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The Existence of Solutions to an Inhomogeneous Higher Order Differential Equation in the Schwartz Space

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The paper deals with the problem of the existence of solutions to an inhomogeneous linear differential equation of higher even order. The problem arises while studying soliton and soliton-like solutions to partial differential equations of integrable type. The theorem on necessary and sufficient conditions of the existence of solutions to the differential equation in the Schwartz space of rapidly decreasing functions is proved by means of theory of pseudodifferential operators.

Key words: existence of solutions, higher order differential equations, the Schwartz space of rapidly decreasing functions, pseudodifferential operators

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1. Introduction and statement of problem

The one-dimensional stationary Shrödinger equation

$$Ly = -\frac{d^2y}{dx^2} + q(x)y = 0, \quad x \in \mathbf{R}, \tag{1.1}$$

is one of basic equations of quantum mechanics. Fundamental properties of spectrum of the operator L and its eigenfunctions are given in [1, 2].

The operator L appears while studying nonlinear partial differential equations of mathematical physics through inverse scattering transform and it is often called the one-dimensional Shrödinger operator [3, 4]. It should be noted that various problems arise for an inhomogeneous differential equation with the one-dimensional Shrödinger operator of the following form:

$$-\frac{d^2y}{dx^2} + q(x)y = f(x), \quad x \in \mathbf{R}. \tag{1.2}$$

In particular, the problem of the existence of solutions to equation (1.2) in the Schwartz space of rapidly decreasing functions appears under constructing asymptotic soliton-like solutions to a number of partial differential equations of integrable type with singular perturbations and variable coefficients [5–7].

The problem of the existence and uniqueness of solutions to differential equations is one of the main problems in the theory of differential equations. In this connection it should be remarked that the problem of the existence of particular solutions with certain asymptotics leads to searching the ones in special functional spaces [8]. In addition, the problem of the existence of solutions to equation (1.2) in the Schwartz space of rapidly decreasing functions was studied in [9] where necessary and sufficient conditions were obtained for it.

On the other hand, while constructing soliton solutions to the higher Korteweg–de Vries equations and the KdV-like equations there appeared the same problem for similar equations with a differential operator of higher order. So, we come to the problem of the existence of a solution to the equation

$$Lv = f \tag{1.3}$$

in the Schwartz space of rapidly decreasing functions.

Here the differential operator L is written as

$$L = \sum_{k=1}^{n} a_{2k} \frac{d^{2k}}{dx^{2k}} + q(x), \quad x \in \mathbf{R}, \tag{1.4}$$

where $a_{2k}, k = \overline{1, n}$, are constant coefficients.

2. Main result

Let $\mathrm{S}(\mathbf{R})$ be the Schwartz space of rapidly decreasing functions.

The main result of the paper is the statement.

Theorem 2.1. *Let*

1°. *the inequalities*

$$(-1)^k a_{2k} \leq 0, \ k = \overline{1, n}, \quad \textit{and} \quad a_{2n} \neq 0$$

hold for the coefficients $a_{2k}, k = \overline{1, n}$;

2°. $q(x) = q_0 + q_1(x)$, *where* $q_0 < 0$ *is a constant,* $q_1(x) \in \mathrm{S}(\mathbf{R})$;

3°. $f \in \mathrm{S}(\mathbf{R})$.

If the kernel of the operator $L : \mathrm{S}(\mathbf{R}) \to \mathrm{S}(\mathbf{R})$ *is trivial, then equation* (1.3) *has a solution in the space* $\mathrm{S}(\mathbf{R})$ *for any function* $f \in \mathrm{S}(\mathbf{R})$.

Otherwise, if the kernel of the operator $L : \mathrm{S}(\mathbf{R}) \to \mathrm{S}(\mathbf{R})$ *is not trivial, then equation* (1.3) *has a solution in the space* $\mathrm{S}(\mathbf{R})$ *iff the function* $f \in \mathrm{S}(\mathbf{R})$ *satisfies the orthogonality condition*

$$\int_{-\infty}^{+\infty} f(x) v_0(x) dx = 0 \tag{2.1}$$

for any $v_0 \in \ker\ L$.

