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Left Invariant Lifted (α, β)-metrics of Douglas Type on Tangent Lie Groups

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In the paper, lifted left invariant (α, β)-metrics of Douglas type on tangent Lie groups are studied. Suppose that g is a left invariant Riemannian metric on a Lie group G, and F is a left invariant (α, β)-metric of Douglas type induced by g. Using vertical and complete lifts, we construct the vertical and complete lifted (α, β)-metrics F^v and F^c on the tangent bundle TG and give necessary and sufficient conditions for them to be of Douglas type. Then the flag curvature of these metrics are studied. Finally, as some special cases, the flag curvatures of F^v and F^c are given for Randers metrics of Douglas type and Kropina and Matsumoto metrics of Berwald type.

Key words: left invariant (α, β)-metric, complete and vertical lifts, flag curvature

Mathematical Subject Classification 2010: 53B21, 22E60, 22E15

1. Introduction

Tangent bundles of differentiable manifolds have great importance in many fields of mathematics and physics. S. Sasaki studied the Riemannian geometry of tangent bundles in his fundamental paper [26] in 1958. He used vertical and horizontal lifts to construct a Riemannian metric on the tangent bundle of a Riemannian manifold (M, g). Another way for constructing Riemannian metrics on the tangent bundle was introduced by Yano and Kobayashi (see [27], [28] and [29]). They used complete and vertical lifts and found many relations between such lifted metrics and the base Riemannian metrics. Asgari and the second author used the same approach to find the relations between the geometry of lifted invariant Riemannian metrics on TG and the left invariant Riemannian metrics on G [4, 5].

Using the lifted invariant Riemannian metrics together with vertical and complete lifts, they constructed two types of left invariant Randers metrics on the tangent bundle of Lie groups and studied their flag curvature in the case of Berwald metric (see [6]). In this work, using the same way, we build left invariant (α, β)-metrics on tangent Lie groups. We give a necessary and sufficient condition for lifted (α, β)-metrics to be of Douglas type and compute their flag curvatures.

Now we give some preliminaries about vertical and complete lifts and also Finsler geometry.

For any m-dimensional smooth manifold M, $\pi : TM \to M$ denotes its tangent bundle. Suppose that X is a vector field on the manifold M and φ_t denotes its flow. Then, on TM, we can define the (local) one-parameter groups of diffeomorphisms in the following two ways:

1. $\phi_t(y) := (T_x\varphi_t)(y)$,
2. $\psi_t(y) := y + tX(x)$

for all $x \in M$ and $y \in T_xM$. The vector field corresponding to the one-parameter groups ϕ_t is named the complete lift of X and denoted by X^c. In a similar way, the one-parameter groups ψ_t determine a vector field X^v on TM which is called the vertical lift of X.

Suppose that $(\pi^{-1}(U), (x^1, \dots, x^n; y^1, \dots, y^n))$ denotes the local coordinates system on TM corresponding to the local coordinates system $(U, (x^1, \dots, x^n))$ of M. If $X|_U = \sum_{i=1}^n \xi^i \frac{\partial}{\partial x^i}$ denotes the local representation of a vector field X on M, then its vertical and complete lifts in terms of local coordinates system (x^i, y^i) are as follows:

$$(X|_U)^v = \sum_{i=1}^n \xi^i \frac{\partial}{\partial y^i},$$
$$(X|_U)^c = \sum_{i=1}^n \xi^i \frac{\partial}{\partial x^i} + \sum_{i,j=1}^n \frac{\partial \xi^i}{\partial x^j} y^j \frac{\partial}{\partial y^i}.$$

In [30], it is shown that if X and Y are any two vector fields on the manifold M, then for the Lie brackets of their vertical and complete lifts on TM we have

$$[X^v, Y^v] = 0, \quad [X^c, Y^c] = [X, Y]^c, \quad [X^v, Y^c] = [X, Y]^v. \tag{1.1}$$

For an arbitrary real n-dimensional connected Lie group G, $\mu : G \times G \to G$, $\iota : G \to G$ and e denote the multiplication map, the inversion map and the identity element, respectively. Also, we use the notation $l_y : G \to G$ for the left translation and $r_y : G \to G$ for the right translation. Then, for all $v \in T_gG$ and $w \in T_hG$, we can see that the tangent map

$$T\mu : T(G \times G) \cong TG \times TG \to TG,$$
$$(v, w) \mapsto T\mu(v, w) = T_h l_g w + T_g r_h v$$

defines a Lie group structure on TG (see [15]). We mention that in the Lie group TG with this structure, $0_e \in T_eG$ is the identity element and $T\iota$ is the inversion map.

Using the above Lie group structure on TG, we can see that the vertical and complete lifts of left invariant vector fields of G are left invariant vector fields of TG (see [16]). Therefore, for any left invariant Riemannian metric g on G, we can define a left invariant Riemannian metric $\tilde{g}$ on TG as follows:

$$\tilde{g}(X^c, Y^c) = g(X, Y), \quad \tilde{g}(X^v, Y^v) = g(X, Y), \quad \tilde{g}(X^c, Y^v) = 0, \tag{1.2}$$

where X and Y are arbitrary vector fields on G. In this work, we study the curvature of left invariant (α, β)-metrics of Douglas type on TG, where α is induced by a lifted left invariant Riemannian metric $\tilde{g}$.
A special type of Finsler metrics, which belongs to the family of (α, β)-metrics, is a Randers metric. G. Randers introduced this family of Finsler metrics in his paper [23] on general relativity in 1941. These metrics have been used in many physical problems. For example, in electromagnetic and gravitational fields, in computation of the Lagrangian function of a test electric charge (see [3, 17, 18] etc).

