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Ricci Solitons and Certain Related Metrics on Almost Co-Kaehler Manifolds

Devaraja Mallesha Naik, V. Venkatesha, and H. Aruna Kumara

In the paper, we study a Ricci soliton and a generalized m-quasi-Einstein metric on almost co-Kaehler manifold M satisfying a nullity condition. First, we consider a non-co-Kaehler (κ, μ)-almost co-Kaehler metric as a Ricci soliton and prove that the soliton is expanding with $\lambda = -2n\kappa$ and the soliton vector field X leaves the structure tensors η, ξ and φ invariant. This result extends Theorem 5.1 of [32]. We construct an example to show the existence of a Ricci soliton on M. Finally, we prove that if M is a generalized (κ, μ)-almost co-Kaehler manifold of dimension higher than 3 such that $h \neq 0$, then the metric of M can not be a generalized m-quasi-Einstein metric, and this recovers the recent result of Wang [37, Theorem 4.1] as a special case.

Key words: almost co-Kaehler manifold, Ricci soliton, generalized m-quasi-Einstein metric, (κ, μ)-nullity distribution

Mathematical Subject Classification 2010: 53C25, 53C15, 53D15

1. Introduction

In 1967, Blair [3] introduced *co-Kaehler manifolds* which are odd dimensional analogues of Kaehler manifolds. This class of manifolds set up one of the three classes of almost contact metric manifolds whose automorphism group attains the maximum dimension (see [33]), and the remaining two classes are Sasakian and Kenmotsu manifolds. Co-Kaehler manifolds were extensively studied by Blair [4], Goldberg and Yano [21], Olszak [28] and many others. In all these papers, they call co-Kaehler manifolds as *cosymplectic manifolds*. This new terminology appeared due to Li in [25], in which the author gave a topology construction of co-Kaehler manifolds via Kaehler mapping tori. According to Li's work, we see that co-Kaehler manifolds are really the odd dimensional analogues of Kaehler manifolds. For more details on topological and geometric properties of co-Kaehler manifolds, see [6].

Recently, many authors have widely studied the generalization of co-Kaehler manifolds called *almost co-Kaehler manifolds*. The products of almost Kaehler manifolds and the real line $\mathbb{R}$ or the circle S^1 are the simplest examples of almost co-Kaehler manifolds. Almost co-Kaehler manifolds, for which the characteristic vector field ξ belongs to the κ-nullity distribution, were first studied by Dacko in [13], and Endo generalized them to (κ, μ)-almost co-Kaehler manifolds in [15].

We arrange this paper as follows: Section 2 consists of basic definitions and notions regarding almost co-Kaehler manifolds. Section 3 is devoted to the study of Ricci solitons on non-co-Kaehler (κ, μ)-almost co-Kaehler manifolds M, and we prove that the soliton is expanding with $\lambda = -2n\kappa$ and the soliton vector field X leaves the structure tensors η, ξ and φ invariant. The existence of the Ricci soliton on M is given at the end of this section with an example. The last section deals with the generalized m-quasi-Einstein metric on generalized (κ, μ)-almost co-Kaehler manifolds M with $h \neq 0$ and we show the non-existence of such metric on M whose dimension is higher than 3. This generalizes Theorem 4.1 of Wang [37].

2. Preliminaries

In this section, we recall the basic definitions and formulas of almost co-Kaehler manifolds and we recommend [6] for more details about it.

A Riemannian manifold M of dimension $2n+1$ is said to have an *almost contact metric structure* if there exists a vector field ξ, a 1-form η, a field of endomorphism φ and a Riemannian metric g satisfying

$$\varphi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \tag{2.1}$$

$$g(\varphi Y, \varphi Z) = g(Y, Z) - \eta(Y)\eta(Z) \tag{2.2}$$

for all $Y, Z \in \mathfrak{X}(M)$. A Riemannian manifold M with (φ, ξ, η, g) structure is called an *almost contact metric manifold*.

An *almost co-Kaehler manifold* is an almost contact metric manifold satisfying $d\eta = d\Phi = 0$, where Φ is the *fundamental 2-form* defined by $\Phi(Y, Z) = g(Y, \varphi Z)$. It is well known that a normal almost co-Kaehler manifold is a *co-Kaehler manifold*.

On an almost co-Kaehler manifold, the tensor fields $h := (1/2)\pounds_\xi \varphi$ and $h' := h \circ \varphi$ are symmetric and satisfy

$$trh = trh' = 0, \quad h\xi = h'\xi = 0, \quad h\varphi + \varphi h = 0, \quad h^2 = h'^2, \tag{2.3}$$

where tr denotes the trace. For an almost co-Kaehler manifold, we also have the following relation (see [15, 28]):

$$\nabla_Y \xi = h'Y, \tag{2.4}$$

$$\nabla_\xi \varphi = 0, \quad \operatorname{div} \xi = 0, \tag{2.5}$$

$$S(\xi, \xi) = -\|h\|^2, \tag{2.6}$$

where S is the Ricci tensor.

