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Fractional Derivatives with Respect to Time for Non-Classical Heat Problem

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We consider the non-classical heat equation with Caputo fractional derivative with respect to the time variable in a bounded domain $\Omega \subset \mathbb{R}^+ \times \mathbb{R}^{d-1}$ for which the energy supply depends on the heat flux on a part of the boundary $S = \{0\} \times \mathbb{R}^{d-1}$ with homogeneous Dirichlet boundary condition on S, the periodicity on the other parts of the boundary and an initial condition. The problem is motivated by the modeling of the temperature regulation in the medium. The existence of the solution to the problem is based on a Volterra integral of second kind in the time variable t with a parameter in $\mathbb{R}^{d-1}$, its solution is the heat flux $(y, \tau) \mapsto V(y, t) = u_x(0, y, t)$ on S, which is also an additional unknown of the considered problem. We establish that a unique local solution exists and can be extended globally in time.

Key words: non-classical d-dimensional heat equation, Caputo fractional derivative, Volterra integral equation, existence and uniqueness solution, integral representation of solution

Mathematical Subject Classification 2010: 26A33, 34A08, 34A12, 34B15, 35K05, 45D05

1. Introduction

Fractional differential equations have large applications in a variety of fields such as electrical networks, signal and image processing, viscoelasticity [13], aerodynamics, economics, etc. Hence, an increasing attention has been recently paid to both theoretical and applied points of view [2, 3] (for more details, see [4, 8, 10, 11, 14, 16]).

In this paper, we generalize to the factional derivative in time the previous results for ordinary partial derivative [17], where the initial boundary value problem for the one-dimensional non-classical heat equation in a slab was considered and extended in [1] in the semi-space $\mathbb{R}^+ \times \mathbb{R}^{n-1}$.

Let $L_1, L_2, \cdots, L_d$ be some strictly positive reals, and $B = (L_2, \cdots, L_d) \in \mathbb{R}^{d-1}$. We denote

$$[[A, B]] = \{y \in \mathbb{R}^{d-1} \text{ such that } \exists \alpha \in [0, 1] : \quad y = A + \alpha(B - A)\}.$$

Then we set the domain $\Omega =]0, L_1[\times]] - B, B[[\subset \mathbb{R}^+ \times \mathbb{R}^{d-1}$ more precisely,

$$\Omega = \left\{(x, y) \in \mathbb{R}^d : \; x = x_1 \in]0, L_1[, \; y = (x_2, \dots, x_d) \in]] - B, B[[\right\},$$

for which the energy supply depends on the heat flux on the part of the boundary defined by

$$S = \{0\} \times]] - B, B[[\subset \{0\} \times \mathbb{R}^{d-1} = \left\{(x, y) \in \mathbb{R}^d; \quad x = 0, \quad y \in \mathbb{R}^{d-1}\right\}.$$

In this paper we study the following non-classical heat problem:

$$\begin{cases} \left({}^cD^\alpha_{0^+,t}u\right)(x, y, t) - \Delta u(x, y, t) = f(u_x(0, y, t)), \quad (x, y) \in \Omega, \quad t > 0 \\ u(0, y, t) = 0; \quad y \in]] - B, B[[\subset \mathbb{R}^{d-1}, \; t > 0 \\ u(x, y, 0) = h(x, y), \quad (x, y) \in \Omega, \\ u \text{ is } L_i\text{-periodic with respect to } x_i \text{ for } i = 1, \dots, d, \end{cases} \tag{1.1}$$

where ${}^cD^\alpha_{0^+,t}$, the Caputo derivative with $0 < \alpha < 1$, $\Delta = \Delta_{x,y}$, denotes the Laplacian in $\mathbb{R}^d$, and $u_x(x, y, t) = \frac{\partial u}{\partial x}(x, y, t)$.

The paper is organized as follows. In Section 2, we provide the basic solution to the d-dimensional heat problem (1.1), which will be used throughout the paper. In Section 3, we show that under certain conditions there exists a unique local solution to problem (1.1), which can be globally extended in time. In section 4, we prove that there exists a unique solution of the integral representation locally in time (4.2), which can be extended globally in time. In the first subsection, we give an existence result by using Schauder's fixed point theorem combined with the diagonalization method. In the second subsection, we reduce the existence of the unique solution to the search for the existence of the unique fixed-point of an appropriate operator using a nonlinear alternative of Leray–Schauder type for contraction maps in Fréchet spaces due to Frigon–Granas [7].

2. Preliminaries

We give now some basic solutions for the d-dimensional heat problem. We begin by recalling the results on the integral representation of solutions to some classical problems of heat distribution in d-dimensional cases. We set here

$$\Omega^- =] - L_1, L_1[\times]] - B, B[[.$$

Consider

$$\begin{cases} \left({}^cD^\alpha_{0^+,t}u\right)(x, y, t) - \Delta u(x, y, t) = 0, \quad (x, y) \in \Omega^- \subset \mathbb{R}^d, \quad t > 0 \\ u(x, y, 0) = h(x, y), \quad (x, y) \in \Omega^-, \\ u \text{ is } L_i\text{-periodic with respect to } x_i \text{ for } i = 1, \dots, d. \end{cases} \tag{2.1}$$

Lemma 2.1. *The solution to the d-dimensional heat problem* (2.1) *is given on any compact set of* $\mathbb{R}^d$ *by the following formula:*

$$u(x,y,t) = \int_{\mathbb{R}^d} G^\alpha(x-\xi, y-\eta, t) h(\xi,\eta)\, d\xi\, d\eta$$

with

$$G^\alpha(x,y,t) = \frac{t^{\frac{-\alpha}{2}}}{2} \phi\left(\frac{-\alpha}{2}, \frac{-\alpha}{2}+1, -|(x,y)|t^{-\frac{\alpha}{2}}\right),$$

$$\phi(\mu,\beta,\rho) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(\mu k, \beta)} \frac{z^k,}{k!}, \quad \mu = -\frac{\alpha}{2}, \quad \beta = -\frac{\alpha}{2}+1, \quad z = -|(x,y)|t^{-\frac{\alpha}{2}},$$

where Γ *is the Euler gamma function,* $|(x,y)| = \sqrt{x_1^2 + x_2^2 + \cdots + x_d^2}$ *denotes the Euclidean norm of the point* (x,y) *such that* $(x = x_1, y = (x_2, \ldots, x_d))$.

Proof. Applying the partial Laplace transform to the two sides of (2.1) with respect to the time variable taking into account the initial condition, we get

$$s^\alpha \mathcal{L}_s(u)(x,y,s) = \Delta \mathcal{L}_s(u)(x,y,s) + s^{\alpha-1} h(x,y). \tag{2.2}$$

Then, applying the partial Fourier transform to the two sides of (2.2) with respect to the space variables (x,y), we get

$$\begin{aligned} s^\alpha \mathcal{F}\left[\mathcal{L}_s(u)\right](\xi,\eta,s) &= \mathcal{F}\left[\Delta \mathcal{L}_s(u)\right](\xi,\eta,s) + s^{\alpha-1}\mathcal{F}\left[h\right](\xi,\eta) \\ &= -|(\xi,\eta)|^2 \mathcal{F}\left[\mathcal{L}_s(u)\right](\xi,\eta,s) + s^{\alpha-1}\mathcal{F}\left[h\right](\xi,\eta). \end{aligned}$$

Thus

$$\mathcal{F}\left[\mathcal{L}_s(u)\right](\xi,\eta,s) = \frac{s^{\alpha-1}}{s^\alpha + \xi^2 + \|\eta\|^2} \mathcal{F}\left[h\right](\xi,\eta) \tag{2.3}$$

with $(\xi,\eta) \in \mathbb{R} \times \mathbb{R}^{d-1}$ and $\|\eta\|^2 = \sum_{j=1}^{d-1} \eta_j^2$. As for $\psi(X,s) = \frac{s^{\frac{\alpha}{2}-1}}{2} e^{-|X|s^{\frac{\alpha}{2}}}$,

$$\mathcal{F}\left[\psi\right](\sigma,s) = \frac{s^{\frac{\alpha}{2}-1}}{2} \frac{2s^{\frac{\alpha}{2}}}{\left(s^{\frac{\alpha}{2}}\right)^2 + |\sigma|^2} = \frac{s^{\alpha-1}}{s^\alpha + |\sigma|^2}, \quad \sigma \in \mathbb{R}^d.$$

Hence, for $\sigma = (\xi,\eta)$, $X = (x,y)$, the relation (2.3) becomes

$$\mathcal{F}\left[\mathcal{L}_s(u)\right](\xi,\eta,s) = \mathcal{F}\left[\psi\right]\mathcal{F}\left[h\right](\xi,\eta,s) = \mathcal{F}\left[\psi * h\right](\xi,\eta,s),$$

where $*$ is the convolution operator. Then, applying the inverse partial Fourier transform, we get

$$\mathcal{L}_s(u)(x,y,s) = (\psi * h)(x,y,s). \tag{2.4}$$

As the function ψ can be developed as series entire with infinite radius, we can write

$$\psi(x,y,s) = \frac{s^{\frac{\alpha}{2}-1}}{2} \sum_{k=0}^{\infty} \frac{\left(-|(x,y)|s^{\frac{\alpha}{2}}\right)^k}{k!} = \frac{1}{2}\sum_{k=0}^{\infty} \frac{(-|(x,y)|)^k}{k!} \frac{1}{s^{-\frac{\alpha}{2}k - \frac{\alpha}{2}+1}}, \tag{2.5}$$

using the Euler gamma function defined by

$$\Gamma(\alpha) = \int_0^{\infty} t^{\alpha-1} e^{-t} dt,$$

which is well defined for $\alpha > 0$. Moreover, the formula

$$\frac{\Gamma(\lambda+1)}{s^{\lambda+1}} = \mathcal{L}_s[t^{\lambda}](s) = \int_0^{+\infty} t^{\lambda} e^{-st} dt \quad \text{for } \mathrm{Re}(\lambda) > -1, \mathrm{Re}(s) > 0,$$

with $\lambda = -\frac{\alpha}{2}k - \frac{\alpha}{2}$, but here the condition $\lambda > -1$ implies that $k < 2\alpha^{-1} - 1$. Notice that $k \to +\infty$ implies that $\alpha \to 0$. Thus, choosing $0 < \alpha < 2(N-1)^{-1} \ll 1$, for large enough $N \in \mathbb{N}$, we obtain that $0 \leq k < N$. Then we can write (2.5) as

$$\begin{aligned}\psi(x,y,s) &= \frac{1}{2}\sum_{k=0}^{\infty} \frac{1}{\Gamma(-\frac{\alpha}{2}k - \frac{\alpha}{2} + 1)} \frac{(-|(x,y)|)^k}{k!} \frac{\Gamma(-\frac{\alpha}{2}k - \frac{\alpha}{2} + 1)}{s^{-\frac{\alpha}{2}k - \frac{\alpha}{2} + 1}} \\ &= \frac{1}{2}\sum_{k=0}^{\infty} \frac{1}{\Gamma(-\frac{\alpha}{2}k - \frac{\alpha}{2} + 1)} \frac{(-|(x,y)|)^k}{k!} \int_0^{\infty} e^{-st} t^{-\frac{\alpha}{2}k - \frac{\alpha}{2}} dt. \end{aligned} \tag{2.6}$$