A generalization of Randers metrics are (α, β)-metrics introduced by M. Matsumoto in [20]. These metrics are important and interesting types of Finsler metrics.

Let (M, g) be a Riemannian manifold and β be a 1-form on M. Assume that $\alpha(x, y) = \sqrt{g_{ij}y^i y^j}$ and $\phi : (-b_0, b_0) \to (\mathbb{R})^+$ is a smooth map. It is shown that $F = \alpha\phi\left(\frac{\beta}{\alpha}\right)$ is a Finsler metric on M, called an (α, β)-metric, if and only if $\|\beta\|_\alpha < b_0$ and $\phi = \phi(s)$ satisfies the following conditions (see [9]):

$$\phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0, \quad |s| \leq b < b_0. \tag{1.3}$$

As some special cases, if $\phi(s) = 1 + s$, $\phi(s) = \frac{1}{s}$ or $\phi(s) = \frac{1}{1-s}$, then we obtain three famous classes of Finsler metrics, which are called Randers metric $\alpha + \beta$, Kropina metric $\frac{\alpha^2}{\beta}$ and Matsumoto metric $\frac{\alpha^2}{\alpha - \beta}$, respectively [9].

It is easy to see that for an arbitrary 1-form β on a Riemannian manifold (M, g), there exists a unique vector field X on M such that for all $x \in M$ and $y \in T_xM$ we have

$$g(y, X(x)) = \beta(x, y). \tag{1.4}$$

This notation is very useful for constructing left invariant (α, β)-metrics on Lie groups. Let g be a left invariant Riemannian metric on a Lie group G and X be a left invariant vector field on G such that $\| X \|_\alpha < b_0$. It is easily seen that the (α, β)-metric, which is defined as above, is a left invariant metric (see [10] and [11]).

In this paper, we study the flag curvature of some special Finsler metrics. This quantity is an important concept in Finsler geometry which is defined by

$$K(P, y) = \frac{g_y(R(u, y)y, u)}{g_y(y, y)g_y(u, u) - g_y^2(u, y)}, \tag{1.5}$$

where $P = \mathrm{span}\{u, y\}$ and

$$g_y(u, v) = \frac{1}{2}\frac{\partial^2}{\partial s \partial t}F^2(y + su + tv)\Big|_{s=t=0}$$

denotes the fundamental tensor. It should be mentioned that in the above definition the Chern connection is used for computing the curvature tensor $R(u, y)y = \nabla_u\nabla_y y - \nabla_y\nabla_u y - \nabla_{[u,y]}y$, (see [8], [9]).

Suppose that F is a Finsler metric on a smooth n-dimensional manifold M. In a standard local coordinates system of TM, the spray coefficients of F are defined by

$$G^i(x,y) := \frac{1}{4}g^{il}\left([F^2]_{x^m y^l}y^m - [F^2]_{x^l}\right), \quad i=1,\dots,n,\ x\in M,\ y\in T_xM. \tag{1.6}$$

The Finsler metric F is called a Douglas metric if the spray coefficients G^i satisfy the following relation:

$$G^i = \frac{1}{2}\Gamma^i_{jk}(x)y^j y^k + P(x,y)y^i, \tag{1.7}$$

where $P(x,y)$ is a local positively homogeneous function of degree one on TM, and F is called of Berwald type if $P(x,y)=0$ (see [7,9]). For an (α,β)-metric, F is known to be of Berwald type if and only if the 1-form β is parallel with respect to the Levi-Civita connection of α [22].

Recently, in [11], there was proven a formula for computing the flag curvature of (α,β)-metrics of Berwald type which nicely shows the relation between the flag curvature of Finsler metric and the sectional curvature of the based Riemannian metric. In this work, we use the following proposition.

Proposition 1.1 ([11, Proposition 3.1]). *Assume that $F=\alpha\phi(s)$, $s=\frac{\beta}{\alpha}$ is an (α,β)-metric of Berwald type on M. Suppose that $P=\mathrm{span}\{u,y\}\subset T_xM$ and $\{u,y\}$ is an orthonormal set with respect to the Riemannian metric g. Then, for any $x\in M$, the flag curvature $K^F(P,y)$ of F is given by*

$$K^F(P,y) = \frac{1}{\phi^2[1+g^2(X,u)D]}K^g(P),$$

where K^g denotes the sectional curvature of the Riemannian metric g and $D=\frac{\phi''}{\phi-s\phi'}$.

In the last decade, many geometric properties of Lie groups equipped with left invariant Finsler metrics, or homogeneous spaces together with invariant Finsler metrics, have been studied (for example, see [12–14,24,25]). In this work, we use the following theorem which was proved by Liu and Deng in [19].

