilde{\eta} = \{\eta_k\}$, $\widetilde{\varphi} = \{\varphi_k\}$, $\widetilde{\xi} = \{\xi_k\}$, respectively, such that the series

$$\sum_{k=1}^{\Bbbk} \|\eta_k\|_-^2, \quad \sum_{k=1}^{\Bbbk} \|\varphi_k\|_+^2, \quad \sum_{k=1}^{\Bbbk} \|\xi_k\|^2$$

converge if $\Bbbk = \infty$, where $\eta_k \in Q_k^-$, $\varphi_k \in Q_k^+$, $\xi_k \in Q_k$. These spaces become Hilbert spaces if we introduce the scalar products by the formulas

$$(\widetilde{\eta}, \widetilde{\sigma})_- = \sum_{k=1}^{\Bbbk} (\eta_k, \sigma_k)_-, \qquad \widetilde{\eta}, \widetilde{\sigma} \in \mathcal{Q}_-,$$

$$(\widetilde{\varphi}, \widetilde{\psi})_+ = \sum_{k=1}^{\Bbbk} (\varphi_k, \psi_k)_+, \qquad \widetilde{\varphi}, \widetilde{\psi} \in \mathcal{Q}_+,$$

$$(\widetilde{\xi}, \widetilde{\zeta}) = \sum_{k=1}^{\Bbbk} (\xi_k, \zeta_k), \qquad \widetilde{\xi}, \widetilde{\zeta} \in \mathcal{Q}.$$

In these spaces, the norms are defined by the equalities

$$\|\widetilde{\eta}\|_-^2 = \sum_{k=1}^{\Bbbk} \|\eta_k\|_-^2, \quad \|\widetilde{\varphi}\|_+^2 = \sum_{k=1}^{\Bbbk} \|\varphi_k\|_+^2, \quad \left\|\widetilde{\xi}\right\|^2 = \sum_{k=1}^{\Bbbk} \|\xi_k\|^2.$$

The spaces $\mathcal{Q}_+$, $\mathcal{Q}_-$ can be treated as spaces with positive and negative norms with respect to $\mathcal{Q}$ (see [2, Chap. 1] and [13, Chap. 2]). So, $\mathcal{Q}_+ \subset \mathcal{Q} \subset \mathcal{Q}_-$ and $\varepsilon_1 \|\widetilde{\varphi}\|_- \leqslant \|\widetilde{\varphi}\| \leqslant \varepsilon_2 \|\widetilde{\varphi}\|_+$, where $\widetilde{\varphi} \in \mathcal{Q}_+$, $\varepsilon_1, \varepsilon_2 > 0$. The "scalar product" $(\widetilde{\eta}, \widetilde{\varphi})$

is defined for all $\widetilde{\varphi} \in \mathcal{Q}_+$, $\widetilde{\eta} \in \mathcal{Q}_-$. If $\widetilde{\eta} \in \mathcal{Q}$, then $(\widetilde{\eta}, \widetilde{\varphi})$ coincides with the scalar product in $\mathcal{Q}$.

Let $\mathcal{M} \subset \mathcal{Q}_-$ be a set of sequences that vanish starting from a certain number (its own for each sequence). The set $\mathcal{M}$ is dense in the space $\mathcal{Q}_-$. The operator $\mathcal{V}_n$ is the restriction of $\mathcal{V}_{n+1}$ to $\widetilde{Q}_n^-$. By $\mathcal{V}'$, denote an operator in $\mathcal{M}$ such that $\mathcal{V}'\widetilde{\eta} = \mathcal{V}_n\widetilde{\eta}_n$ for all $n \in \mathbb{N}$, where $\widetilde{\eta} = (\widetilde{\eta}_n, 0, \ldots)$, $\widetilde{\eta}_n \in \widetilde{Q}_n^-$. It follows from (3.16) that $\mathcal{V}'$ admits an extension by continuity to the space $\mathcal{Q}_-$. By $\mathcal{V}$, denote the extended operator. This operator maps continuously and one-to-one $\mathcal{Q}_-$ onto $\ker(L_{10}^*) \subset \mathfrak{H}_1 \subset \mathfrak{H}$. Moreover, we denote $\widetilde{V}(t)\widetilde{\eta} = (\mathcal{V}\widetilde{\eta})(t)$, where $\widetilde{\eta} = \{\eta_k\} \in \mathcal{Q}_-$. Using (3.16), we get

$$(\mathcal{V}\widetilde{\eta}, \mathcal{V}\widetilde{\sigma})_{\mathfrak{H}} = (\widetilde{\eta}, \widetilde{\sigma})_-, \quad \widetilde{\eta} = \{\eta_k\},\ \widetilde{\sigma} = \{\sigma_k\}, \quad \widetilde{\eta}, \widetilde{\sigma} \in \mathcal{Q}_-. \tag{3.19}$$