Given two real numbers κ and μ, the (κ, μ)-*nullity distribution* (denoted by $N(\kappa, \mu)$) is defined by

$$N_p(\kappa, \mu) = \{W \in T_pM : R(Y, Z)W = \kappa[g(Z, W)Y - g(Y, W)Z]$$

$$+ \mu[g(Z,W)hY - g(Y,W)hZ]\},$$

for any $Y, Z, W \in \mathfrak{X}(M)$. An almost co-Kaehler manifold with $\xi \in N(\kappa, \mu)$, that is,

$$R(Y,Z)\xi = \kappa\{\eta(Z)Y - \eta(Y)Z\} + \mu\{\eta(Z)hY - \eta(Y)hZ\}, \tag{2.7}$$

is called a (κ, μ)-*almost co-Kaehler manifold* (see [15]), and in this case we have

$$h^2 = \kappa\varphi^2. \tag{2.8}$$

It follows from (2.8) that $\kappa \leq 0$. Also, we have $\kappa = 0$ if and only if M is co-Kaehler (see [6]). Also if $h \neq 0$, then $\kappa < 0$. Denoting the Ricci operator by Q, we have

$$QY = \mu hY + 2n\kappa\eta(Y)\xi \tag{2.9}$$

for all $Y \in \mathfrak{X}(M)$. From (2.9), it is easy to obtain

$$S(Y,\xi) = 2n\kappa\eta(Y) \quad (\Rightarrow Q\xi = 2n\kappa\xi), \tag{2.10}$$

$$S(\xi,\xi) = 2n\kappa. \tag{2.11}$$

If $\mu = 0$, then we call the (κ, μ)-almost co-Kaehler manifold as an $N(\kappa)$-*almost co-Kaehler manifold* (see [13]). If κ and μ are smooth functions satisfying the relation (2.7), then we call it as a *generalized* (κ, μ)-*almost co-Kaehler manifold*, and in this case we have the following identities (see [29]):

$$(\nabla_\xi h)(Y) = \mu h'Y, \tag{2.12}$$

$$(\nabla_Y \varphi)Z = g(hY,Z)\xi - \eta(Z)hY, \tag{2.13}$$

$$(\nabla_Y h')Z - (\nabla_Z h')Y = \kappa\{\eta(Z)Y - \eta(Y)Z\} + \mu\{\eta(Z)hY - \eta(Y)hZ\}, \tag{2.14}$$

$$\begin{aligned}(\nabla_Y h)Z - (\nabla_Z h)Y &= \kappa\{2g(\varphi Y,Z)\xi - \eta(Y)\varphi Z + \eta(Z)\varphi Y\} \\ &\quad + \mu\{\eta(Y)h'Z - \eta(Z)h'Y\}\end{aligned} \tag{2.15}$$

for all $Y, Z \in \mathfrak{X}(M)$. Note that the relations (2.12)–(2.15) hold true even for (κ, μ)-almost co-Kaehler manifolds.

3. Ricci solitons on (κ, μ)-almost co-Kaehler manifolds

Let (M, g) be a Riemannian manifold. The Riemannian metric g is said to be a *Ricci soliton* if there exists a vector field $X \in \mathfrak{X}(M)$ and a scalar λ such that

$$\pounds_X g + 2S + 2\lambda g = 0, \tag{3.1}$$

where $\pounds$ denotes the usual Lie derivative. Thus, the Ricci soliton is a generalization of an Einstein metric (that is, $S = ag$ for some constant a). Given a Riemannian manifold (M, g_0), Hamilton's Ricci flow (see [22]) is the one which satisfies $\frac{\partial}{\partial t} g(t) = -2S(t)$ with the initial condition $g = g_0$ at $t = 0$. The Ricci soliton is a special solution to the Ricci flow equation, which is equivalent to the existence of scalars $\sigma(t)$ and diffeomorphisms ψ_t of M such that $g(t) = \sigma(t)\psi_t^* g_0$. We say that the Ricci soliton is *steady* when $\lambda = 0$, *expanding* when $\lambda > 0$

and *shrinking* when $\lambda < 0$. In the framework of contact geometry Ricci solitons were first considered by Sharma in [31]. They were also studied by Ghosh and Sharma [20], Calin and Crasmareanu [5], Ghosh [16], Turan et al. [34], Crasmareanu [12], Cho [11], Ghosh [17], Bejan and Crasmareanu [2], Wang and Liu [40], Naik and Venkatesha [26, 27], Venkatesha et al. [35, 36] and others. In the context of almost co-Kaehler manifolds, Ricci solitons were studied by Wang [38, 39], Chen [10], Suh and De [32] and many others.

We need the following lemma.

Lemma 3.1. *Let M be an $N(\kappa)$-almost co-Kaehler manifold such that $h \neq 0$. If X is a vector field on M satisfying*

$$\pounds_X g = c\{g - \eta \otimes \eta\}, \tag{3.2}$$

where c is a constant, then X is a Jacobi field along the geodesics of ξ and X leaves η, ξ and φ invariant.