Now the series of functions, given by its general term u_k,

$$u_k(x,y,s,t) = \frac{\left(-|(x,y)|t^{-\frac{\alpha}{2}}\right)^k}{\Gamma(-\frac{\alpha}{2}k - \frac{\alpha}{2} + 1)k!},$$

is in the form of that from [12, [p. 54],

$$\phi(\mu,\beta,z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\mu k + \beta))k!},$$

which is absolutely convergent for all $z \in \mathbb{C}$, as $\mu = -\frac{\alpha}{2} > -1$. On all compact set of $\mathbb{R}^d$ the convergence of this series is normal then it is uniform. Thus we can do the following permutation:

$$\begin{aligned}\psi(x,y,s) &= \frac{1}{2}\int_0^{\infty} e^{-st} \left[t^{\frac{\alpha}{2}} \sum_{k=0}^{\infty} \frac{\left(-|(x,y)|t^{-\frac{\alpha}{2}}\right)^k}{\Gamma(-\frac{\alpha}{2}k - \frac{\alpha}{2} + 1)k!} \right] dt \\ &= \int_0^{\infty} e^{-st} \left[\frac{t^{-\frac{\alpha}{2}}}{2} \phi\left(\frac{-\alpha}{2}, \frac{-\alpha}{2} + 1, -|(x,y)|t^{-\frac{\alpha}{2}} \right) \right] dt \\ &= \mathcal{L}_s \left[\frac{t^{-\frac{\alpha}{2}}}{2} \phi\left(\frac{-\alpha}{2}, \frac{-\alpha}{2} + 1, -|(x,y)|t^{-\frac{\alpha}{2}} \right) \right], \end{aligned}$$

then with (2.4):

$$u(x,y,t) = \int_{\mathbb{R}^n} G^{\alpha}(x-\xi, y-\eta, t) h(\xi,\eta)\, d\xi\, d\eta,$$

where

$$G^{\alpha}(x,y,t) = \frac{t^{-\frac{\alpha}{2}}}{2}\phi\left(\frac{-\alpha}{2}, \frac{-\alpha}{2} - |(x,y)|t^{-\frac{\alpha}{2}}\right),$$

which solves problem (2.1) on any compact set of $\mathbb{R}^d$. □

Remark 2.2. This proof leads us to the consideration of the problems in this paper on any compact set, but not on all $\mathbb{R}^d$.

Lemma 2.3. *The solution of the problem*

$$\begin{cases} \left({}^{c}D^{\alpha}_{0^+,t}u\right)(x,y,t) = \Delta_{x,y}u(x,y,t), & (x,y) \in \Omega \subset \mathbb{R}^{d-1},\ t>0, \\ u(0,y,t) = 0, \quad y \in]]-B,B[[,\ t>0, \\ u(x,y,0) = h(x,y), \quad (x,y) \in \Omega, \\ u \text{ is } L_i - \text{periodic with respect to } x_i \text{ for } i=1,\dots,d, \end{cases} \tag{2.7}$$

is given on all compact set $\overline{\Omega}$ *by the formula*

$$u(x,y,t) = \int_{\Omega} K^{\alpha}(x-\xi, y-\eta, t)h(\xi,\eta)\,d\xi\,d\eta,$$

where

$$K^{\alpha}(x-\xi,y-\eta,t-\tau) = G^{\alpha}(x-\xi,y-\eta,t-\tau) - G^{\alpha}(-x-\xi,y-\eta,t-\tau).$$

Proof. Define $\tilde{h}$ on $\Omega\cup\Omega^-$, where $\Omega^- =]-L_1,0]\times]]-B,B[[$, by

$$\tilde{h}(\xi,\eta) = \begin{cases} h(\xi,\eta) & \text{if } (\xi,\eta)\in\Omega, \\ -h(-\xi,\eta) & \text{if } (\xi,\eta)\in\Omega^-. \end{cases}$$

From Lemma 2.1, the solution of the problem

$$\begin{cases} \left({}^{c}D^{\alpha}_{0^+,t}u\right)(x,y,t) = \Delta_{x,y}u(x,y,t), & (x,y)\in\Omega\cup\Omega^-, \quad t>0, \\ u(x,y,0) = \tilde{h}(x,y) \quad (x,y)\in\Omega\cup\Omega^-, \\ u \text{ is } L_i - \text{periodic with respect to } x_i \text{ for } i=1,\cdots,d, \end{cases}$$

is given on all compact set of $\Omega\cup\Omega^-$ by

$$\begin{aligned} u(x,y,t) &= \int_{\Omega\cup\Omega^-} G^{\alpha}(x-\xi,y-\eta,t)\tilde{h}(\xi,\eta)\,d\xi\,d\eta \\ &= \int_{\Omega} G^{\alpha}(x-\xi,y-\eta,t)h(\xi,\eta)\,d\xi\,d\eta \\ &\quad - \int_{\Omega^-} G^{\alpha}(x-\xi,y-\eta,t)h(-\xi,\eta)\,d\xi\,d\eta. \end{aligned}$$

By changing the variables $\xi = -\xi_1$ in Ω^- and $\xi = \xi_1$ in Ω, we get

$$u(x,y,t) = \int_{\Omega} G^{\alpha}(x-\xi_1,y-\eta,t)h(\xi_1,\eta)\,d\xi_1\,d\eta$$

$$+\int_{\Omega^-} G^\alpha(x+\xi_1, y-\eta, t)h(\xi_1,\eta)\,d\xi_1\,d\eta$$
$$=\int_{\Omega}\left[G^\alpha(x-\xi_1, y-\eta, t) - G^\alpha(x+\xi_1, y-\eta, t)\right]h(\xi_1,\eta)\,d\xi_1\,d\eta.$$

Moreover, we have

$$\begin{aligned} G^\alpha(x+\xi_1, y-\eta, t) &= \frac{t^{\frac{-\alpha}{2}}}{2}\sum_{k=0}^{\infty}\frac{(-1)^k}{\Gamma\left(\frac{-\alpha}{2}k-\frac{\alpha}{2}+1\right)}\frac{\left(\left[(x+\xi_1)^2+\|y-\eta\|^2\right]t^\alpha\right)^{\frac{k}{2}}}{k!} \\ &= \frac{t^{\frac{-\alpha}{2}}}{2}\sum_{k=0}^{\infty}\frac{(-1)^k}{\Gamma\left(\frac{-\alpha}{2}k-\frac{\alpha}{2}+1\right)}\frac{\left(\left[(-x-\xi_1)^2+\|y-\eta\|^2\right]t^\alpha\right)^{\frac{k}{2}}}{k!} \\ &= G^\alpha(-x-\xi_1, y-\eta, t), \end{aligned}$$

and

$$K^\alpha(x-\xi, y-\eta, t-\tau) = G^\alpha(x-\xi, y-\eta, t-\tau) - G^\alpha(-x-\xi, y-\eta, t-\tau) = 0 \quad \text{for } x=0,$$

$u(0,y,t)=0$, and thus u is the solution to problem (2.7). □

Now we give some properties in the Fréchet space. Let X be a Fréchet space with a family of semi-norms $\{\|\cdot\|_n\}_{n\in\mathbb{N}}$. Let $Y\subset X$, we say that Y is bounded if for every $n\in\mathbb{N}$ there exists $M_n>0$ such that

$$\|y\|_n \le M_n \quad \text{for all } y\in Y.$$

To X, we associate a sequence of Banach spaces $\{(X^n, \|\cdot\|_n)\}$ as follows. For every $n\in\mathbb{N}$, we consider the equivalence relation $\sim_n$ defined by : $x\sim_n y$ if and only if $\|x-y\|_n=0$ for all $x,y\in X$. We denote by $X^n=(X|_{\sim_n}, \|\cdot\|_n)$ the quotient space, the completion of X^n with respect to $\|\cdot\|_n$. To every $Y\subset X$, we associate a sequence $\{Y^n\}$ of subsets $Y^n\subset X^n$ as follows. For every $x\in X$, we denote by $[x]_n$ the equivalence class of x of subset X^n and define $Y^n=\{[x]_n : x\in Y\}$. We denote by $\overline{Y^n}$, $\text{int}_n(Y^n)$ and $\partial_n Y^n$, respectively, the closure, the interior and the boundary of Y^n with respect to $\|\cdot\|$ in X^n. We assume that the family of semi-norms $\{\|\cdot\|_n\}$ verifies

$$\|x\|_1 \le \|x\|_2 \le \|x\|_3 \le \cdots \quad \text{for every } x\in X.$$

Definition 2.4. [7] A function $f : X\to X$ is said to be a *contraction* if for each $n\in\mathbb{N}$ there exists $k_n\in(0,1)$ such that

$$\|f(x)-f(y)\|_n \le k_n\,\|x-y\|_n \quad \text{for all } x,y\in X.$$

Theorem 2.5 (Nonlinear alternative of Frigon and Granas, [7])**.** *Let X be a Fréchet space, $Y\subset X$ be a closed subset in Y and let $N : Y\to X$ be a contraction such that $N(Y)$ is bounded. Then one of the following statements holds:*

$(S1)$ *N has a unique fixed point,*

$(S2)$ *there exists $\lambda\in[0,1)$, $n\in\mathbb{N}$ and $x\in\partial_n Y^n$ such that $\|x-\lambda\, N\,(x)\|_n=0$.*

3. The first existence result

In this section, we give first the integral representation of the solution of problem (1.1). We should remark that it depends on the heat flow on the boundary S, which is satisfied by a Volterra integral equation that will be specified later.