The adjoint operator $\mathcal{V}^*$ maps continuously $\mathfrak{H}$ onto $\mathcal{Q}_+$. Let us find the form of $\mathcal{V}^*$. Suppose $f \in \mathfrak{H}$, $\widetilde{\eta} \in \mathcal{M}$, $\widetilde{\eta} = \{\widetilde{\eta}_n, 0, \ldots\}$. Then

$$(\widetilde{\eta}, \mathcal{V}^* f) = (\mathcal{V}\widetilde{\eta}, f)_{\mathfrak{H}} = \int_a^{b_0} (d\mathbf{m}(t)\, \widetilde{V}(t)\widetilde{\eta}, f(t)) = \int_a^{b_0} (\widetilde{\eta}, \widetilde{V}^*(t)\, d\mathbf{m}(t) f(t)).$$

Since $\mathcal{V}^* f \in \mathcal{Q}_+$ and the set $\mathcal{M}$ is dense in $\mathcal{Q}_-$, we get

$$\mathcal{V}^* f = \int_a^{b_0} \widetilde{V}^*(t)\, d\mathbf{m}(t) f(t). \tag{3.20}$$

Taking into account Lemmas 3.11 and 3.12, we obtain the following statement.

Lemma 3.13. *The operator $\mathcal{V}$ maps $\mathcal{Q}_-$ onto $\ker(L_{10}^*)$ continuously and one-to-one. A function z belongs to $\ker(L_{10}^*)$ if and only if there exists an element $\widetilde{\eta} = \{\eta_k\} \in \mathcal{Q}_-$ such that $z(t) = (\mathcal{V}\widetilde{\eta})(t) = \widetilde{V}(t)\widetilde{\eta}$. The operator $\mathcal{V}^*$ maps $\mathfrak{H}$ onto $\mathcal{Q}_+$ continuously and acts by the formula (3.20), and $\ker \mathcal{V}^* = \mathfrak{H}_0 \oplus \mathcal{R}(L_{10})$. Moreover, $\mathcal{V}^*$ maps $\ker(L_{10}^*)$ onto $\mathcal{Q}_+$ one-to-one.*

Theorem 3.14. *A pair $\{\widetilde{y}, \widetilde{f}\} \in \mathfrak{H} \times \mathfrak{H}$ belongs to L_0^* if and only if there exists a pair $\{y, f\}$, the functions $y_0, y_0' \in \mathfrak{H}_0$, $\widehat{y}, \widehat{f} \in \mathfrak{H}_1$ and an element $\widetilde{\eta} \in \mathcal{Q}_-$ such that the pairs $\{\widetilde{y}, \widetilde{f}\}$, $\{y, f\}$ are identical in $\mathfrak{H} \times \mathfrak{H}$ and the equalities*

$$y = y_0 + \widehat{y},\ \ f = y_0' + \widehat{f},\ \ \widehat{y}(t) = \widetilde{V}(t)\widetilde{\eta} - \sum_{k=1}^{\Bbbk_1} w_k(t) iJ \int_a^t w_k^*(s)\, d\mathbf{m}(s)\widehat{f}(s) \tag{3.21}$$

hold, where the series in (3.21) converges in $\mathfrak{H}$, $\Bbbk_1$ is the number of intervals $\mathcal{J}_k \in \mathbb{J}$.

Proof. The first two equalities in (3.21) follow from (3.10). Let us prove that the last equality in (3.21) holds. First we prove that if the functions $\widehat{y}$, $\widehat{f}$ satisfy the third equality in (3.21), then the pair $\{\widehat{y}, \widehat{f}\} \in L_{10}^*$. If $\Bbbk_1$ is finite, then this statement follows from Lemmas 3.9 and 3.13. We assume that $\Bbbk_1 = \infty$.

It follows from Lemma 3.13 that $\mathcal{V}\widetilde{\eta} \in \ker(L_{10}^*)$. The function

$$\widehat{y}_k(t) = -w_k(t)\, iJ \int_a^t w_k^*(s)\, d\mathbf{m}(s)\, \widehat{f}(s)$$

$$= -w_k(t)\, iJ \int_{\alpha_k}^{t} w_k^*(s)\Psi_{\mathbf{m}}(s)\widehat{f}(s)\, d\rho_{\mathbf{m}}(s) \tag{3.22}$$

vanishes outside the interval $[\alpha_k, \beta_k)$. (Here $\Psi_{\mathbf{m}}$, $\rho_{\mathbf{m}}$ are the functions from (2.1) in which the measure $\mathbf{P}$ is replaced by $\mathbf{m}$.) We denote $\widehat{f}_k(t) = \chi_{[\alpha_k,\beta_k)}\widehat{f}(t)$. Using (2.1), (2.7), and (3.22), we get

$$\|\widehat{y}_k(t)\| \leqslant \varepsilon_1 \|w_k(t)\| \int_{\alpha_k}^{\beta_k} \|w_k^*(s)\| \left\|\Psi_{\mathbf{m}}^{1/2}(s)\widehat{f}_k(s)\right\| d\rho_{\mathbf{m}}(s)$$