Proof. First, we differentiate (3.2) along Y and make use of (2.4) to obtain

$$(\nabla_Y \pounds_X g)(Z, W) = -c\{g(h'Y, Z)\eta(W) + \eta(Z)g(h'Y, W)\}. \tag{3.3}$$

From Yano [41], we use the formula

$$\begin{aligned}(\pounds_X \nabla_Y g - \nabla_Y \pounds_X g - \nabla_{[X,Y]} g)(Z, W) = \\ - g((\pounds_X \nabla)(Y, Z), W) - g((\pounds_X \nabla)(Y, W), Z)\end{aligned}$$

to obtain

$$(\nabla_Y \pounds_X g)(Z, W) = g((\pounds_X \nabla)(Y, Z), W) + g((\pounds_X \nabla)(Y, W), Z). \tag{3.4}$$

Using the symmetric property of $\pounds_X \nabla$, from (3.4) we derive

$$\begin{aligned} g((\pounds_X \nabla)(Y, Z), W) \\ = \frac{1}{2}(\nabla_Y \pounds_X g)(Z, W) + \frac{1}{2}(\nabla_Z \pounds_X g)(W, Y) - \frac{1}{2}(\nabla_W \pounds_X g)(Y, Z). \end{aligned} \tag{3.5}$$

Next, we feed the expression (3.3) into (3.5) to find

$$(\pounds_X \nabla)(Y, Z) = -2cg(h'Y, Z)\xi. \tag{3.6}$$

Now, we plug $Y = Z = \xi$ in the above relation and recall $h'\xi = 0$ to deduce

$$(\pounds_X \nabla)(\xi, \xi) = 0. \tag{3.7}$$

Then we take $Y = Z = \xi$ in the following identity (see [14]):

$$(\pounds_X \nabla)(Y, Z) = \nabla_Y \nabla_Z X - \nabla_{\nabla_Y Z} X + R(X, Y)Z$$

and use (3.7) and $\nabla_\xi \xi = 0$ to obtain

$$0 = \nabla_\xi \nabla_\xi X + R(X, \xi)\xi,$$

which means that X is a Jacobi field along the geodesics of ξ. Next, we differentiate (3.6) and then make use of (2.4) to infer

$$(\nabla_Y \pounds_X \nabla)(Z, W) = -2c\{g((\nabla_Y h')Z, W)\xi + g(h'Z, W)h'Y\}.$$

Next, we use this in the following equation

$$(\pounds_X R)(Y, Z)W = (\nabla_Y \pounds_X \nabla)(Z, W) - (\nabla_Z \pounds_X \nabla)(Y, W), \tag{3.8}$$

and take the help of (2.14) to deduce

$$\begin{aligned}(\pounds_X R)(Y, Z)W = & -2c\kappa\{\eta(Z)g(Y, W)\xi - \eta(Y)g(Z, W)\xi\} \\ & + 2c\{g(h'Z, W)h'Y - g(h'Y, W)h'Z\}.\end{aligned}$$

We further contract it with respect to Y and employ (2.8) to claim

$$\pounds_X S = 0. \tag{3.9}$$

We Lie-differentiate $S(\xi, \xi) = 2n\kappa$ along X and make use of (3.9), (2.10) and $\kappa < 0$ to find

$$\eta(\pounds_X \xi) = 0. \tag{3.10}$$

On the other hand, the Lie-differentiation of (2.10) gives $S(Y, \pounds_X \xi) = 2n\kappa g(Y, \pounds_X \xi)$ which is equivalent to

$$Q\pounds_X \xi = 2n\kappa \pounds_X \xi.$$

Since $QY = 2n\kappa\eta(Y)\xi$, from (3.10) we have $Q\pounds_X \xi = 0$. As $\kappa < 0$, the above equation yields

$$\pounds_X \xi = 0. \tag{3.11}$$

Now, $\pounds_X \eta = 0$ follows directly by Lie-differentiating $\eta(Y) = g(Y, \xi)$ and further using (3.11) and (3.2). Making use of $\pounds_X \eta = 0$ and (3.11) in the Lie-derivative of $\varphi^2 Y = -Y + \eta(Y)\xi$ gives

$$(\pounds_X \varphi)\varphi Y + \varphi(\pounds_X \varphi)Y = 0 \tag{3.12}$$

for any $Y \in \mathfrak{X}(M)$. Replacing Y by φY in (3.12) gives one equation and operating (3.12) by φ gives another equation. Subtracting the resulting two equations and keeping in mind $\varphi^3 = -\varphi$ shows that

$$\pounds_X \varphi = 0.$$

This completes the proof. □

Now we are prepared to prove the following fruitful result:

Theorem 3.2. *If a non-co-Kaehler (κ, μ)-almost co-Kaehler metric is a Ricci soliton, then the soliton is expanding with the soliton constant $\lambda = -2n\kappa$, the soliton vector field X is a Jacobi field along the geodesics of ξ and X leaves the structure tensors η, ξ and φ invariant.*