Theorem 3.1. *The integral representation of the solution of problem* (1.1) *is given by the expression*

$$u(x,y,t)=u_0(x,y,t)-\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\Gamma\left(\frac{-\alpha}{2}k-\frac{\alpha}{2}+1\right)}\int_0^t(t-\tau)^{\frac{\alpha(k-1)}{2}}I_1(x,y,k)\,d\tau, \quad (3.1)$$

where

$$\begin{aligned} I_1(x,y,k) &= \int_{[[-B,B]]} f(V(\eta,\tau))I_2(x,y,\eta,k)\,d\eta,\\ I_2(x,y,\eta,k) &= \left[\int_0^L \left((x-\xi)^2+\|y-\eta\|^2\right)^{\frac{k}{2}}-\left((x+\xi)^2+\|y-\eta\|^2\right)^{\frac{k}{2}}d\xi\right],\\ u_0(x,y,t) &= \int_\Omega K^\alpha(x-\xi,y-\eta,t)h(\xi,\eta)\,d\xi\,d\eta, \end{aligned}$$

and the function V, (heat flux on the surface $x=0$) defined by $V(y,t)=u_x(0,y,t)$ for $y\in]]-B,B[[\subset\mathbb{R}^{d-1}$, and $t>0$, satisfies the following Volterra integral equation:

$$V(y,t)=V_0(y,t)-\sum_{k=0}^{\infty}\frac{(-1)^k}{(k+1)!\Gamma\left(\frac{-\alpha}{2}k-\alpha+1\right)}\int_0^t(t-\tau)^{\frac{\alpha k}{2}}I_3(y,\eta,k)\,d\tau, \quad (3.2)$$

where

$$I_3(y,\eta,k)=\left[\int_{]]-B,B[[} f(V(\eta,\tau))\left(\left[L^2+\|y-\eta\|^2\right]^{\frac{k+1}{2}}-\|y-\eta\|^{k+1}\right)d\eta\right]$$

in the variable $t>0$, with $y\in]]-B,B[[\subset\mathbb{R}^{d-1}$ being a parameter where

$$V_0(y,t)=\int_\Omega K_x^\alpha(0-\xi,y-\eta,t)h(\xi,\eta)\,d\xi\,d\eta.$$

Proof. As the boundary condition in problem (1.1) is homogeneous, then, following [5] for fractional derivative and [6] for non-fractional derivative, we have the integral representation of its solution given on all compact set $\overline{\Omega}$ by the expression

$$\begin{aligned} u(x,y,t) = &\int_\Omega K^\alpha(x-\xi,y-\eta,t)h(\xi,\eta)\,d\xi\,d\eta\\ &-\int_0^t\int_\Omega K^\alpha(x-\xi,y-\eta,t-\tau)f(V(\eta,\tau))\,d\xi\,d\eta\,d\tau. \end{aligned}$$

Thus,

$$u_x(x,y,t) = \int_\Omega K_x^\alpha(x-\xi, y-\eta, t) h(\xi,\eta)\, d\xi\, d\eta + \int_0^t \int_\Omega K_x^\alpha(x-\xi, y-\eta, t-\tau)\left[-f(V(\eta,\tau))\right] d\xi\, d\eta\, d\tau$$

with

$$K^\alpha(x-\xi, y-\eta, t-\tau) = \frac{(t-\tau)^{\frac{-\alpha}{2}}}{2}\left[\phi\left(-\frac{\alpha}{2}, -\frac{\alpha}{2}+1; z_-\right) - \phi\left(-\frac{\alpha}{2}, -\frac{\alpha}{2}+1; z_+\right)\right],$$

where

$$z_- = -\left[(x-\xi)^2 + \|y-\eta\|^2\right]^{\frac{1}{2}} (t-\tau)^{\frac{\alpha}{2}}$$
$$z_+ = -\left[(x+\xi)^2 + \|y-\eta\|^2\right]^{\frac{1}{2}} (t-\tau)^{\frac{\alpha}{2}}.$$

Let

$$K_x^\alpha(x-\xi, y-\eta, t-\tau) = \frac{\partial}{\partial x} K^\alpha(x-\xi, y-\eta, t-\tau) + \frac{(t-\tau)^{\frac{-\alpha}{2}}}{2}\left[\phi_x\left(-\frac{\alpha}{2}, -\frac{\alpha}{2}+1; z_-\right) - \phi_x\left(-\frac{\alpha}{2}, -\frac{\alpha}{2}+1; z_+\right)\right],$$

$$\phi_x\left(-\frac{\alpha}{2}, -\frac{\alpha}{2}+1; z_-\right) = -\frac{(x-\xi)(t-\tau)^{\frac{\alpha}{2}}}{\left[(x-\xi)^2 + \|y-\eta\|^2\right]^{\frac{1}{2}}}\phi\left(-\frac{\alpha}{2}, -\alpha+1; z_-\right),$$

$$\phi_x\left(-\frac{\alpha}{2}, -\frac{\alpha}{2}+1; z_+\right) = -\frac{(x+\xi)(t-\tau)^{\frac{\alpha}{2}}}{\left[(x+\xi)^2 + \|y-\eta\|^2\right]^{\frac{1}{2}}}\phi\left(-\frac{\alpha}{2}, -\alpha+1; z_+\right).$$

Then

$$K_x^\alpha(x-\xi, y-\eta, t-\tau) = -\frac{(x-\xi)}{2\left[(x-\xi)^2 + \|y-\eta\|^2\right]^{\frac{1}{2}}}\phi\left(-\frac{\alpha}{2}, -\alpha+1; z_-\right) + \frac{(x+\xi)}{2\left[(x+\xi)^2 + \|y-\eta\|^2\right]^{\frac{1}{2}}}\phi\left(-\frac{\alpha}{2}, -\alpha+1; z_+\right),$$

$$K_x^\alpha(0-\xi, y-\eta, t-\tau) = \frac{\xi}{2\left[\xi^2 + \|y-\eta\|^2\right]^{\frac{1}{2}}}\phi\left(-\frac{\alpha}{2}, -\alpha+1; z_{-,x=0}\right) + \frac{\xi}{2\left[\xi^2 + \|y-\eta\|^2\right]^{\frac{1}{2}}}\phi\left(-\frac{\alpha}{2}, -\alpha+1; z_{+,x=0}\right),$$

where

$$z_{-,x=0} = -\left[\xi^2 + \|y-\eta\|^2\right]^{\frac{1}{2}} (t-\tau)^{\frac{\alpha}{2}},$$
$$z_{+,x=0} = -\left[\xi^2 + \|y-\eta\|^2\right]^{\frac{1}{2}} (t-\tau)^{\frac{\alpha}{2}}.$$

Hence,

$$K_x^\alpha(0-\xi, y-\eta, t-\tau) = \frac{\xi}{\left[\xi^2 + \|y-\eta\|^2\right]^{\frac{1}{2}}}\phi\left(-\frac{\alpha}{2}, -\alpha+1; z_{x=0}\right), \tag{3.3}$$

$$z_{x=0} = -\left[\xi^2 + \|y-\eta\|^2\right]^{\frac{1}{2}} (t-\tau)^{\frac{\alpha}{2}}.$$

Using (3.3), we obtain

$$\begin{aligned}
&\int_\Omega K_x^\alpha(0-\xi, y-\eta, t-\tau) f(V(\eta,\tau))\, d\xi\, d\eta \\
&= \int_\Omega \frac{\xi}{[\xi^2 + \|y-\eta\|^2]^{\frac{1}{2}}} \sum_{k=0}^{\infty} \frac{(-1)^k \left[(\xi^2 + \|y-\eta\|^2)^{\frac{1}{2}} (t-\tau)^{\frac{\alpha}{2}}\right]^k}{k!\Gamma\left(-\frac{\alpha}{2}k - \alpha + 1\right)} f(V(\eta,\tau))\, d\xi\, d\eta \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k (t-\tau)^{\frac{\alpha k}{2}}}{k!\Gamma\left(-\frac{\alpha}{2}k - \alpha + 1\right)} \int_\Omega \xi \left[\xi^2 + \|y-\eta\|^2\right]^{\frac{k-1}{2}} f(V(\eta,\tau))\, d\xi\, d\eta \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k (t-\tau)^{\frac{\alpha k}{2}}}{k!\Gamma\left(-\frac{\alpha}{2}k - \alpha + 1\right)} \int_{]]-B,B[[} f(V(\eta,\tau)) \left[\frac{1}{2}\int_0^L 2\xi \left[\xi^2 + \|y-\eta\|^2\right]^{\frac{k-1}{2}} d\xi\right] d\eta.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\int_\Omega K_x^\alpha(0-\xi, y-\eta, t-\tau) f(V(\eta,\tau))\, d\xi\, d\eta \\
&\qquad = \sum_{k=0}^{\infty} \frac{(-1)^k (t-\tau)^{\frac{\alpha k}{2}}}{(k+1)!\Gamma\left(-\frac{\alpha}{2}k - \alpha + 1\right)} I_3(y,\eta,k).
\end{aligned}$$

We have $V(y,t) = u_x(0,y,t)$ and

$$\begin{aligned}
u_x(x,y,t) &= \int_\Omega K_x^\alpha(x-\xi, y-\eta, t) h(\xi,\eta)\, d\xi\, d\eta \\
&\quad - \int_0^t \int_\Omega K_x^\alpha(x-\xi, y-\eta, t-\tau) f(V(\eta,\tau))\, d\xi\, d\eta\, d\tau, \\
u_x(0,y,t) &= \int_\Omega K_x^\alpha(0-\xi, y-\eta, t) h(\xi,\eta)\, d\xi\, d\eta \\
&\quad - \int_0^t \int_\Omega K_x^\alpha(0-\xi, y-\eta, t-\tau) f(V(\eta,\tau))\, d\xi\, d\eta\, d\tau.
\end{aligned}$$