$$\leqslant \varepsilon \left(\int_{\alpha_k}^{\beta_k} \left\|\Psi_{\mathbf{m}}^{1/2}(s)\widehat{f}_k(s)\right\|^2 d\rho_{\mathbf{m}}(s)\right)^{1/2} = \varepsilon \left\|\widehat{f}_k\right\|_{\mathfrak{H}}, \quad \varepsilon_1, \varepsilon > 0.$$

$$\|\widehat{y}_k\|_{\mathfrak{H}}^2 = \int_{\alpha_k}^{\beta_k} (\Psi_{\mathbf{m}}(t)\widehat{y}_k(t), \widehat{y}_k(t))\, d\rho_{\mathbf{m}}(t) \leqslant \varepsilon^2 \rho_{\mathbf{m}}([\alpha_k, \beta_k)) \left\|\widehat{f}_k\right\|_{\mathfrak{H}}^2. \tag{3.23}$$

We denote

$$S_n(t) = \sum_{k=1}^{n} \widehat{y}_k(t)$$

and prove that the sequence $\{S_n\}$ converges in $\mathfrak{H}$. From (3.23), we get

$$\|S_n\|_{\mathfrak{H}}^2 = \sum_{k=1}^{n} \|\widehat{y}_k\|_{\mathfrak{H}}^2 \leqslant \varepsilon^2 \sum_{k=1}^{n} \rho_{\mathbf{m}}([\alpha_k, \beta_k)) \left\|\widehat{f}_k\right\|_{\mathfrak{H}}^2 \leqslant \varepsilon^2 \rho_{\mathbf{m}}([a, b]) \left\|\widehat{f}\right\|_{\mathfrak{H}}^2.$$

Consequently, the sequence $\{S_n\}$ converges to some function $S \in \mathfrak{H}$ and

$$S(t) = -\sum_{k=1}^{\infty} w_k(t) iJ \int_a^t w_k^*(s)\, d\mathbf{m}(s)\widehat{f}(s), \quad \|S\|_{\mathfrak{H}} \leqslant \varepsilon_2 \left\|\widehat{f}\right\|_{\mathfrak{H}}, \quad \varepsilon_2 > 0. \tag{3.24}$$

It follows from Lemma 3.9 that

$$\left\{S_n, \sum_{k=1}^{n} \widehat{f}_k\right\} \in L_{10}^*.$$

The relation L_{10}^* is closed. Therefore, $\{S, \widehat{f}\} \in L_{10}^*$ and $\{\widehat{y}, \widehat{f}\} \in L_{10}^*$.

Now we assume that a pair $\{\widehat{y}, \widehat{f}\} \in L_{10}^*$. For the function $\widehat{f}$, we find a function S by the formula (3.24). Then $\{S, \widehat{f}\} \in L_{10}^*$. Hence $\widehat{y} - S \in \ker L_{10}^*$. By Lemma 3.13, it follows that there exists an element $\widetilde{\eta} \in \mathcal{Q}_-$ such that $\widehat{y} - S = \mathcal{V}\widetilde{\eta}$. Therefore, $\widehat{y}$ has the form (3.21). Now (3.10) implies the desired assertion. The theorem is proved. □

4. The description of dissipative extensions of L_0

By $\mathcal{L}_0$ (by $\mathcal{L}_0^{\perp}$), denote the closure in $\mathfrak{H}$ of the linear span of functions $t \to v_k(t)\eta_k$, where $\eta_k \in Q_k^-$ and v_k has the form (3.15) (form (3.14), respectively). The spaces $\mathcal{L}_0$ and $\mathcal{L}_0^{\perp}$ are orthogonal. Using Lemmas 3.10 and 3.13, we obtain $\mathcal{L}_0 \oplus \mathcal{L}_0^{\perp} = \ker L_{10}^*$. We put $\mathfrak{Q}_- = \mathcal{V}^{-1}\mathcal{L}_0$, $\mathfrak{Q}_-^{\perp} = \mathcal{V}^{-1}\mathcal{L}_0^{\perp}$. By (3.19), it follows that the spaces $\mathfrak{Q}_-$, $\mathfrak{Q}_-^\perp$ are orthogonal in $\mathcal{Q}_-$ and $\mathcal{Q}_- = \mathfrak{Q}_- \oplus \mathfrak{Q}_-^\perp$. We denote $\mathcal{V}_0 = \mathcal{V}P$, $\mathcal{V}_0^\perp = \mathcal{V}(E-P)$, where P is the orthogonal projection onto $\mathfrak{Q}_-$ in $\mathcal{Q}_-$.