Theorem 1.2. *Assume that $F=\alpha\phi\left(\frac{\beta}{\alpha}\right)$ is a homogeneous (α,β)-metric on G/H. Then F is a Douglas metric if and only if either F is a Berwald metric or F is a Douglas metric of Randers type.*

Fortunately, in some special cases, there is a simple way to distinguish left invariant (α,β)-metrics of Berwald type. Suppose that F is a left invariant (α,β)-metric on a Lie group G, which is defined by a left invariant Riemannian metric g and a left invariant vector field X. In Proposition 4.1 of [11] (also see [19]), it is shown that F is a Berwald metric if and only if

$$\begin{cases} g([Y,X],Z)+g([Z,X],Y)=0 & \text{for all } Y,Z\in\mathfrak{g},\\ g([Y,Z],X)=0 & \text{for all } Y,Z\in\mathfrak{g}. \end{cases} \tag{1.8}$$

Moreover, in Theorem 3.2 of [1], An and Deng showed that a left invariant Randers metric on a Lie group G is of Douglas type if and only if

$$g([Y,Z],X)=0 \quad \text{for all } Y,Z\in\mathfrak{g}. \tag{1.9}$$

2. Lifting of (α,β)-metrics on tangent bundles

Let G be a Lie group equipped with a left invariant Riemannian metric g. In [4], Asgari and the second author of this paper proved that for any $X,Y\in\mathfrak{g}=Lie(G)$ the Levi-Civita connection of the lifted left invariant metric $\tilde{g}$ on TG can be computed by the following equations:

$$\begin{cases} \tilde{\nabla}_{X^c}Y^c=(\nabla_XY)^c \\ \tilde{\nabla}_{X^v}Y^v=(\nabla_XY-\frac{1}{2}[X,Y])^c \\ \tilde{\nabla}_{X^c}Y^v=(\nabla_XY+\frac{1}{2}\operatorname{ad}^*_Y X)^v \\ \tilde{\nabla}_{X^c}Y^v=(\nabla_XY+\frac{1}{2}\operatorname{ad}^*_Y X)^v \end{cases} \tag{2.1}$$

In [4], it is also shown that if K and $\tilde{K}$ denote the sectional curvatures of G and TG, respectively, then

$$\begin{cases} \tilde{K}(X^c,Y^c)=K(X,Y) \\ \tilde{K}(X^v,Y^v)=K(X,Y)+g(\nabla_{[X,Y]}Y,X)+\frac{1}{4}\|[X,Y]\|^2 \\ \tilde{K}(X^c,Y^v)=K(X,Y)+\frac{1}{2}g([Y,\nabla_XY],X)-\frac{1}{2}g(\nabla_Y\operatorname{ad}^*_Y X,X) \\ \qquad\qquad\qquad +\frac{1}{4}g([Y,\operatorname{ad}^*_Y X],X)-\frac{1}{2}g([[X,Y],Y),X), \end{cases} \tag{2.2}$$

where X and Y are any two left invariant vector fields on G such that $\{X,Y\}$ is an orthonormal set with respect to g and $\|\cdot\|$ is calculated with respect to g. Assume that g is a left invariant Riemannian metric and X is a left invariant vector field on a Lie group G. Now consider the left invariant (α,β)-metric F defined by

$$F=\sqrt{g(y,y)}\,\phi\left(\frac{g(X(x),y)}{\sqrt{g(y,y)}}\right). \tag{2.3}$$

Then, by using vertical and complete lifts, we can define two types of left invariant Finsler metrics on TG as follows:

$$F^c((x,y),\tilde{z})=\sqrt{\tilde{g}(\tilde{z},\tilde{z})}\,\phi\left(\frac{\tilde{g}(X^c(x,y),\tilde{z})}{\sqrt{\tilde{g}(\tilde{z},\tilde{z})}}\right), \tag{2.4}$$

$$F^v((x,y),\tilde{z})=\sqrt{\tilde{g}(\tilde{z},\tilde{z})}\,\phi\left(\frac{\tilde{g}(X^v(x,y),\tilde{z})}{\sqrt{\tilde{g}(\tilde{z},\tilde{z})}}\right), \tag{2.5}$$

where $x \in G$, $y \in T_xG$ and $\tilde{z} \in T_{(x,y)}TG$.

Since $\|X^c\|_{\tilde{g}} = \|X^v\|_{\tilde{g}} = \|X\|_g < b_0$, F^c and F^v are left invariant (α,β)-metrics on TG. In this section, we use the above (α,β)-metrics F, F^c and F^v to find a necessary and sufficient condition for the Finsler metrics F^c and F^v to be of Douglas type.