Suppose

$$V_0(y,t) = \int_\Omega K_x^\alpha(0-\xi, y-\eta, t) h(\xi,\eta)\, d\xi\, d\eta,$$

then

$$V(y,t) = V_0(y,t) - \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)!\Gamma\left(-\frac{\alpha}{2}k - \alpha + 1\right)}. \int_0^t (t-\tau)^{\frac{\alpha k}{2}} I_3(y,\eta,k)\, d\tau,$$

where we recall that

$$I_3(y,\eta,k) = \int_{]]-B,B[[} f(V(\eta,\tau)) \left[\left(L^2 + \|y-\eta\|^2\right)^{\frac{k+1}{2}} - \|y-\eta\|^{k+1}\right] d\eta.$$

By setting

$$\phi_\alpha(z) = \phi\left(\frac{-\alpha}{2}, \frac{-\alpha}{2}+1; z\right),$$

we have

$$\begin{aligned}
&\int_\Omega K^\alpha(x-\xi, y-\eta, t-\tau) f(V(\eta,\tau))\, d\xi\, d\eta \\
&\quad = \frac{(t-\tau)^{\frac{-\alpha}{2}}}{2} \int_\Omega \left[\phi_\alpha(z_-) - \phi_\alpha(z_+)\right] f(V(\eta,\tau))\, d\xi\, d\eta \\
&\quad = \frac{(t-\tau)^{\frac{-\alpha}{2}}}{2} \left[\int_\Omega \phi_\alpha(z_-) f(V(\eta,\tau)) d\xi d\eta - \int_\Omega \phi_\alpha(z_+) f(V(\eta,\tau))\, d\xi\, d\eta\right] \\
&\quad = \frac{(t-\tau)^{\frac{-\alpha}{2}}}{2} I, \\
&I = I_1 - I_2, \\
&I_1 = \sum_{k=0}^{\infty} \frac{(-1)^k (t-\tau)^{\frac{k\alpha}{2}}}{k!\Gamma\left(-\frac{\alpha}{2}k - \frac{\alpha}{2}+1\right)} \\
&\qquad\qquad \times \int_{[[-B,B]]} f(V(\eta,\tau)) \left[\int_0^L \left[(x-\xi)^2 + \|y-\eta\|^2\right]^{\frac{k}{2}} d\xi\right] d\eta \\
&I_2 = \sum_{k=0}^{\infty} \frac{(-1)^k (t-\tau)^{\frac{k\alpha}{2}}}{k!\Gamma\left(-\frac{\alpha}{2}k - \frac{\alpha}{2}+1\right)} \\
&\qquad\qquad \times \int_{[[-B,B]]} f(V(\eta,\tau)) \left[\int_0^L \left[(x+\xi)^2 + \|y-\eta\|^2\right]^{\frac{k}{2}} d\xi\right] d\eta.
\end{aligned}$$

Thus,

$$I = \sum_{k=0}^{\infty} \frac{(-1)^k (t-\tau)^{\frac{k\alpha}{2}}}{k!\Gamma\left(-\frac{\alpha}{2}k - \frac{\alpha}{2}+1\right)} \int_{]]-B,B[[} f(V(\eta,\tau)) I_4(x,y,\eta,k)\, d\eta,$$

with

$$I_4(x,y,\eta,k) = \int_0^L \left[(x-\xi)^2 + \|y-\eta\|^2\right]^{\frac{k}{2}} - \left[(x+\xi)^2 + \|y-\eta\|^2\right]^{\frac{k}{2}} d\xi.$$

Then

$$\begin{aligned}
&\int_\Omega K^\alpha(x-\xi, y-\eta, t-\tau) f(V(\eta,\tau))\, d\xi\, d\eta \\
&\qquad = \sum_{k=0}^{\infty} \frac{(-1)^k (t-\tau)^{\frac{\alpha(k-1)}{2}}}{k!\Gamma\left(-\frac{\alpha}{2}k - \frac{\alpha}{2}+1\right)} \int_{]]-B,B[[} f(V(\eta,\tau)) I_4(x,y,\eta,k)\, d\eta
\end{aligned}$$

and

$$u(x,y,t) = u_0(x,y,t) - \sum_{k=0}^{\infty} \frac{(-1)^k \left[\int_0^t (t-\tau)^{\frac{\alpha(k-1)}{2}} \int_{[[-B,B]]} f(V(\eta,\tau)) I_4\, d\eta\, d\tau\right]}{k!\Gamma\left(-\frac{\alpha}{2}k - \frac{\alpha}{2}+1\right)},$$

where
$$u_0(x,y,t)=\int_\Omega K^\alpha(x-\xi,y-\eta,t)h(\xi,\eta)\,d\xi\,d\eta. \qquad \square$$

Lemma 3.2. *The simplified form of the Volterra integral equation (3.2) is*

$$V(y,t)=P(y,t)+\int_0^t Q(y,t,\tau,V(y,\tau),\alpha)\,d\tau \tag{3.4}$$

with

$$\begin{aligned}P(y,t)&=\sum_{k=0}^{\infty}\frac{(-1)^k t^{\frac{\alpha k}{2}}}{k!\Gamma\left(-\frac{\alpha}{2}k-\alpha+1\right)}\int_\Omega \xi\left[\xi^2+\|y-\eta\|^2\right]^{\frac{k-1}{2}}h(\xi,\eta)\,d\xi\,d\eta\\&=\int_\Omega\frac{\xi}{\xi^2+\|y-\eta\|^2}\phi\left(\frac{-\alpha}{2},-\alpha+1;-\left[\xi^2+\|y-\eta\|^2\right]^{\frac12}t^{\frac{\alpha}{2}}\right)h(\xi,\eta)\,d\xi\,d\eta\end{aligned}$$

and
$$Q(y,t,\tau,V(y,\tau),\alpha)=\sum_{k=0}^{\infty}\frac{(-1)^{k+1}(t-\tau)^{\frac{\alpha k}{2}}I_3(y,\tau,k)}{(k+1)!\Gamma\left(-\frac{\alpha}{2}k-\alpha+1\right)}.$$

Proof. From (3.3), if $\tau=0$, we get

$$\begin{aligned}K_x^\alpha(-\xi,y-\eta,t)&=\frac{\xi}{\left[\xi^2+\|y-\eta\|^2\right]^{\frac12}}\sum_{k=0}^{\infty}\frac{(-1)^k\left[\xi^2+\|y-\eta\|^2\right]^{\frac{k}{2}}t^{\frac{k\alpha}{2}}}{k!\Gamma\left(-\frac{\alpha}{2}k-\alpha+1\right)}\\&=\xi\sum_{k=0}^{\infty}\frac{(-1)^k\left[\xi^2+\|y-\eta\|^2\right]^{\frac{k-1}{2}}t^{\frac{k\alpha}{2}}}{k!\Gamma\left(-\frac{\alpha}{2}k-\alpha+1\right)}.\end{aligned}$$

Then (3.2) becomes

$$\begin{aligned}V(y,t)&=V_0(y,t)-\sum_{k=0}^{\infty}\frac{(-1)^k}{(k+1)!\Gamma\left(-\frac{\alpha}{2}k-\alpha+1\right)}\int_0^t(t-\tau)^{\frac{\alpha k}{2}}I_3(y,\tau,k)\,d\tau,\\V_0(y,t)&=\int_\Omega K_x^\alpha(-\xi,y-\eta,t)h(\xi,\eta)\,d\xi\,d\eta.\end{aligned}$$

Hence,

$$\begin{aligned}V(y,t)=&\sum_{k=0}^{\infty}\frac{(-1)^k t^{\frac{\alpha k}{2}}}{k!\Gamma\left(-\frac{\alpha}{2}k-\alpha+1\right)}\int_\Omega\xi\left[\xi^2+\|y-\eta\|^2\right]^{\frac{k-1}{2}}h(\xi,\eta)\,d\xi\,d\eta\\&-\sum_{k=0}^{\infty}\frac{(-1)^k}{(k+1)!\Gamma\left(-\frac{\alpha}{2}k-\alpha+1\right)}\int_0^t(t-\tau)^{\frac{\alpha k}{2}}I_3(y,\tau,k)\,d\tau,\end{aligned}$$

where we recall

$$I_3(y,\tau,k)=\int_{B]]-B,B[[}f(V(\eta,\tau))\left[\left(L^2+\|y-\eta\|^2\right)^{\frac{k+1}{2}}-\|y-\eta\|^{k+1}\right]d\eta,$$

and thus $V(y,t)$ is given by (3.4). $\square$

Now we prove, under some assumption on the data, that there exists a unique solution of problem (1.1) locally in time which can be extended globally in time.

Theorem 3.3. *Assume that $h \in C(\mathbb{R}^{d-1} \times \mathbb{R}^+)$, $f \in C(\mathbb{R})$ and there exists $\lambda > 0$ such that*

$$\left|f(X) - f(\bar{X})\right| \leq \lambda |X - \bar{X}| \quad \textit{for each } X, \bar{X} \in \mathfrak{B} \subset \mathbb{R}$$

holds. Then there exists a unique solution of problem (1.1) locally in time, which can be extended globally in time.

Proof. Theorem 3.1 yields that to prove the existence and uniqueness of the solution (3.1) of problem (1.1), it is enough to solve the Volterra integral equation (3.4). We have to check the conditions (H1)–(H4) in Theorem 1.1, p. 87, (H5)–(H6) in Theorem 1.2, p. 91, and Theorem 2.3, p. 97 in [15].

- The function P is defined and continuous for all $(y,t) \in \mathbb{R}^{d-1} \times \mathbb{R}^+$ since $h \in C(\mathbb{R}^{d-1} \times \mathbb{R}^+)$, and thus (H1) holds.

- The function Q is measurable in (y, t, τ, X, α) for all $0 < \alpha < 1$, $0 \leq \tau \leq t < \infty$, $X \in \mathbb{R}$, $y \in \mathbb{R}^{d-1}$ and continuous in X for all (y, t, τ, α) in $\mathbb{R}^{d-1} \times \mathbb{R}^+ \times \mathbb{R}^+ \times (0, 1)$, $Q(y, t, \tau, X, \alpha) = 0$ if $\tau > t$, and thus (H2) holds.