It follows from Lemma 3.13 that $\mathcal{V}^* f$ ($f \in \mathfrak{H}$) is an element of the space $\mathcal{Q}_+ \subset \mathcal{Q}$, i.e., a sequence with elements of the form

$$w_n^*(\tau_k)\mathbf{m}(\{\tau_k\})f(\tau_k), \qquad \int_a^{b_0} \chi_{[\alpha_k,\beta_k)\setminus\mathcal{S}_\mathbf{m}} w_k^*(t)\, d\mathbf{m}(t)\, f(t) \tag{4.1}$$

(and possibly with zeros), where $\tau_k \in (\mathcal{S}_\mathbf{m} \setminus \overline{\mathcal{S}}_p) \cap \mathcal{J}_n$; $(\alpha_k,\beta_k) = \mathcal{J}_k$; $\mathcal{J}_n, \mathcal{J}_k \in \mathbb{J}$. The element $\mathcal{V}_0^* f$ is a sequence with elements of the first form in (4.1) (and possibly with zeros), and $(\mathcal{V}_0^\perp)^* f$ is a sequence with elements of the second form in (4.1) (and possibly with zeros). Therefore,

$$(\mathcal{V}^* f, \mathcal{V}_0^* g) = (\mathcal{V}_0^* f, \mathcal{V}_0^* g), \quad f,g \in \mathfrak{H}. \tag{4.2}$$

Using (3.12), we obtain

$$\begin{aligned}(iJw_n^*(\tau_k)\mathbf{m}(\{\tau_k\})f(\tau_k), w_n^*(\tau_k)\mathbf{m}(\{\tau_k\})g(\tau_k))\\ = (iJ\mathbf{m}(\{\tau_k\})f(\tau_k), \mathbf{m}(\{\tau_k\})g(\tau_k)), \quad f,g \in \mathfrak{H}.\end{aligned} \tag{4.3}$$

We denote $\mathbf{H}_- = \mathfrak{H}_0 \times \mathcal{Q}_-$, $\mathbf{H}_+ = \mathfrak{H}_0 \times \mathcal{Q}_+$. Suppose a pair $\{\widetilde{y}, \widetilde{f}\} \in L_0^*$. By Theorem 3.14, there exists a pair $\{y, f\}$ such that the pairs $\{\widetilde{y}, \widetilde{f}\}$, $\{y, f\}$ are identical in $\mathfrak{H} \times \mathfrak{H}$ and the equalities

$$y = y_0 + \widehat{y}, \quad f = y_0' + \widehat{f}, \quad \{\widehat{y}, \widehat{f}\} \in L_{10}^* \tag{4.4}$$

hold, where $y_0, y_0' \in \mathfrak{H}_0$ and $\widehat{y}$ has the form (3.21). With each pair $\{y, f\}$ we associate a pair of boundary values $\{Y, Y'\} \in \mathbf{H}_- \times \mathbf{H}_+$ by the formulas

$$Y = \{y_0, Y_{10}\} \in \mathbf{H}_- = \mathfrak{H}_0 \times \mathcal{Q}_-, \quad Y' = \{y_0', Y_{10}'\} \in \mathbf{H}_+ = \mathfrak{H}_0 \times \mathcal{Q}_+, \tag{4.5}$$

where

$$Y_{10} = \widetilde{\eta} - 2^{-1} i\widetilde{J}\mathcal{V}^*\widehat{f} + 2^{-1} i\widetilde{J}\mathcal{V}_0^*\widehat{f}, \quad Y_{10}' = \mathcal{V}^*\widehat{f}, \tag{4.6}$$

$\widetilde{J}$ is the operator in $\mathcal{Q}$ acting as $\widetilde{J}\widetilde{\xi} = \{J\xi_k\}$, $\widetilde{\xi} = \{\xi_k\} \in \mathcal{Q}$.

Let Γ denote the operator that takes each pair $\{y, f\} \in L_0^*$ to the ordered pair $\{Y, Y'\}$ of boundary values Y, Y', i.e., $\Gamma\{y, f\} = \{Y, Y'\}$. We put $\Gamma_1\{y, f\} = Y$, $\Gamma_2\{y, f\} = Y'$. It follows from Lemma 3.13 that if pairs $\{\widetilde{y}, \widetilde{f}\}$, $\{y, f\}$ are identical in $\mathfrak{H} \times \mathfrak{H}$, then their boundary values coincide.

Theorem 4.1. *The range $\mathcal{R}(\Gamma)$ of the operator Γ coincides with $\mathbf{H}_- \times \mathbf{H}_+$ and "the Green formula"*

$$(f, z)_\mathfrak{H} - (y, g)_\mathfrak{H} = (Y', Z) - (Y, Z') \tag{4.7}$$

holds, where $\{y, f\}, \{z, g\} \in L_0^*$, $\Gamma\{y, f\} = \{Y, Y'\}$, $\Gamma\{z, g\} = \{Z, Z'\}$.