Proof. First, we differentiate the Ricci soliton equation (3.1) along Y to get

$$(\nabla_Y \pounds_X g)(Z, W) = -2(\nabla_Y S)(Z, W), \tag{3.13}$$

which is combined with (3.5) giving

$$g((\pounds_X \nabla)(Y, Z), W) = (\nabla_W S)(Y, Z) - (\nabla_Y S)(Z, W) - (\nabla_Z S)(Y, W). \tag{3.14}$$

Note that (2.9) is equivalent to

$$S(Y, Z) = \mu g(hY, Z) + 2n\kappa\eta(Y)\eta(Z) \tag{3.15}$$

for all $Y, Z \in \mathfrak{X}(M)$. Now, we differentiate this along W and then make use of (3.14) to produce

$$\begin{aligned} g((\pounds_X \nabla)(Y, Z), W) = \mu g((\nabla_W h)Y - (\nabla_Y h)W, Z) - \mu g((\nabla_Z h)Y, W) \\ - 2n\kappa\{g(h'Y, Z)\eta(W) + h(h'Z, Y)\eta(W)\}. \end{aligned} \tag{3.16}$$

Now, we feed (2.15) into (3.16) to show

$$\begin{aligned} (\pounds_X \nabla)(Y, Z) = -\mu\kappa\{g(\varphi Y, Z)\xi + \eta(Y)\varphi Z + 2\eta(Z)\varphi Y\} \\ - \mu^2\{\eta(Y)h'Z - g(h'Y, Z)\xi\} - \mu(\nabla_Z h)Y \\ - 2n\kappa\{g(h'Y, Z)\xi + g(h'Z, Y)\xi\}. \end{aligned}$$

We set $Z = \xi$ in the aforementioned equation and make use of (2.12) to obtain

$$(\pounds_X \nabla)(Y, \xi) = -2\mu\kappa\varphi Y - \mu^2 h'Y, \tag{3.17}$$

and we further differentiate it along Z to deduce

$$\begin{aligned} (\nabla_Z \pounds_X \nabla)(Y, \xi) + (\pounds_X \nabla)(Y, h'Z) \\ = -2\mu\kappa\{g(hZ, Y)\xi - \eta(Y)hZ\} - \mu^2(\nabla_Z h')Y, \end{aligned}$$

where we used (2.13). Next, we use the aforesaid equation in (3.8) to obtain

$$\begin{aligned} (\pounds_X R)(Y, Z)\xi = -2\mu\kappa\{\eta(Y)hZ - \eta(Z)hY\} - \mu^2\{(\nabla_Y h')Z - (\nabla_Z h')Y\} \\ - (\pounds_X \nabla)(Z, h'Y) + (\pounds_X \nabla)(Y, h'Z). \end{aligned}$$

Plugging $Z = \xi$ in the preceding equation and with the aid of (2.14), (2.3) and (3.17), we arrive at

$$(\pounds_X R)(Y, \xi)\xi = 4\mu\kappa hY - 2\kappa\mu^2 Y + 2\mu^2\kappa\eta(Y)\xi - \mu^3 hY. \tag{3.18}$$

Contracting (3.18) over Y, and since $trh = 0$, leads to

$$(\pounds_X S)(\xi, \xi) = -4n\kappa\mu^2. \tag{3.19}$$

Now, applying the Lie-derivative to $S(\xi, \xi) = 2n\kappa$ and recalling that $S(\cdot, \xi) = 2n\kappa\eta(\cdot)$, we have

$$(\pounds_X S)(\xi, \xi) = -4n\kappa\eta(\pounds_X \xi). \tag{3.20}$$

On the other hand, (3.1) gives

$$\eta(\pounds_X\xi) = \lambda + 2n\kappa. \tag{3.21}$$

Due to (3.21), equation (3.20) takes the form

$$(\pounds_X S)(\xi,\xi) = -4n\kappa(\lambda + 2n\kappa). \tag{3.22}$$

Now, we compare (3.22) with (3.19) to deduce

$$\mu^2 = \lambda + 2n\kappa. \tag{3.23}$$

Now, we plug $Z = \xi$ in (2.7) to find $R(Y,\xi)\xi = \kappa(Y - \eta(Y)\xi) + \mu hY$. Taking the Lie-derivative of this along X gives

$$\begin{aligned}(\pounds_X R)(Y,\xi)\xi + R(Y,\pounds_X\xi)\xi + R(Y,\xi)\pounds_X\xi \\ = -\kappa\{(\pounds_X\eta)(Y)\xi - \eta(Y)\pounds_X\xi\} + \mu(\pounds_X h)Y.\end{aligned} \tag{3.24}$$