Lemma 2.1. *Let F be an arbitrary left invariant (α,β)-metric defined by (2.3). F is of Douglas type if and only if F^c is of Douglas type.*

Proof. Suppose that F is a Douglas metric. Then (1.2) shows that F is a Berwald metric or a Douglas metric of Randers type. If F is of Berwald type, then the relations (1.8) show that for all $Y, Z \in \mathfrak{g}$ we have

$$g([Z,Y],X) = g(\mathrm{ad}_Y^* X, Z) = 0. \tag{2.6}$$

So, $\mathrm{ad}_Y^* X = 0$. Now the formula (2.1) proves $\tilde{\nabla}_{Y^c}X^c = \tilde{\nabla}_{Y^v}X^c = 0$, which shows that F^c is a Berwald metric. If F is a Douglas metric of Randers type, then, by the relation (1.9), for all $Y, Z \in \mathfrak{g}$ we have $g([Z,Y],X) = 0$. On the other hand, we have the following relations:

$$\tilde{g}([Z^c,Y^c],X^c) = g([Z,Y],X),\ \tilde{g}([Z^v,Y^c],X^c) = 0,\ \tilde{g}([Z^v,Y^v],X^c) = 0. \tag{2.7}$$

Thus the relation (1.9) says F^c is a Douglas metric of Randers type.

Conversely, let F^c be of Douglas type. If F^c is a Berwald metric, then for any $Y \in \mathfrak{g}$ we have $\tilde{\nabla}_{Y^c}X^c = \tilde{\nabla}_{Y^v}X^c = 0$. So, for any $Y \in \mathfrak{g}$ we have $\nabla_Y X = 0$, which means that F is of Berwald type. If F^c is a Randers metric of Douglas type, then $g([Z,Y],X) = \tilde{g}([Z^c,Y^c],X^c) = 0$, which shows that F is of Douglas type. □

Lemma 2.2. *Assume that F is an arbitrary left invariant (α,β)-metric defined by (2.3). Then the left invariant (α,β)-metric F^v is a Berwald metric if and only if the following two conditions hold:*

1. $\mathrm{ad}_X^* = \mathrm{ad}_X$,
2. *for any* $Y \in \mathfrak{g}$, $\nabla_X Y = \frac{1}{2}[X,Y]$.

Proof. We know that F^v is a Berwald metric if and only if for any $Y \in \mathfrak{g}$, $\tilde{\nabla}_{Y^c}X^v = \tilde{\nabla}_{Y^v}X^v = 0$. Now formula (2.1) completes the proof. □

Remark 2.3. For the Lie algebra $\mathfrak{g}$, suppose that $z(\mathfrak{g})$ denotes the center of Lie algebra. If F is a Berwald metric, then the previous lemma, together with the formula (2.1), shows that F^v is a Berwald metric if and only if $X \in z(\mathfrak{g})$.

Lemma 2.4. *Let F be a left invariant Randers metric on a Lie group G. Then F is a Douglas metric if and only if F^v is a Douglas metric.*

Proof. If F is a Douglas metric of Randers type, then by the relation (1.9), for all $Y, Z \in \mathfrak{g}$, $g([Z,Y],X) = 0$. Thus F^v is a Douglas metric because

$$\tilde{g}([Z^v,Y^v],X^v) = 0,\ \tilde{g}([Z^c,Y^c],X^v) = 0,\ \tilde{g}([Z^v,Y^c],X^v) = g([Z,Y],X). \tag{2.8}$$

Conversely, let F^v be a Douglas metric. Then the above equations show that F is a Douglas metric because by the relation (1.9) we have

$$\tilde{g}([Z^v, Y^c], X^v) = 0 \quad \text{for all } Y, Z \in \mathfrak{g}. \qquad \square$$

Let F be a Douglas metric. Here we compute the flag curvature formulas of F^c and F^v.

Theorem 2.5. *Suppose that F is a Berwaldian left invariant (α, β)-metric defined by (2.3) on a Lie group G. Suppose that P is a two-dimensional subspace of $\mathfrak{g}$ and $\{Y, V\}$ is an orthonormal set in P with respect to g. If K denotes the sectional curvature of g and $D = \frac{\phi''}{\phi - s\phi'}$, then the flag curvature K^{F^c} of the Finsler metric F^c can be computed as follows:*

1. $\tilde{P} = \operatorname{span}\{Y^c, V^c\}$,
$$K^{F^c}(\tilde{P}, Y^c) = \frac{1}{\phi^2(g(X,Y))[(1 + g^2(X,V)D)]} K(V,Y);$$

2. $\tilde{P} = \operatorname{span}\{Y^c, V^v\}$,
$$K^{F^c}(\tilde{P}, Y^c) = \frac{1}{\phi^2(g(X,Y))} \Big\{ K(V,Y) + \frac{1}{2} g([V, \nabla_Y V], Y) - \frac{1}{2} g(\nabla_V \operatorname{ad}_V^* Y, Y) + \frac{1}{4} g([V, \operatorname{ad}_V^* Y], Y) - \frac{1}{2} g([[Y,V],V],Y) \Big\};$$

3. $\tilde{P} = \operatorname{span}\{Y^v, V^c\}$,
$$K^{F^c}(\tilde{P}, Y^v) = \frac{1}{\phi^2(0)[(1 + g^2(X,V)D)]} \Big\{ K(V,Y) + \frac{1}{2} g([Y, \nabla_V Y], U) - \frac{1}{2} g(\nabla_Y \operatorname{ad}_Y^* V, V) + \frac{1}{4} g([Y, \operatorname{ad}_Y^* V], V) - \frac{1}{2} g([[V,Y],Y],V) \Big\};$$