- For each real number $T > 0$ and each bounded set $\mathfrak{B}$ in $\mathbb{R}$, we have

$$\begin{aligned}|Q(y,t,\tau,V(y,\tau),\alpha)| &= \left|\sum_{k=0}^{\infty} \frac{(-1)^{k+1}(t-\tau)^{\frac{\alpha k}{2}} I_3(y,\tau,k)}{(k+1)!\Gamma\left(-\frac{\alpha}{2}k - \alpha + 1\right)}\right| \\ &\leq \int_{[[-B,B]]} |f(V(\eta,\tau))| \frac{1}{(t-\tau)^{\frac{\alpha}{2}}} \left|\phi_-(t) - \phi_+(t)\right| d\eta,\end{aligned}$$

where

$$\phi_{\pm}(t) = \phi\left(\frac{-\alpha}{2}, \frac{-\alpha}{2} + 1; \pm(t-\tau)^{\frac{\alpha}{2}} \left(L^2 + \|y - \eta\|^2\right)^{\frac{1}{2}}\right).$$

Thus,

$$\begin{aligned}|Q(y,t,\tau,V(y,\tau),\alpha)| &\leq \sup_{X \in \mathfrak{B}} |f(X)| 2|\phi| \frac{1}{(t-\tau)^{\frac{\alpha}{2}}} \int_{[[-B,B]]} d\eta \\ &\leq \sup_{X \in \mathfrak{B}} |f(X)| 2|\phi| \frac{1}{(t-\tau)^{\frac{\alpha}{2}}} \mu([[-B,B]])\end{aligned}$$

$\mu([[-B,B]])$ measure of $[[-B,B]]$. So, there exists a measurable function m given by

$$m(t,\tau) = \sup_{X \in \mathfrak{B}} |f(X)| 2|\phi| \frac{1}{(t-\tau)^{\frac{\alpha}{2}}} \mu([[-B,B]])$$

such that

$$|Q(y,t,\tau,V(y,\tau),\alpha)| \leq m(t,\tau)$$

and satisfies

$$\sup_{t\in[0,T]}\int_0^t m(t,\tau)d\tau = \sup_{X\in\mathfrak{B}}|f(X)|2|\phi|\mu([[-B,B]])\sup_{t\in[0,T]}\int_0^t \frac{1}{(t-\tau)^{\frac{\alpha}{2}}}\,d\tau$$
$$= \sup_{X\in\mathfrak{B}}|f(X)|2|\phi|\mu([[-B,B]])\sup_{t\in[0,T]}\left[\frac{2}{2-\alpha}t^{\frac{-\alpha}{2}+1}\right] < \infty.$$

Moreover, we also have

$$\lim_{t\to 0^+}\int_0^t m(t,\tau)\,d\tau$$
$$= \sup_{X\in\mathfrak{B}}|f(X)|2|\phi|\mu([[-B,B]])\lim_{t\to 0^+}\left[\frac{2}{2-\alpha}t^{\frac{-\alpha}{2}+1}\right] = 0, \quad (3.5)$$

and as

$$\int_\nu^{\nu+t} m(t,\tau)\,d\tau$$
$$= \sup_{X\in\mathfrak{B}}|f(X)|2|\phi|\mu([[-B,B]])\lim_{t\to 0^+}\left[\frac{-2}{2-\alpha}\left(-\nu^{\frac{-\alpha}{2}+1} - (t-\nu)^{\frac{-\alpha}{2}+1}\right)\right],$$

then

$$\lim_{t\to 0^+}\int_\nu^{\nu+t} m(t,\tau)\,d\tau = 0. \quad (3.6)$$

Thus (H3) holds.

- For each compact subinterval J of $\mathbb{R}^+$, each bounded set $\mathfrak{B}$ in $\mathbb{R}$ and each $t_0 \in \mathbb{R}^+$, let

$$\bar{Q}(t,y,V(y),\alpha) = \int_J |Q(y,t,\tau,V(y,\tau),\alpha) - Q(y,t_0,\tau,V(y,\tau),\alpha)|\,d\tau$$
$$= \int_J\left|\int_{[[-B,B]]} f(V(\eta,\tau))\left(\frac{\phi_-(t)-\phi_+(t)}{(t-\tau)^{\frac{\alpha}{2}}} - \frac{\phi_-(t_0)-\phi_+(t_0)}{(t_0-\tau)^{\frac{\alpha}{2}}}\right)d\eta\right|d\tau,$$

as the function $\tau \to V(\eta,\tau)$ is continuous and is in the compact $\mathfrak{B}\subset\mathbb{R}$ for all $\eta\in[[-B,B]]$. So, by the continuity of f, we have $f(V(\eta,\tau))\subset f(\mathfrak{B})$. Then there exists a constant $M>0$ such that $|f(V(\eta,\tau))|\le M$ $(\eta,\tau)\in[[-B,B]]\times\mathbb{R}^+$. Suppose

$$z(t,\tau,L,y,\eta) = -(t-\tau)^{\frac{\alpha}{2}}\left(L^2+\|y-\eta\|^2\right)^{\frac{1}{2}},$$
$$\bar{z}(t,\tau,y,\eta) = -(t-\tau)^{\frac{\alpha}{2}}\|y-\eta\|,$$
$$|\bar{Q}(t,y,V(y),\alpha)|$$
$$\le M\int_J\int_{[[-B,B]]}\left|\frac{\phi_-(t,L)-\phi_+(t,0)}{(t-\tau)^{\frac{\alpha}{2}}} - \frac{\phi_-(t_0,L)-\phi_+(t_0,0)}{(t_0-\tau)^{\frac{\alpha}{2}}}\right|d\eta\,d\tau.$$

As the right-hand side of the above inequality tends to zero for $t \to t_0$, then

$$\lim_{t\to t_0} \sup_{V(y)\in C(J,\mathfrak{B})} \int_J |Q(y,t,\tau,V(y,\tau),\alpha) - Q(y,t_0,\tau,V(y,\tau),\alpha)|\, d\tau = 0.$$

Thus (H4) holds.

- For each compact interval $I \subset \mathbb{R}^+$, each continuous function $\varphi : I \to \mathbb{R}$ and all $t_0 > 0$,

$$\int_J |Q(y,t,\tau,\varphi(\tau),\alpha) - Q(y,t_0,\tau,\varphi(\tau),\alpha)|\, d\tau$$
$$= \int_J \left| \int_{[[-B,B]]} f(\varphi(\tau)) \left[\frac{[\phi(z,L,t) - \phi(\bar{z},0,t]}{(t-\tau)^{\frac{\alpha}{2}}} - \frac{[\phi(z,L,t_0) - \phi(\bar{z},0,t_0]}{(t_0-\tau)^{\frac{\alpha}{2}}} \right] d\eta\, d\tau \right|,$$

where

$$\phi(z,L,t) = \phi\left(\frac{-\alpha}{2}, \frac{-\alpha}{2}+1; z(t,\tau,L,y,\eta)\right).$$

As $f \in C(\mathbb{R})$ and $\varphi \in C(I,\mathbb{R})$, then there exists a real $M > 0$ such that $|f(\varphi(\tau))| \le M$ for all $\tau \in I$. Then

$$\int_J |Q(y,t,\tau,\varphi(\tau),\alpha) - Q(y,t_0,\tau,\varphi(\tau),\alpha)|\, d\tau$$
$$\le M \int_J \int_{[[-B,B]]} \left| \frac{[\phi(z,L,t) - \phi(\bar{z},0,t]}{(t-\tau)^{\frac{\alpha}{2}}} - \frac{[\phi(z,L,t_0) - \phi(\bar{z},0,t_0]}{(t_0-\tau)^{\frac{\alpha}{2}}} \right| d\eta\, d\tau.$$

As the right-hand side of the above inequality tends to zero for $t \to t_0$, then

$$\lim_{t\to t_0} \int_J |Q(y,t,\tau,\varphi(\tau),\alpha) - Q(y,t_0,\tau,\varphi(\tau),\alpha)|\, d\tau = 0.$$

Thus (H5) holds.

- For each constant $T > 0$ and each bounded set $\mathfrak{B} \subset \mathbb{R}$, there exists a measurable function $k(t,\tau,\alpha)$ such that

$$\left|Q(y,t,\tau,X,\alpha) - Q(y,t,\tau,\bar{X},\alpha)\right| \le k(t,\tau,\alpha)|X - \bar{X}|,$$

whenever $0 \le \tau \le t \le T$ and both X and $\bar{X}$ are in $\mathfrak{B}$. Then we have

$$\left|Q(y,t,\tau,X,\alpha) - Q(y,t,\tau,\bar{X},\alpha)\right| = \left| \int_{[[-B,B]]} \frac{[f(X) - f(\bar{X})][\phi(z) - \phi(\bar{z})]}{(t-\tau)^{\frac{\alpha}{2}}}\, d\eta \right|$$
$$\le \int_{[[-B,B]]} \frac{[f(X) - f(\bar{X})][\phi(z)| + |\phi(\bar{z})]}{(t-\tau)^{\frac{\alpha}{2}}}\, d\eta.$$

Since f is locally Lipschitz function in $\mathbb{R}$, then

$$\left|Q(y,t,\tau,X,\alpha) - Q(y,t,\tau,\bar{X},\alpha)\right| \le \frac{1}{(t-\tau)^{\frac{\alpha}{2}}} 2|\phi|\mu([[-B,B]])\lambda|X - \bar{X}|,$$

and thus

$$k(t,\tau,\alpha) = \frac{\lambda 2|\phi|\mu([[-B,B]])}{(t-\tau)^{\frac{\alpha}{2}}}.$$

We also have for each $t \in [0,T]$ the function $k \in L^1(0,T)$ as a function of τ, and

$$\begin{aligned}\int_t^{t+\nu} k(t+\nu,\tau,\alpha)\,d\tau &= \lambda 2|\phi|\mu([[-B,B]]) \int_t^{t+\nu} \frac{1}{(t+\nu-\tau)^{\frac{\alpha}{2}}}\,d\tau \\ &= \frac{4\lambda|\phi|\mu([[-B,B]])}{2-\alpha}(\nu)^{\frac{-\alpha}{2}+1},\end{aligned}$$

then

$$\lim_{\nu\to 0}\int_t^{t+\nu} k(t+\nu,\tau,\alpha)\,d\tau = \lim_{\nu\to 0}\frac{4\lambda|\phi|\mu([[-B,B]])}{2-\alpha}(\nu)^{\frac{-\alpha}{2}+1} = 0.$$

Thus (H6) holds. Consequently, all the conditions from (H1) to (H6) are satisfied with (3.5) and (3.6).