Then we take the Lie-derivative of $\eta(Y) = g(\xi, Y)$ along X and make use of (3.1) and (2.10) to find

$$(\pounds_X\eta)(Y) = g(\pounds_X\xi, Y) - 2(\lambda + 2n\kappa)\eta(Y).$$

Using this in (3.24), we have

$$\begin{aligned}(\pounds_X R)(Y,\xi)\xi = 2\kappa(\lambda + 2n\kappa)\{-Y + \eta(Y)\xi\} - 2\mu(\lambda + 2n\kappa)hY \\ + \mu\{\eta(Y)h\pounds_X\xi + g(hY,\pounds_X\xi)\xi + (\pounds_X h)Y\}.\end{aligned} \tag{3.25}$$

Comparing (3.25) with (3.18) and using (3.23), we obtain

$$4\mu\kappa hY = \mu\{\eta(Y)h\pounds_X\xi + g(hY,\pounds_X\xi)\xi + \mu(\pounds_X h)Y\}. \tag{3.26}$$

Now, we change Y to hY in the above equation to find

$$4\mu\kappa h^2 Y = \mu\{g(h^2 Y,\pounds_X\xi)\xi + (\pounds_X h)hY\}.$$

We operate (3.26) by h on both sides to deduce

$$4\mu\kappa h^2 Y = \mu\{\eta(Y)h^2\pounds_X\xi + h(\pounds_X h)Y\}.$$

Adding the above two equations and using (2.8) and (3.21), we get

$$\begin{aligned}8\mu\kappa h^2 Y = \kappa\mu\{-\eta(Y)\pounds_X\xi + 2(\lambda + 2n\kappa)\eta(Y)\xi - g(Y,\pounds_X\xi)\xi\} \\ + \mu\{(\pounds_X h)hY + h(\pounds_X h)Y\}.\end{aligned} \tag{3.27}$$

Note that (2.8) gives $h^2 Y = \kappa(-Y + \eta(Y)\xi)$, and taking the Lie-derivative of this along X provides

$$(\pounds_X h)hY + h(\pounds_X h)Y = \kappa\{g(\pounds_X\xi, Y)\xi - 2(\lambda + 2n\kappa)\eta(Y)\xi + \eta(Y)\pounds_X\xi\}.$$

The substitution of this in (3.27) gives $8\mu\kappa h^2 Y = 0$, which with the help of (2.8) yields $8\mu\kappa^2\varphi^2 Y = 0$. Tracing this and taking into consideration that M is non-co-Kaehler, i.e., $\kappa < 0$, we have that $\mu = 0$. Thus, from (3.23), we have $\lambda = -2n\kappa$, and since $\kappa < 0$, the soliton is expanding. Now using (3.15) with $\mu = 0$ in the soliton equation (3.1), we have

$$(\pounds_X g)(Y, Z) = -4n\kappa\{g(Y, Z) - \eta(Y)\eta(Z)\}.$$

The rest of the proof follows from Lemma 3.1. □

Remark 3.3. Very recently, Suh and De [32] studied the (κ, μ)-almost co-Kaehler manifold M admitting the Ricci soliton g with soliton vector field $X = \xi$ and proved that the soliton is expanding (see Theorem 5.1 in [32]). Thus, our Theorem 3.2 is stronger than Theorem 5.1 of [32].

Here we give an example for a 3-dimensional non-co-Kaehler $(-1, 0)$-almost co-Kaehler manifold which admits a Ricci soliton. However, as we shall prove at the end of Section 4, it can not be compact.

Example 3.4. Following Dacko [13], we define an almost co-Kaehler structure (φ, ξ, η, g) on $M = \mathbb{R}^3$ as:

$$\varphi(\partial_x) = e^{2z}\partial_y, \quad \varphi(\partial_y) = -e^{-2z}\partial_x, \quad \varphi(\partial_z) = 0, \quad \xi = \partial_z, \eta = dz,$$

$$(g_{ij}) = \begin{pmatrix} e^{2z} & 0 & 0 \\ 0 & e^{-2z} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where $\partial_x = \frac{\partial}{\partial x}, \partial_y = \frac{\partial}{\partial y}$ and $\partial_z = \frac{\partial}{\partial z}$. From Koszul's formula, we find the Levi-Civita connection ∇ as given below:

$$\begin{array}{lll} \nabla_{\partial_x}\partial_x = -e^{2z}\partial_z, & \nabla_{\partial_x}\partial_y = 0, & \nabla_{\partial_x}\partial_z = \partial_x, \\ \nabla_{\partial_y}\partial_x = 0, & \nabla_{\partial_y}\partial_y = e^{-2z}\partial_z, & \nabla_{\partial_y}\partial_z = -\partial_y, \\ \nabla_{\partial_z}\partial_x = \partial_x, & \nabla_{\partial_z}\partial_y = -\partial_y, & \nabla_{\partial_z}\partial_z = 0. \end{array} \tag{3.28}$$