Proof. The equality $\mathcal{R}(\Gamma) = \mathbf{H}_- \times \mathbf{H}_+$ follows from Lemma 3.13 and the formulas (3.10), (4.5), (4.6). Let us prove (4.7). Suppose that a pair $\{y, f\}$ has the form (3.21) and a pair $\{z, g\}$ has the form

$$z = z_0 + \widehat{z}, \quad g = z_0' + \widehat{g}, \quad \{\widehat{z}, \widehat{g}\} \in L_{10}^*,$$

where $z_0, z_0' \in \mathfrak{H}_0$,

$$\widehat{z}(t) = \widetilde{V}(t)\widetilde{\zeta} - \sum_{k=1}^{\Bbbk_1} w_k(t) iJ \int_a^t w_k^*(s)\, d\mathbf{m}(s)\widehat{g}(s), \quad \widetilde{\zeta} \in \mathcal{Q}_-, \ \widehat{g} \in \mathfrak{H}_1. \tag{4.8}$$

Then

$$(f, z)_{\mathfrak{H}} - (y, g)_{\mathfrak{H}} = (y_0', z_0)_{\mathfrak{H}} - (y_0, z_0')_{\mathfrak{H}} + (\widehat{f}, \widehat{z})_{\mathfrak{H}} - (\widehat{y}, \widehat{g})_{\mathfrak{H}}.$$

Thus, it is enough to prove the equality

$$(\widehat{f}, \widehat{z})_{\mathfrak{H}} - (\widehat{y}, \widehat{g})_{\mathfrak{H}} = (Y_{10}', Z_{10}) - (Y_{10}, Z_{10}'). \tag{4.9}$$

Using (4.6), we get

$$(\widehat{f}, \mathcal{V}\widetilde{\zeta})_{\mathfrak{H}} = (\mathcal{V}^*\widehat{f}, \widetilde{\zeta}) = (\mathcal{V}^*\widehat{f}, Z_{10} + 2^{-1} i\widetilde{J}\mathcal{V}^*\widehat{g} - 2^{-1} i\widetilde{J}\mathcal{V}_0^*\widehat{g}), \tag{4.10}$$

$$(\mathcal{V}\widetilde{\eta}, \widehat{g})_{\mathfrak{H}} = (\widetilde{\eta}, \mathcal{V}^* g) = (Y_{10} + 2^{-1} i\widetilde{J}\mathcal{V}^*\widehat{f} - 2^{-1} i\widetilde{J}\mathcal{V}_0^*\widehat{f}, \mathcal{V}^*\widehat{g}). \tag{4.11}$$

In (3.21) and (4.8), we denote

$$\widetilde{F}(t) = -\sum_{k=1}^{\Bbbk_1} w_k(t)\, iJ \int_a^t w_k^*(s)\, d\mathbf{m}(s)\, \widehat{f}(s),$$

$$\widetilde{G}(t) = -\sum_{k=1}^{\Bbbk_1} w_k(t)\, iJ \int_a^t w_k^*(s)\, d\mathbf{m}(s)\, \widehat{g}(s).$$

We define the functions F_k, G_k by the equalities

$$F_k(t) = -w_k(t)\, iJ \int_{\alpha_k}^t w_k^*(s)\, d\mathbf{m}(s)\, \widehat{f}(s), \ G_k(t) = -w_k(t)\, iJ \int_{\alpha_k}^t w_k^*(s)\, d\mathbf{m}(s)\, \widehat{g}(s).$$

It follows from Lemma 2.2 that the functions F_k, G_k are the solutions of equation (2.8) on $[\alpha_k, \beta_k)$ for $x_0 = 0$ (G_k is the solution if f is replaced by g in (2.8)). Using (3.12) and Lemma 2.1, for $\mathbf{p}_1 = \mathbf{p}_2 = \mathbf{p}_0$, $\mathbf{q} = \mathbf{m}$, $c_1 = \alpha_k$, $c_2 = \beta < \beta_k$, we obtain

$$\begin{aligned}
&\int_{\alpha_k}^{\beta} (\widehat{f}(s), d\mathbf{m}(s)\, G_k(s)) - \int_{\alpha_k}^{\beta} (F_k(s), d\mathbf{m}(s)\widehat{g}(s)) \\
&\quad = \left(iJ w_k(\beta)\, iJ \int_{\alpha_k}^{\beta} w_k^*(s)\, d\mathbf{m}(s)\, \widehat{f}(s), w_k(\beta)\, iJ \int_{\alpha_k}^{\beta} w_k^*(s)\, d\mathbf{m}(s)\, \widehat{g}(s) \right) \\
&\qquad - \sum_{\tau \in \mathcal{S}_{\mathbf{m}} \cap [\alpha_k, \beta)} (iJ\mathbf{m}(\{\tau\})\widehat{f}(\tau), \mathbf{m}(\{\tau\})\widehat{g}(\tau))
\end{aligned}$$

$$= \left(iJ \int_{\alpha_k}^{\beta} w_k^*(s)\, d\mathbf{m}(s)\, \widehat{f}(s), \int_{\alpha_k}^{\beta} w_k^*(s)\, d\mathbf{m}(s)\, \widehat{g}(s) \right) - \sum_{\tau \in \mathcal{S}_{\mathbf{m}} \cap [\alpha_k, \beta)} (iJ\mathbf{m}(\{\tau\})\widehat{f}(\tau), \mathbf{m}(\{\tau\})\widehat{g}(\tau)). \tag{4.12}$$