From [15] (see Theorem 1.1, p. 87, Theorem 1.2, p. 91 and Theorem 2.3, p. 97), it follows that there exists a unique local-in-time solution of the Volterra integral equation (3.2) which can be extended globally in time. Then the proof of this theorem is complete. □

4. The second existence result

We consider the following problem:

$$\begin{cases} \left({}^cD^{\alpha}_{0^+,t}u\right)(x,y,t) = \Delta_{x,y}u(x,y,t) - f\left(t,u(x,y,t)\right), \ (x,y)\in\Omega,\ t>0 \\ u(0,y,t)=0; \quad y \in]]-B,B[[\subset \mathbb{R}^{d-1},\ t>0 \\ u(x,y,0)=h(x,y), \quad (x,y)\in\Omega, \\ u \text{ is } L_i-\text{periodic with respect to } x_i \text{ for } i=1,\cdots,d. \end{cases} \tag{4.1}$$

As the boundary condition in problem (4.1) is homogeneous, so, following [5] for fractional derivative, and [6] for non-fractional derivative, we have the integral representation of its solution given on all compact set $\overline{\Omega}$ by the expression

$$\begin{aligned} u(x,y,t) &= \int_{\mathbb{R}^+\times\mathbb{R}^{d-1}} K^{\alpha}(x-\xi,y-\eta,t)h(\xi,\eta)\,d\xi\,d\eta \\ &\quad - \int_0^t\int_{\mathbb{R}^+\times\mathbb{R}^{d-1}} K^{\alpha}(x-\xi,y-\eta,t-\tau)f(\tau,u(\xi,\eta,\tau))\,d\xi\,d\eta\,d\tau, \end{aligned} \tag{4.2}$$

where

$$\begin{aligned} &K^{\alpha}(x-\xi,y-\eta,t-\tau) \\ &\qquad = \frac{(t-\tau)^{\frac{-\alpha}{2}}}{2}\left[\phi\left(-\frac{\alpha}{2},-\frac{\alpha}{2}+1;z_-\right)-\phi\left(-\frac{\alpha}{2},-\frac{\alpha}{2}+1;z_+\right)\right] \\ &\qquad z_- = -\left[(x-\xi)^2+\|y-\eta\|^2\right]^{\frac{1}{2}}(t-\tau)^{\frac{\alpha}{2}}, \end{aligned}$$

$$z_+ = -\left[(x+\xi)^2 + \|y-\eta\|^2\right]^{\frac{1}{2}} (t-\tau)^{\frac{\alpha}{2}},$$

with $\|\cdot\|$ being the norm of $\mathbb{R}^{d-1}$. This integral representation (4.2) is also an integral equation. So we can also prove that there exists a unique solution of the integral representation (4.2) locally in time which can be extended globally in time by using Theorem 1.1, page 87 and Theorem 1.2, page 91 in [15]. In the first subsection, we prove the existence result using Schauder's fixed point theorem [9] and then, in the second subsection, we prove the uniqueness of its solution using a consequence of the nonlinear alternative theorem [7].

4.1. The existence result obtained by using the diagonalization method. Let $J = [0,\infty)$ and $J_n = [0,n]$, $n \in \mathbb{N}$. We will need the following hypotheses which are assumed hereafter :

(H7) $f : J \times \mathbb{R}^d \to \mathbb{R}^d$ is a Carathéodory function such that for any positive constant $C > 0$ we have

$$|f(t,u(x,y,t))| \leq \frac{1}{2nC}\int_0^t \|u(s)\|_\infty \, ds,$$

where $\|u(s)\|_\infty = \sup_{(x,y)\in\Omega} |u(x,y,s)|$.

(H8) Let $h \in L^1(\Omega)$.

Theorem 4.1. *Assume that (H7), (H8) and the condition*

$$\|\phi\|_\infty < C, \tag{4.3}$$

where $\|\phi\|_\infty = \sup_{\Omega\times J} |\phi(x,t)|$, *hold. Then problem* (4.1) *has at least one solution.*

Proof. Fix $n \in \mathbb{N}$ and recall problem (4.1) for $t \in J_n$:

$$\begin{cases} \left({}^cD^\alpha_{0+,t}u\right)(x,y,t) = \Delta_{x,y}u(x,y,t) - f\left(\tau, u(x,y,t)\right), \ (x,y)\in\Omega, \ t\in J_n, \\ u(0,y,t) = 0; \quad y \in]]-B,B[[, \quad t \in J_n, \\ u(x,y,0) = h(x,y), \quad (x,y)\in\Omega, \\ u \text{ is } L_i - \text{periodic with respect to } x_i \text{ for } i = 1,\dots,d. \end{cases} \tag{4.4}$$

The proof will be given in two parts. In the first part we show that problem (4.4) has a solution $u_n \in C_n(\Omega\times J_n, \mathbb{R}^d)$. We define the operator $N : C_n(\Omega \times J_n, \mathbb{R}^d) \to C_n(\Omega\times J_n, \mathbb{R}^d)$ by

$$\begin{aligned} N(u)(x,y,t) = &\int_\Omega K^\alpha(x-\xi, y-\eta, t)h(\xi,\eta)\, d\xi\, d\eta \\ &- \int_0^t \int_\Omega K^\alpha(x-\xi,y-\eta,t-\tau) f(\tau, u(\xi,\eta,\tau))\, d\xi\, d\eta\, d\tau. \end{aligned}$$

Step 1. *N maps $B_n = \left\{u \in C_n(\Omega \times J_n, \mathbb{R}^d) : \|u\|_\star \le r_n\right\}$ into itself.* Consider the norm

$$\|u\|_\star := \frac{1}{n}\int_0^n \|u(t)\|_\infty dt.$$

For u in B_n from conditions (H7), (H8) we get

$$\begin{aligned}|N(u)(x,y,t)| \le{}& \frac{t^{\frac{-\alpha}{2}}}{2} 2\|\phi\|_\infty \|h\|_{L^1(\Omega)} \\ &+ 2\|\phi\|_\infty \int_0^t \frac{(t-\tau)^{\frac{-\alpha}{2}}}{2} \int_\Omega |f(\tau, u(\xi,\eta,\tau))|\, d\xi\, d\eta \\ \le{}& t^{\frac{-\alpha}{2}} \|\phi\|_\infty \|h\|_{L^1(\Omega)} \\ &+ \|\phi\|_\infty \int_0^n (t-\tau)^{\frac{-\alpha}{2}} \left(\frac{1}{2nC}\int_0^\tau \|u(s)\|_\infty ds\right) d\tau,\end{aligned}$$

by the Fubini theorem,

$$\begin{aligned}&\int_0^n (t-\tau)^{\frac{-\alpha}{2}} \left(\frac{1}{2nC}\int_0^\tau \|u(s)\|_\infty ds\right) d\tau \\ &\qquad = \frac{1}{2nC}\int_0^\tau \|u(s)\|_\infty \left(\int_0^n (t-\tau)^{\frac{-\alpha}{2}} d\tau\right) ds.\end{aligned}$$

Moreover, for $0<\alpha<1$, we obtain

$$\left|\int_0^n (t-\tau)^{\frac{-\alpha}{2}} d\tau\right| = \left|\frac{n^{1-\frac{\alpha}{2}}}{1-\frac{\alpha}{2}} - \frac{(t-n)^{1-\frac{\alpha}{2}}}{1-\frac{\alpha}{2}}\right| \le \left|\frac{n^{1-\frac{\alpha}{2}}}{1-\frac{\alpha}{2}} + \frac{|t-n|^{1-\frac{\alpha}{2}}}{1-\frac{\alpha}{2}}\right| < 2n$$

and also

$$\int_0^n t^{-\frac{\alpha}{2}}\, dt = \frac{n^{1-\frac{\alpha}{2}}}{1-\frac{\alpha}{2}} \le 2n.$$

Thus,

$$\int_0^n \|N(u)(t)\|_\infty\, dt \le 2n\|\phi\|_\infty)\|h\|_{L^1(\Omega)} + \|\phi\|_\infty \frac{1}{2C} 2n\|u\|_\star,$$

then

$$\frac{1}{n}\int_0^n \|N(u)(t)\|_\infty\, dt \le 2\|\phi\|_\infty \|h\|_{L^1(\Omega)} + \|\phi\|_\infty \frac{1}{C}\|u\|_\star.$$

So,

$$\|N(u)\|_\star \le 2\|\phi\|_\infty \|h\|_{L^1(\Omega)} + \frac{1}{C}\|\phi\|_\infty \|u\|_\star.$$

Under condition (4.3), we deduce that there exists $r_n > 0$ such that $\|Nu\|_\star \le r_n$, with

$$r_n \ge \frac{2\|\phi\|_\infty \|h\|_{L^1(\Omega)}}{1 - \|\phi\|_\infty / C},$$

which means that $N(B_n) \subset B_n$.