We employ (3.28) to find the following:

$$\begin{array}{ll} R(\partial_x, \partial_y)\partial_z = R(\partial_y, \partial_z)\partial_x = 0, & R(\partial_x, \partial_z)\partial_y = 0, \\ R(\partial_x, \partial_z)\partial_x = e^{2z}\partial_z, & R(\partial_y, \partial_z)\partial_y = e^{-2z}\partial_z, \\ R(\partial_x, \partial_y)\partial_x = -e^{2z}\partial_y, & R(\partial_y, \partial_z)\partial_z = -\partial_y, \\ R(\partial_x, \partial_z)\partial_z = -\partial_x, & R(\partial_x, \partial_y)\partial_y = e^{-2z}\partial_x. \end{array} \tag{3.29}$$

Using (3.29), one can easily show that

$$R(Y, Z)\xi = -\{\eta(Z)Y - \eta(Y)Z\}$$

for all $Y, Z \in \mathfrak{X}(M)$, and thus M is a non-co-Kaehler $(-1, 0)$-almost co-Kaehler manifold. We employ (3.29) to find the Ricci tensor

$$(S_{ij}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \tag{3.30}$$

If we choose the vector field

$$X = (2x - ax)\frac{\partial}{\partial x} + (2y + ay)\frac{\partial}{\partial y} + a\frac{\partial}{\partial z}, \tag{3.31}$$

where a is a real constant, then from (3.28) and (3.30) it is not hard to show that

$$\pounds_X g + 2S + 4g = 0. \tag{3.32}$$

Hence g is a Ricci soliton with soliton constant $\lambda = 2$.

4. Generalized (κ,μ)-almost co-Kaehler metric as generalized m-quasi-Einstein metric

The *m-Bakry-Emery Ricci tensor* is a natural extension of the Ricci tensor to smooth metric measure spaces and is given by

$$S_f^m = S + \operatorname{Hess} f - \frac{1}{m} df \otimes df,$$

where f is a smooth function and m is an integer such that $0 < m \leq \infty$. A complete Riemannian manifold (M, g, f) with a potential function f is called *m-quasi-Einstein* if its associated m-Bakry-Emery Ricci tensor is a constant multiple of the metric g (see [7, 23] and the references therein). Recently, in [8], Catino introduced *generalized quasi-Einstein manifolds* which extend the notion of m-quasi-Einstein manifolds. More precisely, a complete Riemannian n-manifold (M, g) $(n \geq 3)$ is a *generalized quasi-Einstein metric* if there exist smooth functions f, γ and μ satisfying

$$S + \operatorname{Hess} f - \mu df \otimes df = \gamma g. \tag{4.1}$$

As a particular case of (4.1), we give the following definition which is due to Barros and Ribeiro [1].

Definition 4.1. A Riemannian manifold (M, g) is said to be a *generalized m-quasi-Einstein metric* if there exist two smooth functions f and γ on M satisfying

$$S + \operatorname{Hess} f - \frac{1}{m} df \otimes df = \gamma g, \tag{4.2}$$

where $0 < m \leq \infty$ is an integer.

It is important to point out that (4.2) corresponds to the well-known metrics such as:

- gradient Ricci soliton if $m = \infty$ and $\gamma = \text{const.}$;
- gradient almost Ricci soliton if $m = \infty$ and $\gamma \in C^\infty(M)$;
- m-quasi-Einstein metric if $\gamma = \text{const.}$;
- (m, ρ)-quasi-Einstein metric [24] if $\gamma = \rho r + \lambda$, where $\rho, \lambda \in \mathbb{R}$;

- gradient ρ-Einstein soliton [9] if $m = \infty$ and$\gamma = \rho r + \lambda$, where $\rho, \lambda \in \mathbb{R}$.

Ghosh considered (m, ρ)-quasi-Einstein metrics and generalized m-quasi-Einstein metrics on contact geometry in [18] and [19], respectively. Here, we consider this within the framework of generalized (κ, μ)-almost co-Kaehler manifolds.

First, we start with the following lemma.

Lemma 4.2. *Let (M, g, f, m) be a generalized m-quasi-Einstein manifold. If g is a generalized (κ, μ)-almost co-Kaehler metric, then*

$$R(Y,Z)Df = (\nabla_Z Q)Y - (\nabla_Y Q)Z + \frac{1}{m}[Y(f)QZ - Z(f)QY] + \frac{\gamma}{m}[Z(f)Y - Y(f)Z] + [Y(\gamma)Z - Z(\gamma)Y]. \tag{4.3}$$

Proof. In light of (4.2), it follows that

$$\nabla_Y Df + QY = \frac{1}{m} g(Y, Df)Df + \gamma Y, \tag{4.4}$$

from which we compute (4.3). □

We employ the above lemma to prove the following fruitful result.