3. Necessary definitions and statements

To prove the theorem, we need to recall some notations, definitions, and auxiliary results. For any function $h \in \mathrm{S}(\mathbf{R})$ there is its Fourier transform denoted as

$$F[h](\xi) = \int_{-\infty}^{+\infty} e^{-i\xi x} h(x)\, dx.$$

Due to the properties of the Fourier transform, for any differential operator

$$p\left(x, \frac{d}{dx}\right) = \sum_{k=0}^{n} a_k(x) \frac{d^k}{dx^k}, \quad x \in \mathbf{R},$$

its action on a function $h \in \mathrm{S}(\mathbf{R})$ can be defined by the formula

$$p\left(x, \frac{d}{dx}\right) h(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{ix\xi}\, p(x,\xi) F[h](\xi)\, d\xi. \tag{3.1}$$

Here

$$p(x,\xi) = \sum_{k=0}^{n} a_k(x)(-i\xi)^k, \quad x, \xi \in \mathbf{R},$$

is called the symbol of the differential operator $p\left(x, \frac{d}{dx}\right)$.

Further we will use the following notations [10,11]. Let S^m be a set of symbols $p(x,\xi) \in \mathrm{C}^\infty(\mathbf{R}^2)$ such that for any integers k, $l \in \mathbf{N} \cup \{0\}$ the inequality

$$\left| p_{(l)}^{(k)}(x,\xi) \right| \le C_{kl} \left(1 + |\xi|\right)^{m-k}, \quad (x,\xi) \in \mathbf{R}^2,$$

is true, where

$$p_{(l)}^{(k)}(x,\xi) = \frac{\partial^{k+l}}{\partial \xi^k \partial x^l}\, p(x,\xi), \quad (x,\xi) \in \mathbf{R}^2,$$

and C_{kl}, k, $l \in \mathbf{N} \cup \{0\}$, are some constants.

Let S_0^m be a set of symbols $p(x,\xi) \in S^m$ such that

$$|p(x,\xi)| \le M(x) \left(1 + |\xi|\right)^m,$$

where $M(x) \to 0$ as $|x| \to +\infty$.

By $\overline{C}_0^\infty(\mathbf{R})$, we denote a space of infinitely differentiable functions $u(x)$, $x \in \mathbf{R}$, satisfying the relation $d^n u(x)/dx^n \to 0$ as $|x| \to +\infty$ for any non-negative integer n.

Let $\mathrm{H}_s(\mathbf{R})$, $s \in \mathbf{R}$, be a Sobolev space [11], i.e., the space of distributions $g \in \mathrm{S}^*(\mathbf{R})$ such that their Fourier transform $F[g](\xi)$ satisfies the condition

$$\|g\|_s^2 = \int_{-\infty}^{+\infty} (1 + |\xi|^2)^s \, |F[g](\xi)|^2 \, d\xi < \infty. \tag{3.2}$$

It is worthy to recall the following theorem.

Theorem 3.1 (Grushin, [11]). *Let $p(x,\xi) \in S^m$ be a symbol such that $\partial^l p(x,\xi)/\partial x^l \in S_0^m$, $l \in \mathbf{N}$, and the inequality*

$$\varliminf_{(x,\xi)\to\infty} \frac{|p(x,\xi)|}{(1+|\xi|)^m} > 0$$

is true.

Then the differential operator $p\left(x, \frac{d}{dx}\right) : H_{s+m}(\mathbf{R}) \to H_s(\mathbf{R})$ defined by formula (3.1) is the Noether operator for any $s \in \mathbf{R}$.

4. Proof of Theorem 2.1

Now we proceed to proving Theorem 2.1. It contains two steps. Firstly, we show that the operator $L : \mathrm{H}_{s+2n}(\mathbf{R}) \to \mathrm{H}_s(\mathbf{R})$ of the form (1.4) is the Noether operator for any $s \in \mathbf{R}$. Later we prove that equation (1.3) is solvable in the Schwartz space $\mathrm{S}(\mathbf{R})$.

Let us consider a symbol of the differential operator L having the form

$$p(x,\xi) = \sum_{k=1}^{n} (-1)^k a_{2k}\xi^{2k} + q(x). \tag{4.1}$$

It is obvious that $p(x,\xi)$ belongs to the set S^{2n} due to the inequality

$$\left|\frac{\partial^{k+l}}{\partial\xi^k \partial x^l}\, p(x,\xi)\right| \le C_{kl}(1+|\xi|)^{2n-k}, \quad k,l \in \mathbf{N}\cup\{0\}.$$

Moreover,

$$\frac{\partial^l}{\partial x^l}\, p(x,\xi) \in S_0^{2n}, \quad l \in \mathbf{N}.$$

According to the assumptions of Theorem 2.1, the operator $L : \mathrm{H}_{s+2n}(\mathbf{R}) \to \mathrm{H}_s(\mathbf{R})$ satisfies all conditions of Theorem 3.1 for any $s \in \mathbf{R}$. So, it is the Noether operator. As a consequence, the operator $L : \mathrm{H}_{s+2n}(\mathbf{R}) \to \mathrm{H}_s(\mathbf{R})$ is normally solvable.