4. $\tilde{P} = \operatorname{span}\{Y^v, V^v\}$,
$$K^{F^c}(\tilde{P}, Y^v) = \frac{1}{\phi^2(0)} \Big\{ K(V,Y) + g(\nabla_{[V,Y]} Y, V) + \frac{1}{4} \|[V,Y]\|^2 \Big\}.$$

Proof. By Lemma 2.1, F^c is a Berwald metric. So, the Chern connection of F^c is the same as the Levi-Civita connection of $\tilde{g}$. Now, by using the relations (2.2) and Proposition 1.1, the proof can be completed. For example, 1 follows from the fact that $\tilde{g}(X^c, Y^c) = g(X,Y)$ and $\tilde{g}(X^c, V^c) = g(X,V)$. Moreover, by the relations (2.2), $\tilde{K}(V^c, Y^c) = K(V,Y)$. So, by Proposition 1.1, we have

$$\begin{aligned} K^{F^c}(\tilde{P}, Y^c) &= \frac{1}{\phi^2(\tilde{g}(X^c, Y^c))[1 + \tilde{g}^2(X^c, V^c)D]} \tilde{K}(V^c, Y^c) \\ &= \frac{1}{\phi^2(g(X,Y))[(1 + g^2(X,V)D)]} K(V,Y). \end{aligned}$$

The other equations can be verified in a similar way. $\square$

Theorem 2.6. *Assume that G is a Lie group equipped with a left invariant Riemannian metric g. Suppose that*

$$F = \sqrt{g(y,y)} + g(X(x), y) \tag{2.9}$$

is the induced left invariant Randers metric of Douglas type on G defined by g and a left invariant vector field X. Then, for the flag curvature of the left invariant Randers metric F^c on TG, we have:

1. $\tilde{P} = \mathrm{span}\{Y^c, V^c\}$,

$$K^{F^c}(\tilde{P}, Y^c) = \frac{1}{(1+g(X,Y))^2} K(V,Y) + \frac{1}{4(1+g(X,Y))^2}\left\{3g([X,Y],Y) \right.$$
$$\left. -4(1+g(X,Y))g\left(U\left(Y, \sum_{i=1}^m \eta_i X_i\right), X\right)\right\};$$

2. $\tilde{P} = \mathrm{span}\{Y^c, U^v\}$,

$$K^{F^c}(\tilde{P}, Y^c) = \frac{1}{(1+g(X,Y))^2}\left\{K(V,Y) + \frac{1}{2}g([V, \nabla_Y V], Y) \right.$$
$$\left. -\frac{1}{2}g(\nabla_V \operatorname{ad}_V^* Y, Y) + \frac{1}{4}g([V, \operatorname{ad}_V^* Y], Y) - \frac{1}{2}g([[Y,V],V],Y)\right\}$$
$$+\frac{1}{4(1+g(X,Y))^2}\left\{3g^2([X,Y],Y) \right.$$
$$\left. -4(1+g(X,Y))g\left(U\left(Y, \sum_{i=1}^m \eta_i X_i\right), X\right)\right\};$$

3. $\tilde{P} = \mathrm{span}\{Y^v, V^c\}$,

$$K^{F^c}(\tilde{P}, Y^v) = K(V,Y) + \frac{1}{2}g([Y, \nabla_V Y], V) - \frac{1}{2}g(\nabla_Y \operatorname{ad}_Y^* V, V)$$
$$+\frac{1}{4}g([Y, \operatorname{ad}_Y^* V], V) - \frac{1}{2}g([[V,Y],Y],V)$$
$$+\frac{1}{4}\left\{3g^2([Y,X],Y) + 4g\left(U\left(Y, \sum_{i=1}^m \mu_j X_j\right), X\right)\right\};$$

4. $\tilde{P} = \mathrm{span}\{Y^v, V^v\}$,

$$K^{F^c}(\tilde{P}, Y^v) = K(V,Y) + g(\nabla_{[V,Y]} Y, V) + \frac{1}{4}\|[V,Y]\|^2$$
$$+\frac{1}{4}\left\{3g^2([Y,X],Y) + 4g\left(U\left(Y, \sum_{i=1}^m \mu_j X_j\right), X\right)\right\},$$

where $\{X_i \mid i = 1, \ldots, m\}$ is a basis for the Lie algebra $\mathfrak{g}$ of G and $U : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is a symmetric function defined by the following equation:

$$2g(U(v_1, v_2), v_3) = g([v_3, v_1], v_2) + g([v_3, v_2], v_1). \tag{2.10}$$

Proof. Lemma 2.1 shows that F^c is of Douglas type. It is sufficient to use Theorem 2.4 of [4] and the following formula of the flag curvature given in Theorem 2.1 of [13],

$$K^{F^c}(P, Y^c) = \frac{\tilde{g}(Y^c, Y^c)}{F^c(Y^c)^2}\tilde{K}(\tilde{P}) + \frac{1}{4F^c(Y^c)^4}\{3\tilde{g}(\tilde{U}(Y^c, Y^c), X^c)$$
$$- 4F\tilde{g}(\tilde{U}(Y^c, \tilde{U}(Y^c, Y^c), X^c),$$