Step 2. *N is continuous.* let $\{u_k\}_{k\in\mathbb{N}}$ be a sequence such that $u_k \to u$ in B_n as $k \to \infty$. Then

$$\begin{aligned}|(Nu_k - Nu)(x,y,t)| &\le \int_0^t \int_\Omega |K^\alpha(x-\xi, y-\eta, t-\tau)||f(\tau,u_k) - f(\tau,u)|\, d\xi\, d\eta\, d\tau \\ &\le \int_0^t \left| \frac{(t-\tau)^{\frac{-\alpha}{2}}}{2} \right| \int_\Omega |\phi_- - \phi_+|\, |f(\tau,u_k) - f(\tau,u)|\, d\xi\, d\eta\, d\tau,\end{aligned}$$

where

$$\begin{aligned}f(\tau,u_k) - f(\tau,u) &= f(\tau,u_k(\xi,\eta,\tau)) - f(\tau,u(\xi,\eta,\tau)) \\ \phi_- - \phi_+ &= \phi\left(\frac{-\alpha}{2}, -\frac{\alpha}{2}+1; z_-\right) - \phi\left(\frac{-\alpha}{2}, -\frac{\alpha}{2}+1; z_+\right).\end{aligned}$$

Bearing in the mind (H7), we have by the Lebesgue dominated convergence theorem that $|(Nu_k)(x,y,t) - (Nu)(x,y,t)| \to 0$, as $k \to \infty$, for all $(x,y,t) \in \Omega \times J_n$. Consequently, $\lim_{k\to\infty} \|Nu_k - Nu\|_\infty = 0$, which implies that

$$\|Nu_k - Nu\|_\star = \frac{1}{n}\int_0^n \|Nu_k(t) - Nu(t)\|_\infty\, dt \le \|Nu_k - Nu\|_\infty.$$

So, $\lim\limits_{k\to\infty} \|Nu_k - Nu\|_\star = 0$, and thus N is continuous for $\|\cdot\|_\star$.

Step 3. *$N(B_n)$ is equicontinuous.* Indeed, let $(x_1,y_1,t_1),(x_2,y_2,t_2) \in \Omega \times J_n$ with $x_1 < x_2$, $y_1 < y_2$, $t_1 < t_2$ and let $u \in B_n$. Then

$$\begin{aligned}(N(u)(x_2,y_2,t_2) - (Nu)(x_1,y_1,t_1) &= \int_\Omega [K_2^\alpha(t_2) - K_1^\alpha(t_1)]\, h(\xi,\eta)\, d\xi\, d\eta \\ &\quad - \int_0^{t_2}\int_\Omega K_2^\alpha(t_2-\tau) f(\tau, u(\xi,\eta,\tau))\, d\xi\, d\eta\, d\tau \\ &\quad + \int_0^{t_1}\int_\Omega K_1^\alpha(t_1-\tau) f(\tau, u(\xi,\eta,\tau))\, d\xi\, d\eta\, d\tau,\end{aligned}$$

where $K_i^\alpha(t_i) = K^\alpha(x_i - \xi, y_i - \eta, t_i)$ for $i = 1,2$, so

$$\begin{aligned}(N(u)(x_2,y_2,t_2) - (Nu)(x_1,y_1,t_1) &= \int_\Omega [K_2^\alpha(t_2) - K_1^\alpha(t_1)]\, h(\xi,\eta)\, d\xi\, d\eta \\ &- \int_0^{t_1}\int_\Omega [K_2^\alpha(t_2-\tau) - K_1^\alpha(t_1-\tau)]\, f(\tau, u(\xi,\eta,\tau))\, d\xi\, d\eta\, d\tau \\ &+ \int_{t_1}^{t_2}\int_\Omega K_2^\alpha(t_2-\tau) f(\tau, u(\xi,\eta,\tau))\, d\xi\, d\eta\, d\tau.\end{aligned}$$

Thus,

$$\begin{aligned}|N(u)(x_2,y_2,t_2) - (Nu)(x_1,y_1,t_1)| &\le \int_\Omega |K_2^\alpha(t_2) - K_1^\alpha(t_1)||h(\xi,\eta)|\, d\xi\, d\eta \\ &\quad + \int_0^{t_1}\int_\Omega |K_2^\alpha(t_2-\tau) - K_1^\alpha(t_1-\tau)||f(\tau, u(\xi,\eta,\tau)|\, d\xi\, d\eta\, d\tau\end{aligned}$$

$$
\begin{aligned}
&+\int_{t_1}^{t_2}\int_\Omega |K_2^\alpha(t_2-\tau))f(\tau,u(\xi,\eta,\tau))|\,d\xi\,d\eta\,d\tau\\
\le& \sup_{(\xi,\eta)\in\Omega}|K_2^\alpha(t_2)-K_1^\alpha(t_1)|\|h\|_{L^1(\Omega)}\\
&+n\sup_{(\xi,\eta)\in\Omega}|K_2^\alpha(t_2-\tau)-K_1^\alpha(t_1-\tau)|\int_\Omega |f(\tau,u(\xi,\eta,\tau)|\,d\xi\,d\eta\\
&+(t_2-t_1)\sup_{(\xi,\eta)\in\Omega}|K_2^\alpha(t_2-\tau)|\int_\Omega |f(\tau,u(\xi,\eta,\tau))|\,d\xi\,d\eta\\
\le& \sup_{(\xi,\eta)\in\Omega}|K_2^\alpha(t_2)-K_1^\alpha(t_1)|\|h\|_{L^1(\Omega)}\\
&+n\sup_{(\xi,\eta)\in\Omega}|K_2^\alpha(t_2-\tau)-K_1^\alpha(t_1-\tau)|\frac{1}{2nC}\int_0^t\|u(s)\|_\infty\,ds,\\
&+(t_2-t_1)\sup_{(\xi,\eta)\in\Omega}|K_2^\alpha(t_2-\tau)|\frac{1}{2nC}\int_0^t\|u(s)\|_\infty\,ds,\\
\le& \sup_{(\xi,\eta)\in\Omega}|K_2^\alpha(t_2)-K_1^\alpha(t_1)|\|h\|_{L^1(\Omega)}\\
&+\frac{\|u\|_*}{2C}\Bigg(n\sup_{(\xi,\eta)\in\Omega}|K_2^\alpha(t_2-\tau)-K_1^\alpha(t_1-\tau)|\\
&\qquad\qquad+(t_2-t_1)\sup_{(\xi,\eta)\in\Omega}|K_2^\alpha(t_2-\tau)|\Bigg)\\
\le& \sup_{(\xi,\eta)\in\Omega}|K_2^\alpha(t_2)-K_1^\alpha(t_1)|\|h\|_{L^1(\Omega)}\\
&+\frac{r_n}{2C}\Bigg(n\sup_{(\xi,\eta)\in\Omega}|K_2^\alpha(t_2-\tau)-K_1^\alpha(t_1-\tau)|\\
&\qquad\qquad+(t_2-t_1)\sup_{(\xi,\eta)\in\Omega}|K_2^\alpha(t_2-\tau)|\Bigg). \qquad (4.5)
\end{aligned}
$$

The right-hand side of the above inequality tends to zero as $(x_1,y_1,t_1)\to(x_2,y_2,t_2)$. Thus $N(B_n)$ is equicontinuous.

As a consequence of steps 1 - 3, together with the Arzela–Ascoli theorem, we can conclude that the operator N is completely continuous. Therefore, we deduce from Schauder's fixed point theorem that N has a fixed point $u_n\in B_n$ which is a solution of problem (4.4).

In the second part we use the following diagonalization process. For $k\in\mathbb{N}$, let

$$
u_k(x,y,t)=\begin{cases}u_k(x,y,t), & t\in(0,n_k]\\ u_k(x,y,n_k), & t\in[n_k,\infty).\end{cases}
$$

Here $\{n_k\}_k$ is a sequence of $n_k\in\mathbb{N}^*$ satisfying

$$
0<n_1<n_2<\cdots<n_k<\cdots\uparrow\infty.
$$

Let $\mathcal{S} = \{u_k\}_{k=1}^{\infty}$. For $k \in \mathbb{N}^*$, $s < t$ with s and $t \in (0, n_1]$, we have

$$u_{n_k}(x,y,t) = \int_\Omega K^\alpha(x-\xi, y-\eta, t)h(\xi,\eta)\,d\xi\,d\eta + \int_0^s \int_\Omega K^\alpha(x-\xi, y-\eta, s-\tau) f(\tau, u_{n_k}(\xi,\eta,\tau))\,d\xi\,d\eta\,d\tau.$$

Then we get

$$|u_{n_k}(x,y,t) - u_{n_k}(x,y,s)| \le \int_\Omega |K_2^\alpha(t) - K_1^\alpha(s)||h(\xi,\eta)|\,d\xi\,d\eta + \int_0^s \int_\Omega |K_2^\alpha(t-\tau) - K_1^\alpha(s-\tau)||f(\tau, u_{n_k}(\xi,\eta,\tau)|\,d\xi\,d\eta\,d\tau + \int_s^t \int_\Omega |K_2^\alpha(t-\tau)) f(\tau, u_{n_k}(\xi,\eta,\tau))|\,d\xi\,d\eta\,d\tau.$$

As for (4.5), using (H7) and using the Arzela–Ascoli theorem which guarantees that there is a subsequence $\tilde{\mathbb{N}}_1^*$ of $\mathbb{N}$ and a function $v_1 \in C(\Omega \times J_{n_1}, \mathbb{R}^d)$ with $u_{n_k} \to v_1$ in $C(\Omega \times J_{n_1}, \mathbb{R}^d)$ as $k \to \infty$ through $\tilde{\mathbb{N}}_1^*$. Let $\tilde{\mathbb{N}}_1 = \tilde{\mathbb{N}}_1^* \backslash \{1\}$. Notice that

$$\|u_{n_k}\|_\star \le r_{n_2} \quad \text{for } t \in (0, n_2], \quad k \in \mathbb{N}.$$

Also, for $k \in \mathbb{N}$ and $t, s \in (0, n_2]$, we can use the Arzela–Ascoli theorem which guarantees that there is a subsequence $\tilde{\mathbb{N}}_2^*$ of $\tilde{\mathbb{N}}_1$ and a function $v_2 \in C(\Omega \times J_{n_2}, \mathbb{R}^d)$ with $u_{n_k} \to v_2$ in $C(\Omega \times J_{n_2}, \mathbb{R}^d)$ as $k \to \infty$ through $\tilde{\mathbb{N}}_2^*$. Notice that $v_1 = v_2$ on $(0, n_1]$ since $\tilde{\mathbb{N}}_2^* \subseteq \tilde{\mathbb{N}}_1$. Let $\tilde{\mathbb{N}}_2 = \tilde{\mathbb{N}}_2^* \backslash \{2\}$. Proceed by induction to obtain for $m \in \{3, 4, \ldots\}$ a subsequence $\tilde{\mathbb{N}}_m^*$ of $\tilde{\mathbb{N}}_{m-1}$ and a function $v_m \in C(\Omega \times J_{n_m}, \mathbb{R}^d)$ with $u_{n_k} \to v_m$ in $C(\Omega \times J_{n_m}, \mathbb{R}^d)$ as $k \to \infty$ through $\tilde{\mathbb{N}}_m^*$. Let $\tilde{\mathbb{N}}_m = \tilde{\mathbb{N}}_m^* \backslash \{m\}$.