Passing to the limit as $\beta \to \beta_k - 0$ in (4.12), we obtain that (4.12) will remain true if β is replaced by β_k. Therefore,

$$\int_{\alpha_k}^{\beta_k} (\widehat{f}(s), d\mathbf{m}(s)\, G_k(s)) - \int_{\alpha_k}^{\beta_k} (F_k(s), d\mathbf{m}(s)\, \widehat{g}(s)) = \left(iJ \int_{\alpha_k}^{\beta_k} w_k^*(s)\, d\mathbf{m}(s)\, \widehat{f}(s), \int_{\alpha_k}^{\beta_k} w_k^*(s)\, d\mathbf{m}(s)\, \widehat{g}(s) \right) - \sum_{\tau \in \mathcal{S}_{\mathbf{m}} \cap [\alpha_k, \beta_k)} (iJ\mathbf{m}(\{\tau\})\widehat{f}(\tau), \mathbf{m}(\{\tau\})\widehat{g}(\tau)).$$

Taking into account (3.20), (4.1), and (4.3), we get

$$(\widehat{f}, \widetilde{G})_{\mathfrak{H}} - (\widetilde{F}, \widehat{g})_{\mathfrak{H}} = (i\widetilde{J}\mathcal{V}^*\widehat{f}, \mathcal{V}^*\widehat{g}) - (i\widetilde{J}\mathcal{V}_0^*\widehat{f}, \mathcal{V}_0^*\widehat{g}).$$

Then equalities (4.10) and (4.11) imply

$$\begin{aligned}(\widehat{f}, \widehat{z})_{\mathfrak{H}} - (\widehat{y}, \widehat{g})_{\mathfrak{H}} = {} & (\mathcal{V}^*\widehat{f}, Z_{10}) - 2^{-1}(i\widetilde{J}\mathcal{V}^*\widehat{f}, \mathcal{V}^*\widehat{g}) + 2^{-1}(i\widetilde{J}\mathcal{V}^*\widehat{f}, \mathcal{V}_0^*\widehat{g}) \\ & - (Y_{10}, \mathcal{V}^*\widehat{g}) - 2^{-1}(i\widetilde{J}\mathcal{V}^*\widehat{f}, \mathcal{V}^*\widehat{g}) + 2^{-1}(i\widetilde{J}\mathcal{V}_0^*\widehat{f}, \mathcal{V}^*\widehat{g}) \\ & + (i\widetilde{J}\mathcal{V}^*\widehat{f}, \mathcal{V}^*\widehat{g}) - (i\widetilde{J}\mathcal{V}_0^*\widehat{f}, \mathcal{V}_0^*\widehat{g}).\end{aligned}$$

Now, using (4.2) and (4.6), we obtain (4.9). The theorem is proved. □

From the theory of spaces with positive and negative norms (see [2, Chap. 1] and [13, Chap. 2]), it follows that there exist isometric operators $\delta_- : \mathcal{Q}_- \to \mathcal{Q}$, $\delta_+ : \mathcal{Q}_+ \to \mathcal{Q}$ such that the equality $(\widetilde{\eta}, \widetilde{\varphi}) = (\delta_-\widetilde{\eta}, \delta_+\widetilde{\varphi})$ holds for all $\widetilde{\eta} \in \mathcal{Q}_-$, $\widetilde{\varphi} \in \mathcal{Q}_+$. We denote $\mathcal{H} = \mathfrak{H}_0 \times \mathcal{Q}$. Suppose $\{\widetilde{y}, \widetilde{f}\} \in L_0^*$. According to Theorem 3.14, there exists a pair $\{y, f\}$ such that the pairs $\{\widetilde{y}, \widetilde{f}\}$, $\{y, f\}$ are identical in $\mathfrak{H} \times \mathfrak{H}$ and equalities (4.4) hold. To each pair $\{y, f\}$ assign a pair of boundary values $\gamma\{y, f\} = \{\mathcal{Y}, \mathcal{Y}'\} \in \mathcal{H} \times \mathcal{H}$ by the formulas

$$\mathcal{Y} = \gamma_1\{y, f\} = \{y_0, \delta_- Y_{10}\}, \quad \mathcal{Y}' = \gamma_2\{y, f\} = \{y_0', \delta_+ Y_{10}'\}.$$

By Theorem 4.1, it follows that the operator γ maps L_0^* onto $\mathcal{H} \times \mathcal{H}$ and the equality

$$(f, z)_{\mathfrak{H}} - (y, g)_{\mathfrak{H}} = (\mathcal{Y}', \mathcal{Z}) - (\mathcal{Y}, \mathcal{Z}') \tag{4.13}$$

holds, where $\{y, f\}, \{z, g\} \in L_0^*$, $\gamma\{y, f\} = \{\mathcal{Y}, \mathcal{Y}'\}$, $\gamma\{z, g\} = \{\mathcal{Z}, \mathcal{Z}'\}$. This implies that the ordered triple $(\mathcal{H}, \gamma_1, \gamma_2)$ is a space of boundary values (a boundary triplet in another terminology) for L_0 in the sense of papers [4, 5, 16] (see also [13, Chap. 3]).