Theorem 4.3. *If M is a generalized (κ, μ)-almost co-Kaehler manifold of dimension higher than 3 such that $h \neq 0$, then the metric of M can not be a generalized m-quasi-Einstein metric.*

Proof. Note that under the hypothesis of present theorem, Q satisfies (see [37])

$$QY = 2n\kappa\eta(Y)\xi + \mu hY, \tag{4.5}$$

where $\kappa = \text{const} \neq 0$ and the smooth function μ satisfies $d\mu \wedge \eta = 0$. We differentiate (4.5) and use (2.4) to find

$$(\nabla_Y Q)Z = 2n\kappa\{g(h'Y, Z)\xi + \eta(Z)h'Y\} + Y(\mu)hZ + \mu(\nabla_Y h)Z. \tag{4.6}$$

Now, we interchange Y and Z in (4.6) and then subtract it by (4.6) to yield

$$(\nabla_Y Q)Z - (\nabla_Z Q)Y = \mu\{(\nabla_Y h)Z - (\nabla_Z h)Y\} + Y(\mu)hZ - Z(\mu)hY + 2n\kappa\{\eta(Z)h'Y - \eta(Y)h'Z\}.$$

We substitute the aforementioned equation and (2.15) into (4.3) to deduce

$$\begin{aligned} R(Y,Z)Df = {} & \kappa\mu\{\eta(Y)\varphi Z - \eta(Z)\varphi Y + 2g(Y, \varphi Z)\xi\} + Z(\mu)hY - Y(\mu)hZ \\ & - \mu^2\{\eta(Y)h'Z - \eta(Z)h'Y\} + 2n\kappa\{\eta(Y)h'Z - \eta(Z)h'Y\} \\ & + \frac{\gamma}{m}\{Z(f)Y - Y(f)Z\} + \frac{1}{m}\{Y(f)QZ - Z(f)QY\} \\ & + \{Y(\gamma)Z - Z(\gamma)Y\}. \end{aligned} \tag{4.7}$$

Next, we take the scalar product of (4.7) with ξ to deduce

$$\begin{aligned} g(R(Y,Z)Df,\xi) = 2\kappa\mu g(Y,\varphi Z) + \frac{\gamma}{m}\{Z(f)\eta(Y) - Y(f)\eta(Z)\} \\ + \frac{2n\kappa}{m}\{Y(f)\eta(Z) - Z(f)\eta(Y)\} \\ + \{Y(\gamma)\eta(Z) - Z(\gamma)\eta(Y)\}. \end{aligned}$$

Then setting $Y = \varphi Y$ and $Z = \varphi Z$ gives

$$g(R(\varphi Y,\varphi Z)Df,\xi) = 2\kappa\mu g(\varphi Y,\varphi^2 Z).$$

From (2.7), we see that the left side of aforesaid equation is zero, and thus it follows that $2\kappa\mu g(\varphi Y, Z) = 0$ for any $Y, Z \in \mathfrak{X}(M)$. Replacing Z by φZ and since $\kappa < 0$, it follows that $\mu\varphi^2 Y = 0$. Tracing this gives $\mu = 0$.

Thus, (4.7) becomes

$$\begin{aligned} R(Y,Z)Df = 2n\kappa\{\eta(Y)h'Z - \eta(Z)h'Y\} + \frac{\gamma}{m}\{Z(f)Y - Y(f)Z\} \\ + \frac{1}{m}\{Y(f)QZ - Z(f)QY\} + \{Y(\gamma)Z - Z(\gamma)Y\}. \end{aligned} \tag{4.8}$$

Replace Y by ξ in (4.8) and then take the scalar product with ξ to obtain

$$g(R(\xi,Z)Df,\xi) = \left(\frac{\gamma - 2n\kappa}{m}\right)\{Z(f) - \xi(f)\eta(Z)\} + \{\xi(\gamma)\eta(Z) - Z(\beta)\}. \tag{4.9}$$

On the other hand, from (2.7) we find

$$g(R(\xi,Z)Df,\xi) = \kappa\{Z(f) - \xi(f)\eta(Z)\}. \tag{4.10}$$

Comparing the previous two equations, we can see that

$$\left(\frac{\gamma - 2n\kappa}{m} - \kappa\right)(Df - \xi(f)\xi) + (\xi(\gamma)\xi - D\gamma) = 0. \tag{4.11}$$

Now, we contract (4.8) over Y to find

$$\left(1 - \frac{1}{m}\right)S(Z,Df) = \left(\frac{2n\gamma - 2n\kappa}{m}\right)g(Z,Df) - 2ng(Z,D\gamma),$$

which gives

$$2nD\gamma = \left(\frac{2n\gamma - 2n\kappa}{m}\right)Df - \left(1 - \frac{1}{m}\right)QDf.$$

Using this in (4.11), we obtain

$$\kappa\left(\frac{1-2n}{m} - 1\right)Df + \left(\frac{2n\kappa}{m}\right)\xi(f)\xi + \frac{1}{2n}\left(1 - \frac{1}{m}\right)QDf = 0. \tag{4.12}$$