Define a function w as follows. Fix $t \in (0, \infty)$ and let $m \in \mathbb{N}$ with $s \le n_m$. Subsequently define $w(x,y,t) = v_m(x,y,t)$, then $w \in C(\Omega \times J_\infty, \mathbb{R}^d)$. Again fix $t \in (0, \infty)$ and let $m \in \mathbb{N}$ with $s \le n_m$. Then, for $n \in \tilde{\mathbb{N}}_m$, we have

$$u_{n_k}(x,y,t) = \int_\Omega K^\alpha(x-\xi, y-\eta, t)h(\xi,\eta)\,d\xi\,d\eta - \int_0^t \int_\Omega K^\alpha(x-\xi, y-\eta, t-\tau) f(\tau, u_{n_k}(\xi,\eta,\tau))\,d\xi d\eta\,d\tau.$$

Let $n_k \to \infty$ through $\tilde{\mathbb{N}}_m$ to obtain

$$v_m(x,y,t) = \int_\Omega K^\alpha(x-\xi, y-\eta, t)h(\xi,\eta)\,d\xi\,d\eta - \int_0^t \int_\Omega K^\alpha(x-\xi, y-\eta, t-\tau) f(\tau, v_m(\xi,\eta,\tau))\,d\xi\,d\eta\,d\tau.$$

Thus,

$$w(x,y,t) = \int_\Omega K^\alpha(x-\xi, y-\eta, t)h(\xi,\eta)\,d\xi\,d\eta$$

$$- \int_0^t \int_\Omega K^\alpha(x-\xi, y-\eta, t-\tau) f(\tau, w(\xi,\eta,\tau))\, d\xi\, d\eta\, d\tau.$$

This method can be used for each $s \in (0, n_m]$ and for each $m \in \mathbb{N}$. Thus the constructed function u is a solution of (4.1). This completes the proof of the theorem. □

4.2. The existence and uniqueness in the Fréchet space. Let $M_x, M_t \in \mathbb{N}$ and $M_y \in \mathbb{N}^{d-1}$. Then $C_M((0, M_x)\times]] - M_y, M_y[[\times(0, M_t); \mathbb{R}^d)$ is the Fréchet space equipped with the family of seminorms

$$\|u\|_{C_M} = \int_0^{M_t} \sup\left\{|u(x,y,t)|dt : \quad (x,y) \in (0, M_x)\times]] - M_y, M_y[[\right\}.$$

Further, we present the conditions for the existence and uniqueness of a global solution of problem (4.1). We will need the following hypotheses which are assumed hereafter:

(H9) $h : \Omega \to \mathbb{R}^d$ is a continuous function with

$$h^* = \|h\|_\infty =: \sup_{(x,y)\in\Omega} |h(x,y)| < \infty.$$

(H10) $f : \mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{R}^d$ is continuous and for each $M \in \mathbb{N}$ there exists l_M in $C((0, M_t], \mathbb{R}^+)$ such that for each $(x,y,t) \in (0, M_x)\times]] - M_y, M_y[[\times(0, M_t)$,

$$|f(t, u(x,y,t)) - f(t, \bar{u}(x,y,t))| \le l_M(t)\|u - \bar{u}\|_{C_M}, \qquad u, \bar{u} \in C_M,$$

and $l_M^* = \sup_{t\in(0,M_t]} |l_M(t)|$.

Theorem 4.2. *Let hypotheses (H9) and (H10) hold, and for $M_t > 0$,*

$$\frac{2-\alpha}{4\ell_M^\star} > M_t^{2-\alpha}\|\Phi\|_\infty. \tag{4.6}$$

Then there exists a unique solution for problem (4.1) on $\Omega \times \mathbb{R}^+$.

Proof. We transform problem (4.1) into a fixed point problem considering the operator

$$N : C_M\left((0, M_x)\times]] - M_y, M_y[[\times(0, M_t); \mathbb{R}^d\right) \to C_M\left((0, M_x)\times]] - M_y, M_y[[\times(0, M_t); \mathbb{R}^d\right)$$

defined by

$$N(u)(x,y,t) = \int_\Omega K^\alpha(x-\xi, y-\eta, t) h(\xi,\eta)\, d\xi\, d\eta - \int_0^t \int_\Omega K^\alpha(x-\xi, y-\eta, t-\tau) f(\tau, u(\xi,\eta,\tau))\, d\xi d\eta\, d\tau.$$

Clearly, the fixed points of the operator N are solutions of problem (4.1). We have

$$N(u)(x,y,t) = \int_{\Omega} K^{\alpha}(x-\xi, y-\eta, t)h(\xi,\eta)\, d\xi\, d\eta \\ - \int_0^t \int_{\Omega} K^{\alpha}(x-\xi, y-\eta, t-\tau) f(\tau, u(\xi,\eta,\tau))\, d\xi\, d\eta\, d\tau,$$

A priori estimate. We should start by showing the existence of a priori estimate of the solution of the equation $u = \lambda N(u)$ with $\lambda \in (0,1)$. Thus,

$$|u(x,y,t)| = |\lambda N(u)(x,y,t)| \le \int_{\Omega} |K^{\alpha}(x-\xi, y-\eta, t)|\, |h(\xi,\eta)|\, d\xi\, d\eta \\ + \int_0^t \int_{\Omega} |K^{\alpha}(x-\xi, y-\eta, t-\tau)| \{|f(\tau, u(\xi,\eta,\tau)) - f(\tau,0)| + |f(\tau,0)|\}\, d\xi\, d\eta\, d\tau.$$

From conditions (H9) and (H10) we obtain

$$\sup_{(0,M_x)\times]]-M_y,M_y[[} |N(u)(x,y,s)| \\ \le \frac{t^{-\frac{\alpha}{2}}}{2} 2\|\phi\|_{\infty} h^* + 2\|\phi\|_{\infty} \left(l_M^* \|u\|_{C_M} + f^*\right) \frac{2M_t^{\frac{2-\alpha}{2}}}{2-\alpha}.$$

By integration in times for $\tau \in (0, M_t)$, we have

$$\|u\|_{C_M} \le \frac{2M_t^{\frac{2-\alpha}{2}}}{2-\alpha} \|\phi\|_{\infty} h^* + 2\|\phi\|_{\infty} \left(l_M^* \|u\|_{C_M} + f^*\right) \frac{2M_t^{\frac{4-\alpha}{2}}}{2-\alpha},$$

where $f^* = \sup_{t \in (0,M_t]} |f(t,0)|$. Let $\|u\|_{C_M} \le R_M$,

$$R_M = \frac{\frac{2M_t^{\frac{2-\alpha}{2}}}{2-\alpha} \|\phi\|_{\infty} h^* + 2\|\phi\|_{\infty} f^* \frac{2M_t^{\frac{4-\alpha}{2}}}{2-\alpha}}{1 - \frac{4\|\phi\|_{\infty} M_t^{\frac{4-\alpha}{2}} l_M^*}{2-\alpha}}.$$

Set

$$C_R = \left\{ u \in C_M((0,M_x)\times]]-M_y, M_y[[\times(0,M_t); \mathbb{R}^d), \quad \|u\|_{C_M} \le R_M + 1 \right\}.$$

We will show that $N : C_R \to C_M$ is a contraction map. Indeed, consider $u, \bar{u}$ in C_R. Then, for each $(x,y,t) \in (0,M_x)\times]]-M_y, M_y[[\times(0,M_t)$, we have

$$|N(u)(x,y,t) - N(\bar{u})(x,y,t)| \\ \le \int_0^t \int_{\Omega} |K^{\alpha}|\, |f(\tau, u(\xi,\eta,\tau)) - f(\tau, \bar{u}(\xi,\eta,\tau))|\, d\xi\, d\eta\, d\tau \\ \le \frac{4\|\phi\|_{\infty} M_t^{\frac{2-\alpha}{2}} l_M^*}{2-\alpha} \|u - \bar{u}\|_{C_M},$$

where $K^\alpha = K^\alpha(x-\xi, y-\eta, t-\tau)$. Thus,

$$\|N(u)-N(\bar{u})\|_{C_M} \leq \frac{4\|\phi\|_\infty M_t^{\frac{2-\alpha}{2}} l_M^*}{2-\alpha}\|u-\bar{u}\|_{C_M}.$$

Hence, by (4.6), $N : C_R \to C_M$ is a contraction. By our choice of C_R, there is no $u \in \partial_M C_R^M$ such that $u = \lambda N(u)$ for $\lambda \in (0,1)$. As a consequence of the nonlinear alternative Theorem (2.5) [7], we deduce that N has a unique fixed point u in C_R which is a solution to problem (4.1). □

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Дробові похідні за часом для некласичної задачі теплопровідності

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Розглядається некласичне рівняння теплопровідності з дробовою похідною Капуто за часовою змінною в обмеженій області $\Omega \subset \mathbb{R}^+ \times \mathbb{R}^{d-1}$, для якого постачання енергії залежить від теплового потоку на частині межі $S = \{0\} \times \mathbb{R}^{d-1}$ з однорідною граничною умовою Діріхле на S, періодичністю на інших частинах границі та початковою умовою. Задача мотивована моделюванням регулювання температури в середовищі. Існування розв'язку задачі базується на інтегралі Вольтерра другого роду за часовою змінною t з параметром в $\mathbb{R}^{d-1}$, її розв'язком є тепловий потік $(y, \tau) \mapsto V(y, t) = u_x(0, y, t)$ на S, що також є додатковим невідомим задачі, що розглядається. Установлено, що існує єдиний локальний розв'язок, який можна продовжити глобально у часі.

Ключові слова: некласичне d-вимірне рівняння теплопровідності, дробова похідна Капуто, інтегральне рівняння Вольтерра, існування та єдність розв'язку задачі, інтегральне зображення розв'язку