Let θ be a linear relation, $\theta \subset \mathcal{H} \times \mathcal{H}$. By L_θ, denote a linear relation such that $L_0 \subset L_\theta \subset L_0^*$ and $\gamma L_\theta = \theta$. It follows from (4.13) that both relations L_θ

and θ are maximal dissipative (or maximal accumulative, or maximal symmetric, or self-adjoint). From here, taking into account the description of self-adjoint relations (see [20]), of dissipative relations (see [14]), we obtain the following assertion.

Theorem 4.2. *If U is a contraction on $\mathfrak{H}$, then the restriction of the relation L_0^* to the set of pairs $\{y, f\} \in L_0^*$ satisfying the condition*

$$(U - E)\Gamma_2 f + (U + E)\Gamma_1 f = 0 \tag{4.14}$$

or

$$(U - E)\Gamma_2 f - (U + E)\Gamma_1 f = 0 \tag{4.15}$$

is a maximal dissipative, respectively, maximal accumulative extension of L_0. Conversely, any maximal dissipative (maximal accumulative) extension of L_0 is the restriction of L_0^* to the set of pairs $\{y, f\} \in L_0^*$ satisfying (4.14) (or (4.15)), where a contraction U is uniquely determined by an extension. The maximal symmetric extensions of the relation L_0 on $\mathfrak{H}$ are described by the conditions (4.14), (or (4.15)), where U is an isometric operator. These conditions define a self-adjoint extension if U is unitary.

Let us consider some examples.

Example 4.3. Suppose $\mathbf{p} = \mathbf{p}_0$ is a continuous measure, $\mathbf{m} = \mu$ is the usual Lebesgue measure on $[a, b]$ (i.e., $\mu([\alpha, \beta)) = \beta - \alpha$, where $a \leqslant \alpha < \beta \leqslant b$ (we write ds instead of $d\mu(s)$)). In this case, L_0, L_0^* are operators, $\Bbbk_1 = \Bbbk = 1$, $\mathfrak{H}_0 = \{0\}$, $Q_{1,0} = \{0\}$, $Q_1 = H = \mathcal{Q}_- = \mathcal{Q}_+$. Equality (3.21) has the form

$$y(t) = W(t)\eta - W(t)\, iJ \int_a^t W^*(s) f(s)\, ds, \quad f = L_0^* y, \quad \eta \in H.$$

By direct calculations, we obtain

$$Y = 2^{-1}(y(a) + W^{-1}(b)y(b)); \;\; Y' = iJ(W^{-1}(b)y(b) - y(a)). \tag{4.16}$$

Now we assume that the measures $\mathbf{p}$, $\mathbf{m}$ are continuous. Generally, then L_0, L_0^* are not operators. In this case, $\Bbbk_1 = \Bbbk = 1$, $\mathfrak{H}_0 = \{0\}$. In general, $Q_1 \neq H$, $Q_1 \neq Q_1^-$. If a pair $\{y, f\} \in L_0^*$ is such that $y(a) \in Q_1$, then equalities (4.16) hold.

Suppose that $\mathbf{m} = \mu$ and the set $\mathcal{S}_\mathbf{p}$ of single-point atoms of the measure $\mathbf{p}$ can be arranged as an increasing sequence converging to b. For this case the space of boundary values was constructed in [11].

Example 4.4. Suppose that $\mathcal{S}_\mathbf{m} \neq \varnothing$ and $\mathbf{m} = \mu + \overline{\mathbf{m}}$, where $\mu = \mathbf{m}_0$ is the usual Lebesgue measure on $[a, b]$ and $\mu(\Delta) = \mathbf{m}(\Delta)$ for all Borel sets such that $\Delta \cap \mathcal{S}_\mathbf{m} = \varnothing$. So, $\mathcal{S}_\mathbf{m} = \mathcal{S}_{\overline{\mathbf{m}}}$ and $\mathbf{m}(\{\beta\}) = \overline{\mathbf{m}}(\{\beta\})$ for all $\beta \in \mathcal{S}_\mathbf{m}$. We denote $\widehat{Q}_{k,0} = \ker \mathbf{m}(\{\tau_k\})$, $\widehat{Q}_k = H \ominus \widehat{Q}_{k,0}$, where $\tau_k \in \mathcal{S}_\mathbf{m}$. Let $\mathbf{m}_k$ be the restriction of the operator $\mathbf{m}(\{\tau_k\})$ to $\widehat{Q}_k$. The operator $\mathbf{m}_k$ is self-adjoint and $\mathcal{R}(\mathbf{m}_k) \subset \widehat{Q}_k$. By $\widehat{Q}_k^-$, denote the completion of $\widehat{Q}_k$ with respect to the norm $\|\xi\|_- = (\mathbf{m}_k\xi, \xi)^{1/2}$, where $\xi \in \widehat{Q}_k$. Let $\widehat{\mathcal{Q}}_-$ be the linear space of sequences $\widetilde{\eta} = \{\eta_k\}$ such that the series $\sum_{k=1}^{\infty} \|\eta_k\|_-^2$ converges if $\Bbbk_2 = \infty$, where $\Bbbk_2$ is the number of elements in $\mathcal{S}_{\mathbf{m}}$. Then $\mathfrak{H} = L_2(H; a, b) \oplus \widehat{\mathcal{Q}}_-$.