Since $\mu = 0$, (4.5) shows that $QY = 2n\kappa\eta(Y)\xi$, which gives $QDf = 2n\kappa\xi(f)\xi$. Hence, (4.12) takes the form

$$\kappa\left(\frac{1-2n}{m} - 1\right)(Df - \xi(f)\xi) = 0. \tag{4.13}$$

Since $h \neq 0$, it follows that $\kappa < 0$. Thus, (4.13) gives

$$\left(\frac{1-2n}{m} - 1\right)(Df - \xi(f)\xi) = 0.$$

If $m = \infty$, then clearly $Df - \xi(f)\xi = 0$. Let m be an integer such that $0 < m < \infty$. Then $\frac{1-2n}{m} - 1 = 0$ if and only if $m = 1 - 2n$, and in such a case $m < 0$, which will be a contradiction. Hence, $\frac{1-2n}{m} - 1 \neq 0$, and consequently we have

$$Df - \xi(f)\xi = 0.$$

Next, we differentiate this along Y and then make use of (4.4) and (2.4) to arrive at

$$QY = \gamma Y + (2n\kappa - \gamma)\eta(Y)\xi - \xi(f)h'Y - g(Df, h'Y)\xi. \tag{4.14}$$

Tracing the above equation, using $r = 2n\kappa$ and (2.3), we have $2n\gamma = 0$ and so $\gamma = 0$. Thus, (4.14) becomes

$$QY = 2n\kappa\eta(Y)\xi - \xi(f)h'Y - g(Df, h'Y)\xi = 0,$$

which, by virtue of $QY = 2n\kappa\eta(Y)\xi$, leads to

$$\xi(f)h'Y + g(Df, h'Y)\xi = 0. \tag{4.15}$$

By considering the ξ-component of (4.15), we see that $g(Df, h'Y) = 0$. Using this in (4.15), we get $\xi(f)h'Y = 0$ for all $Y \in \mathfrak{X}(M)$. Then replacing Y by $h'Y$ and recalling the fourth relation of (2.3), we obtain

$$\xi(f)\|h\|^2 = 0.$$

We employ (2.6) and (2.11) in the above equation to get $2n\kappa\xi(f) = 0$. Since $\kappa < 0$, it follows that $\xi(f) = 0$ and thus $Df = 0$. Hence, (4.4) leads to $QY = 0$, which is a contradiction to (4.5) as $\kappa < 0$. This is what we wanted to prove. □

As a direct consequence, we have:

Corollary 4.4. *If M is a generalized (κ,μ)-almost co-Kaehler manifold of dimension higher than 3 such that $h \neq 0$, then the metric of M can not be a gradient almost Ricci soliton.*

Clearly, both Theorem 4.3 and Corollary 4.4 generalize the result of Wang (Theorem 4.1 in [37]).

We have already noticed that (4.5) is true even for a $(2n+1)$-dimensional (κ,μ)-almost co-Kaehler manifold. Then, following the same approach as in the proof of Theorem 4.3, we have

Theorem 4.5. *If M is a (κ,μ)-almost co-Kaehler manifold such that $h \neq 0$, then the metric of M can not be a generalized m-quasi-Einstein metric.*

As a consequence, a (κ,μ)-almost co-Kaehler manifold such that $h \neq 0$ does not admit a gradient Ricci soliton. According to Perelman [30]: *A Ricci soliton on a compact manifold is a gradient Ricci soliton*. Thus we have the following:

Corollary 4.6. *If M is a compact (κ,μ)-almost co-Kaehler manifold such that $h \neq 0$, then the metric of M can not be a Ricci soliton.*

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Devaraja Mallesha Naik,

Department of Mathematics, CHRIST (Deemed to be University), Bengaluru-560029, Karnataka, India,
E-mail: devaraja.mallesha@christuniversity.in

V. Venkatesha,

Department of Mathematics, Kuvempu University, Shankaraghatta, Karnataka 577 451, India,
E-mail: vensmath@gmail.com

H. Aruna Kumara,

Department of Mathematics, Kuvempu University, Shankaraghatta, Karnataka 577 451, India,
E-mail: arunmathsku@gmail.com

Солітони Річчі та деякі пов'язані з ними метрики на майже ко-келерових многовидах

Devaraja Mallesha Naik, V. Venkatesha, and H. Aruna Kumara

У статті вивчаються солітони Річчі та узагальнена m-квазіейнштейнова метрика на майже ко-келеровому многовиду M, що задовольняє нуль-умову. Спочатку ми розглядаємо не ко-келерову (κ, μ)-майже ко-келерову метрику як солітон Річчі і доводимо, що солітон розширюється з $\lambda = -2n\kappa$, а векторне поле солітона X залишає структурні тензори η, ξ and φ інваріантними. Даний результат узагальнює Теорему 5.1 з [32]. Побудовано приклад існування солітона Річчі на M.

Наприкінці ми доводимо що, якщо M це узагальнений (κ,μ)-майже ко-келеровий многовид розмірності більшої за 3, такий що $h \neq 0$, то тоді метрика M не може бути узагальненою m-квазі-ейнштейнновою метрикою, і це включає результат нещодавно отриманий Вангом [37, Theorem 4.1] як окремий випадок.

Ключові слова: майже ко-келеровий многовид, солітон Річчі, узагальнена m-квазі-ейнштейнова метрика, розподіл (κ,μ)-обнулення