where $\tilde{U} : \tilde{\mathfrak{g}} \times \tilde{\mathfrak{g}} \to \tilde{\mathfrak{g}} = Lie(TG)$ satisfies the formula (2.10). Hence,

$$\tilde{g}(\tilde{U}(Y^c, Y^c), X^c) = g([X, Y], Y),$$

$$\tilde{g}(\tilde{U}(Y^c, \tilde{U}(Y^c, Y^c)), X^c) = g\left(U\left(Y, \sum_{i=1}^{m} \eta_i X_i\right), X\right),$$

$$\tilde{g}(\tilde{U}(Y^v, Y^v), X^c) = g([Y, X], Y),$$

$$\tilde{g}(\tilde{U}(Y^v, \tilde{U}(Y^v, Y^v)), X^c) = -g\left(U\left(Y, \sum_{j=1}^{m} \mu_j X_j\right), X\right),$$

where $\tilde{U}(Y^c, Y^c) = \sum_{i=1}^m \eta_i X_i^c + \sum_{i=1}^m \delta_i X_i^v$ and $\tilde{U}(Y^v, Y^v) = \sum_{j=1}^m \lambda_j X_j^c + \sum_{j=1}^m \mu_j X_j^v$. □

In the following theorems we compute the flag curvature of F^v.

Theorem 2.7. *Suppose that F is a left invariant (α, β)-metric on a Lie group G defined by (2.3), and F^v is a Berwald metric. Assume that P is a two-dimensional subspace of $\mathfrak{g}$ and $\{Y, V\}$ is an orthonormal set in P with respect to g. If K denotes the sectional curvature of g, then the flag curvature K^{F^v} of the (α, β)-metric F^v can be computed as follows:*

1. $\tilde{P} = \mathrm{span}\{Y^c, V^c\}$,
$$K^{F^v}(\tilde{P}, Y^c) = \frac{1}{\phi^2(0)} K(V, Y);$$

2. $\tilde{P} = \mathrm{span}\{Y^c, V^v\}$,
$$K^{F^v}(\tilde{P}, Y^c) = \frac{1}{\phi^2(0)[1 + g^2(X, V)D]}\Big\{K(V, Y) + \frac{1}{2} g([V, \nabla_Y V], Y) - \frac{1}{2} g(\nabla_V \operatorname{ad}_V^* Y, Y) + \frac{1}{4} g([V, \operatorname{ad}_V^* Y], Y) - \frac{1}{2} g([[Y, V], V], Y)\Big\};$$

3. $\tilde{P} = \mathrm{span}\{Y^v, V^c\}$,
$$K^{F^v}(\tilde{P}, Y^v) = \frac{1}{\phi^2(g(X, Y))}\Big\{K(V, Y) + \frac{1}{2} g([Y, \nabla_V Y], U) - \frac{1}{2} g(\nabla_Y \operatorname{ad}_Y^* V, V) + \frac{1}{4} g([Y, \operatorname{ad}_Y^* V], V) - \frac{1}{2} g([[V, Y], Y], V)\Big\};$$

4. $\tilde{P} = \mathrm{span}\{Y^v, V^v\}$,
$$K^{F^v}(\tilde{P}, Y^v) = \frac{1}{\phi^2(g(X, Y))[1 + g^2(X, V)D]}\Big\{K(V, Y) + g(\nabla_{[V,Y]} Y, V) + \frac{1}{4}\|[V, Y]\|^2\Big\}.$$

Proof. The using of the relations (2.2) and the curvature formula of Proposition 1.1 completes the proof. For example, for 2, we have

$$K^{F^v}(\tilde{P}, Y^c) = \frac{1}{\phi^2(\tilde{g}(X^v, Y^c))[1 + \tilde{g}^2(X^v, V^v)D]} \tilde{K}(V^v, Y^c)$$

$$= \frac{1}{\phi^2(0)[1+g^2(X,V)D]}\Big\{K(V,Y)+\frac{1}{2}g([V,\nabla_Y V],Y)$$
$$-\frac{1}{2}g(\nabla_V \operatorname{ad}^*_V Y,Y)+\frac{1}{4}g([V,\operatorname{ad}^*_V Y],Y)-\frac{1}{2}g([[Y,V],V],Y)\Big\}.$$

The other equations can be verified in a similar way. □

Theorem 2.8. *Let G be a Lie group equipped with a left invariant Riemannian metric g and*

$$F=\sqrt{g(y,y)}+g(X(x),y) \tag{2.11}$$

be a left invariant Randers metric of Douglas type on G defied by g and a left invariant vector field X. Then, for the flag curvature of the left invariant (α,β)-metric F^v on TG, we have:

1. $\tilde{P}=\operatorname{span}\{Y^c,V^c\}$,
$$K^{F^v}(\tilde{P},Y^c)=K(V,Y)-\frac{1}{2}g([X,Y],\sum_{j=1}^m \delta_j X_j);$$