Suppose $\mathbf{p} = 0$ and $a, b \notin \mathcal{S}_{\mathbf{m}}$. (The case of an arbitrary continuous measure $\mathbf{p}$ can be considered in a similar way.) If $\mathbf{p} = 0$, then $\mathfrak{H}_0 = \{0\}$, $W(t) = E$, and $\mathcal{Q}_- = H \oplus \widehat{\mathcal{Q}}_-$. It follows from Lemma 3.3 and (3.1) that a pair $\{y, f\} \in L_0$ if and only if

$$y(t) = -iJ \int_a^t f(s)\, ds, \qquad y(b) = 0, \qquad \mathbf{m}(\beta) f(\beta) = 0, \quad \beta \in \mathcal{S}_{\mathbf{m}}.$$

Using Theorem 3.14, we obtain that a pair $\{y, f\} \in L_0^*$ if and only if

$$y(t) = \eta_0 + \sum_{\tau_k \leqslant t} \chi_{\{\tau_k\}}\eta_k - iJ \int_a^t d\mathbf{m}(s)\, f(s), \tag{4.17}$$

where $\eta_0 \in H$, $\tau_k \in \mathcal{S}_{\mathbf{m}}$, $\eta_k \in \widehat{Q}_k^-$, and the sequence $\widetilde{\eta} = \{\eta_0, \eta_k\}$ belongs to $\mathcal{Q}_-$ (here $k \in \mathbb{N}$ if $\Bbbk_2 = \infty$, and $1 \leqslant k \leqslant \Bbbk_2$ if $\Bbbk_2$ is finite).

It follows from (4.5), (4.6), and (4.1) that the boundary values Y, Y' are the sequence of the form

$$Y = \left\{ \eta_0 - 2^{-1} iJ \int_a^b f(s)\, ds,\ \eta_k \right\},$$
$$Y' = \left\{ \int_a^b f(s)\, ds,\ \mathbf{m}(\{\tau_k\}) f(\tau_k) \right\}, \qquad k = 1, 2, \ldots$$

Suppose that the set $\mathcal{S}_{\mathbf{m}}$ of single-point atoms τ_k of measure $\mathbf{m}$ can be arranged as an increasing sequence; $\tau_1 < \tau_2 < \ldots$. In this case, we find η_0, η_k. Using (4.17), we get

$$y(t) = \eta_0 + \sum_{\tau_k \leqslant t} \chi_{\{\tau_k\}}\eta_k - iJ \int_a^t f(s) ds - iJ \sum_{\tau_k < t} \mathbf{m}(\{\tau_k\}) f(\tau_k). \tag{4.18}$$

From (4.18), by direct calculations we obtain

$$\eta_0 = y(a),$$
$$\eta_1 = y(\tau_1) - y(a) + iJ \int_a^{\tau_1} f(s)\, ds,$$
$$\eta_k = y(\tau_k) - y(\tau_{k-1}) + iJ \int_{\tau_{k-1}}^{\tau_k} f(s) ds + iJ\mathbf{m}(\{\tau_{k-1}\}) f(\tau_{k-1}).$$

Thus, the boundary values Y, Y' are expressed through the values of the functions y, f and the integrals of f.

Acknowledgment. The author thanks the reviewer for useful comments.

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Received October 26, 2019, revised December 10, 2019.

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Дисипативні розширення лінійних відношень, породжених інтегральними рівняннями з операторними мірами

Vladislav M. Bruk

У статті визначено мінімальне відношення L_0, яке породжене інтегральним рівнянням з операторними мірами, і надано опис спряженого відношення L_0^*. Для цього мінімального відношення побудовано простір граничних значень (гранична трійка), що задовольняє абстрактну “формулу Гріна”, і одержано опис максимального дисипативного (акумулятивного) відношення, а також самоспряжених розширень мінімального відношення.

Ключові слова: гільбертів простір, лінійне відношення, інтегральне рівняння, дисипативне розширення, самоспряжене розширення, граничне значення, операторна міра