By L^*, denote the operator adjoint to the operator L. If $\ker L^*$ is trivial, then equation (1.3) has a solution in the space $\mathrm{H}_{s+2n}(\mathbf{R})$ for any $f \in \mathrm{S}(\mathbf{R})$ because $L : \mathrm{H}_{s+2n}(\mathbf{R}) \to \mathrm{H}_s(\mathbf{R})$ is the Noether operator.

If $\ker L^*$ is non-trivial, then equation (1.3) is solvable in $\mathrm{H}_{s+2n}(\mathbf{R})$ iff the orthogonality condition

$$< f, \ker L^* >= 0 \tag{4.2}$$

holds.

Remark in [11] implies the inclusion $\ker L^* \subset \bigcap\limits_{s\in\mathbf{R}} \mathrm{H}_s(\mathbf{R})$.

Applying Sobolev embedding theorems for the spaces $\mathrm{H}_s(\mathbf{R})$, $s \in \mathbf{R}$, we have $v_0^* \in \overline{\mathrm{C}}_0^\infty(\mathbf{R})$ for any element $v_0^* \in \ker L^*$.

As a consequence of the orthogonality condition (4.2) and Theorem 3.1, one easily obtains that the solution $v(x)$ of equation (1.3) belongs to the space $\bigcap\limits_{s\in\mathbf{R}} \mathrm{H}_s(\mathbf{R})$. Applying again Sobolev embedding theorems, we get $v \in \overline{\mathrm{C}}_0^\infty(\mathbf{R})$.

Now let us show that $v \in \mathrm{S}(\mathbf{R})$. Indeed, since the function $v \in \overline{\mathrm{C}}_0^\infty(\mathbf{R})$ and it satisfies the equation

$$\sum_{k=1}^{n} a_{2k} \frac{d^{2k} v}{dx^{2k}} + q_0 v = f_1, \tag{4.3}$$

where

$$f_1 = -q_1(x)v + f \in \mathrm{S}(\mathbf{R}),$$

then, due to the properties of elliptic pseudodifferential operators with polynomial coefficients [12], we deduce that any solution to equation (4.3) from the space $\mathrm{S}^*(\mathbf{R})$ belongs to the space $\mathrm{S}(\mathbf{R})$. Thus, $v \in \mathrm{S}(\mathbf{R})$.

Continuing this line of reasoning we find $v_0^* \in \mathrm{S}(\mathbf{R})$. It follows from the properties of the operator L^* that the equality $v_0^* = v_0$ holds. It means that the orthogonality condition (4.2) is equivalent to (2.1).

From the above consideration it also follows that if the kernel of the operator $L : \mathrm{S}(\mathbf{R}) \to \mathrm{S}(\mathbf{R})$ is trivial, i.e., the homogeneous equation $Lv = 0$ has the only trivial solution in the space $\mathrm{S}(\mathbf{R})$, then equation (1.3) has a solution in the space $\mathrm{S}(\mathbf{R})$ for any $f \in \mathrm{S}(\mathbf{R})$.

Theorem 2.1 is proved.

Corollary 4.1. *Let condition 1^0 of Theorem 2.1 be true and $q(x) = q_0 < 0$. Then equation (1.3) has a solution in the space $\mathrm{S}(\mathbf{R})$ for any $f \in \mathrm{S}(\mathbf{R})$.*

5. Conclusions

The theorem on the existence of a solution to the linear higher order inhomogeneous differential equation in the Schwartz space of rapidly decreasing functions is proved. The theorem can be used while studying soliton solutions to integrable systems of modern mathematical physics and asymptotic soliton-like solutions to singular perturbed higher order partial differential equations of integrable type [7]. The theorem generalizes the statement on the existence of a solution to the inhomogeneous equation with the one-dimensional Schrödinger operator in the Schwartz space [9].

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Існування розв'язків неоднорідного диференціального рівняння вищого порядку в просторі Шварца

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У статті розглядається задача про існування розв'язків для неоднорідного лінійного диференціального рівняння вищого парного порядку. Така задача виникає при вивченні солітонних та солітоноподібних розв'язків рівнянь з частинними похідними інтегровного типу. Теорему про необхідні та достатні умови існування розв'язків згаданого рівняння у просторі Шварца швидко спадних функцій доведено з використанням методів теорії псевдодиференціальних операторів.

Ключові слова: існування розв'язків, диференціальні рівняння вищого порядку, простір Шварца швидко спадних функцій, псевдодиференціальні оператори