2. $\tilde{P}=\operatorname{span}\{Y^c,V^v\}$,
$$K^{F^v}(\tilde{P},Y^c)=K(V,Y)+\frac{1}{2}g([V,\nabla_Y V],Y)-\frac{1}{2}g(\nabla_V \operatorname{ad}^*_V Y,Y)$$
$$+\frac{1}{4}g([V,\operatorname{ad}^*_V Y],Y)-\frac{1}{2}g([[Y,V],V],Y)$$
$$-\frac{1}{2}g\left([X,Y],\sum_{j=1}^m \delta_j X_j\right);$$

3. $\tilde{P}=\operatorname{span}\{Y^v,U^c\}$,
$$K^{F^v}(\tilde{P},Y^v)=\frac{1}{(1+g(X,y))^2}\Big\{K(U,Y)+\frac{1}{2}g([Y,\nabla_U Y],U)$$
$$-\frac{1}{2}g(\nabla_Y \operatorname{ad}^*_Y U,U)+\frac{1}{4}g([Y,\operatorname{ad}^*_Y U],U)-\frac{1}{2}g([[U,Y],Y],U)\Big\}$$
$$-\frac{1}{2(1+g(X,y))^4}g\left(\left[X,\sum_{j=1}^m \eta_j X_j\right],Y\right);$$

4. $\tilde{P}=\operatorname{span}\{Y^v,U^v\}$,
$$K^{F^v}(\tilde{P},Y^v)=\frac{1}{(1+g(X,y))^2}\Big\{K(U,Y)+g(\nabla_{[U,Y]}Y,U)+\frac{1}{4}\|[U,Y]\|^2\Big\}$$
$$-\frac{1}{2(1+g(X,y))^4}g\left(\left[X,\sum_{j=1}^m \eta_j X_j\right],Y\right).$$

Proof. Lemma 2.4 shows that F^v is of Douglas type. We know that

$$\tilde{g}(\tilde{U}(Y^c,Y^c),X^v)=\tilde{g}(\tilde{U}(Y^v,Y^v),X^v)=0,$$
$$\tilde{g}(\tilde{U}(Y^c,\tilde{U}(Y^c,Y^c)),X^v)=\frac{1}{2}g([X,Y],\sum_{j=1}^m \delta_j X_j),$$

$$\tilde{g}(\tilde{U}(Y^v,\tilde{U}(Y^v,Y^v)),X^v)=\frac{1}{2}g([X,\sum_{j=1}^{m}\eta_jX_j],Y).$$

By using the similar method as in Theorem 2.6, we can complete the proof of this theorem. □

3. Examples

In this section, we study the flag curvature of two important families of (α,β)-metrics which are called Matsumoto and Kropina metrics. These metrics, as well as the Randers metric, have physical application (see [2] and [21]).

Example 3.1. Let G be a Lie group equipped with a left invariant Riemannian metric g and let

$$F=\frac{g(y,y)}{\sqrt{g(y,y)}-g(X(x),y)} \tag{3.1}$$

be the Berwaldian left invariant Matsumoto metric on G defined by g and a left invariant vector field X which is parallel with respect to the Levi-civita connection of g. Then, for the flag curvature of the left invariant Matsumoto Metric F^c on TG, we have:

1. $\tilde{P}=\mathrm{span}\{Y^c,U^c\}$,
$$K^{F^c}(\tilde{P},Y^c)=\frac{(1-g(X,Y))^3(1-2g(X,Y))}{1+2g^2(X,U)+2g^2(X,Y)-3g(X,Y)}K(U,Y);$$

2. $\tilde{P}=\mathrm{span}\{Y^c,U^v\}$,
$$K^{F^c}(\tilde{P},Y^c)=(1-g(X,Y))^2\Big\{K(U,Y)+\frac{1}{2}g([U,\nabla_YU],Y)$$
$$-\frac{1}{2}g(\nabla_U\operatorname{ad}_U^*Y,Y)+\frac{1}{4}g([U,\operatorname{ad}_U^*Y],Y)-\frac{1}{2}g([[Y,U],U],Y)\Big\};$$

3. $\tilde{P}=\mathrm{span}\{Y^v,U^c\}$,
$$K^{F^c}(\tilde{P},Y^v)=\frac{1}{2g^2(X,U)+1}\Big\{K(U,Y)+\frac{1}{2}g([Y,\nabla_UY],U)$$
$$-\frac{1}{2}g(\nabla_Y\operatorname{ad}_Y^*U,U)+\frac{1}{4}g([Y,\operatorname{ad}_Y^*U],U)-\frac{1}{2}g([[U,Y],Y],U)\Big\};$$

4. $\tilde{P}=\mathrm{span}\{Y^v,U^v\}$,
$$K^{F^c}(\tilde{P},Y^v)=K(U,Y)+g(\nabla_{[U,Y]}Y,U)+\frac{1}{4}\|[U,Y]\|^2,$$

where the assumptions are similar to those from the previous section and ∇ denotes the Levi-Civita connection of g. In this case, the formulas for the flag curvature of F^v are as follows:

1. $\tilde{P}=\mathrm{span}\{Y^c,U^c\}$,
$$K^{F^v}(\tilde{P},Y^c)=K(U,Y);$$

2. $\tilde{P} = \text{span}\{Y^c, U^v\}$,
$$K^{F^v}(\tilde{P}, Y^c) = \frac{1}{1+2g^2(X,U)}\Big\{K(U,Y) + \frac{1}{2}g([U,\nabla_Y U],Y)$$
$$-\frac{1}{2}g(\nabla_U \operatorname{ad}_U^* Y, Y) + \frac{1}{4}g([U, \operatorname{ad}_U^* Y], Y) - \frac{1}{2}g([[Y,U],U],Y)\Big\};$$

3. $\tilde{P} = \text{span}\{Y^v, U^c\}$,
$$K^{F^v}(\tilde{P}, Y^v) = (1-g(X,Y))^2\Big\{K(U,Y) + \frac{1}{2}g([Y,\nabla_U Y],U)$$
$$-\frac{1}{2}g(\nabla_Y \operatorname{ad}_Y^* U, U) + \frac{1}{4}g([Y, \operatorname{ad}_Y^* U], U) - \frac{1}{2}g([[U,Y],Y],U)\Big\};$$

4. $\tilde{P} = \text{span}\{Y^v, U^v\}$,
$$K^{F^v}(\tilde{P}, Y^v) = \frac{(1-g(X,Y))^3(1-2g(X,Y))}{1+2g^2(X,U)+2g^2(X,Y)-3g(X,Y)}$$
$$\times\Big\{K(U,Y) + g(\nabla_{[U,Y]}Y, U) + \frac{1}{4}\|[U,Y]\|^2\Big\}.$$

Example 3.2. Suppose that G is a Lie group equipped with a left invariant Riemannian metric g, X is a left invariant vector field and

$$F = \frac{g(y,y)}{g(X(x),y)} \tag{3.2}$$

is the left invariant Kropina metric of Berwald type on G defined by g and X. Then, for the flag curvature of the left invariant Kropina Metric F^c on TG, we have:

1. $\tilde{P} = \text{span}\{Y^c, U^c\}$,
$$K^{F^c}(\tilde{P}, Y^c) = \frac{g^4(X,Y)}{g^2(X,U)+g^2(X,Y)}K(U,Y);$$

2. $\tilde{P} = \text{span}\{Y^c, U^v\}$,
$$K^{F^c}(\tilde{P}, Y^c) = (g(X,Y))^2\Big\{K(U,Y) + \frac{1}{2}g([U,\nabla_Y U],Y)$$
$$-\frac{1}{2}g(\nabla_U \operatorname{ad}_U^* Y, Y) + \frac{1}{4}g([U, \operatorname{ad}_U^* Y], Y) - \frac{1}{2}g([[Y,U],U],Y)\Big\};$$

3. $\tilde{P} = \text{span}\{Y^v, U^c\}$
$K^{F^c}(\tilde{P}, Y^v)$ is not defined;

4. $\tilde{P} = \text{span}\{Y^v, U^v\}$,
$K^{F^c}(\tilde{P}, Y^v)$ is not defined.

Also the flag curvature formulas of the Finsler metric F^v are as follows:

1. $\tilde{P} = \text{span}\{Y^c, U^c\}$,
$K^{F^v}(\tilde{P}, Y^c)$ is not defined;

2. $\tilde{P} = \mathrm{span}\{Y^c, U^v\}$,
   $K^{F^v}(\tilde{P}, Y^c)$ is not defined;

3. $\tilde{P} = \mathrm{span}\{Y^v, U^c\}$

$$K^{F^v}(\tilde{P}, Y^v) = g^2(X,Y)\bigg\{K(U,Y) + \frac{1}{2}g([Y,\nabla_U Y],U) - \frac{1}{2}g(\nabla_Y \mathrm{ad}_Y^* U, U) + \frac{1}{4}g([Y,\mathrm{ad}_Y^* U],U) - \frac{1}{2}g([[U,Y],Y],U)\bigg\};$$

4. $\tilde{P} = \mathrm{span}\{Y^v, U^v\}$,

$$K^{F^v}(\tilde{P}, Y^v) = \frac{g^4(X,Y)}{g^2(X,U)+g^2(X,Y)} \times \bigg\{K(U,Y) + g(\nabla_{[U,Y]}Y,U) + \frac{1}{4}\|[U,Y]\|^2\bigg\}.$$

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Лівоінваріантні підняті (α, β)-метрики типу Дугласа на дотичних групах Лі

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У статті розглядаються підняті лівоінваріантні (α, β)-метрики типу Дугласа на дотичних групах Лі. Припускається, що g є лівоінваріантною рімановою метрикою на групі Лі G, а F є лівоінваріантною (α, β)-метрикою типу Дугласа, що індукована метрикою g. За допомогою вертикального і повного підняття ми будуємо вертикальні і повні підняті (α, β)-метрики F^v і F^c на дотичному розшаруванні TG і доводимо необхідні й достатні умови для того, щоб вони були метриками типу Дугласа. Також вивчаються флагові кривини цих метрик. Нарешті, в якості особливих випадків пораховані флагові кривини F^v і F^c для метрик Рандерса типу Дугласа та метрик Кропіної і Мацумото типу Бервальда.

Ключові слова: лівоінваріантна (α, β)-метрика, повне і вертикальне підняття, флагова кривина