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Para-Complex Norden Structures in Cotangent Bundle Equipped with Vertical Rescaled Cheeger–Gromoll Metric

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In the paper, a deformation (in the vertical bundle) of the Cheeger–Gromoll metric on the cotangent bundle T^*M over an m-dimensional Riemannian manifold (M, g), called the vertical rescaled Cheeger–Gromoll metric, is considered. The para-Nordenian properties of the vertical rescaled Cheeger–Gromoll metric are studied.

Key words: cotangent bundles, horizontal lift, vertical lift, vertical rescaled Cheeger–Gromoll metric, para-complex structure, pure metric

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1. Introduction

The geometry of the cotangent bundle T^*M has been studied by many authors: A.A. Salimov and F. Agca [18,19], K. Yano and S. Ishihara [24], F. Agca [1], F. Ocak and S. Kazimova [16], F. Ocak [15], A. Gezer and M. Altunbas [9] and others.

The notion of almost para-complex structure (or almost product structure) on a smooth manifold was introduced in [12], and a survey of further results on para-complex geometry (including para-Hermitian and para-Kähler geometry) can be found, for instance, in [3, 5]. Also, other further significant developments are to be found in [2, 22]. Some aspects concerning the geometry of tangent and cotangent bundles are presented in [8–10, 15, 17, 18].

In this paper, we introduce the vertical rescaled Cheeger–Gromoll metric on the cotangent bundle T^*M as a new natural metric with respect to the metric g. First we study the geometry of the vertical rescaled Cheeger–Gromoll metric. We construct almost para-complex Norden structures on a cotangent bundle equipped with the vertical rescaled Cheeger–Gromoll metric and investigate conditions for these structures to be para-Kähler–Norden, quasi-para-Kähler–Norden. Finally, we describe some properties of almost para-complex Norden structures in the context of almost product Riemannian manifolds.

2. Cotangent bundles T^*M

Let (M^m, g) be an m-dimensional Riemannian manifold, T^*M be its cotangent bundle and $\pi : T^*M \to M$ be the natural projection. A local chart $(U, x^i)_{i=\overline{1,m}}$ on M induces a local chart $(\pi^{-1}(U), x^i, x^{\bar{i}} = p_i)_{i=\overline{1,m}, \bar{i}=m+i}$ on T^*M, where p_i is the component of covector p in each cotangent space T^*_xM, $x \in U$, with respect to the natural coframe dx^i. Let $C^\infty(M)$ (respectively, $C^\infty(T^*M)$) be the ring of real-valued C^∞ functions on M(respectively, T^*M) and $\Im^r_s(M)$ (respectively, $\Im^r_s(T^*M)$) be the module over $C^\infty(M)$ (respectively, $C^\infty(T^*M)$) of C^∞ tensor fields of type (r, s).

Denote by Γ^k_{ij} the Christoffel symbols of g and by ∇, the Levi-Civita connection of g.

We have two complementary distributions on T^*M, the vertical distribution $VT^*M = Ker(d\pi)$ and the horizontal distribution HT^*M that define a direct sum decomposition

$$TT^*M = VT^*M \oplus HT^*M. \tag{2.1}$$

Let $X = X^i \frac{\partial}{\partial x^i}$ and $\omega = \omega_i dx^i$ be local expressions in $U \subset M$ of a vector and covector fields $X \in \Im^1_0(M)$ and $\omega \in \Im^0_1(M)$, respectively. Then the horizontal and the vertical lifts of X and ω are defined respectively by

$$X^H = X^i \frac{\partial}{\partial x^i} + p_h \Gamma^h_{ij} X^j \frac{\partial}{\partial p_i}, \tag{2.2}$$

$$\omega^V = \omega_i \frac{\partial}{\partial p_i} \tag{2.3}$$

with respect to the natural frame $\{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial p_i}\}$, where Γ^h_{ij} are components of the Levi-Civita connection ∇ on M (see [24] for more details).

Lemma 2.1 ([24]). *Let (M, g) be a Riemannian manifold, ∇ be the Levi-Civita connection and R be the Riemannian curvature tensor. Then the Lie bracket of the cotangent bundle T^*M of M satisfies the following:*

1. $[\omega^V, \theta^V] = 0,$
2. $[X^H, \theta^V] = (\nabla_X \theta)^V,$
3. $[X^H, Y^H] = [X, Y]^H - (pR(X, Y))^V,$

for all $X, Y \in \Im^1_0(M)$ and $\omega, \theta \in \Im^0_1(M)$, such that $pR(X, Y) = p_a R^a_{ijk} X^i Y^j \, dx^k$, where R^a_{ijk} are local components of R on (M, g).

Let (M, g) be a Riemannian manifold. We define the map

$$\Im^0_1(M) \to \Im^1_0(M)$$
$$\omega \mapsto \widetilde{\omega}$$

for all $X \in \Im^1_0(M)$, $g(\widetilde{\omega}, X) = \omega(X)$. Locally, for all $\omega = \omega_i \, dx^i \in \Im^0_1(M)$, we have $\widetilde{\omega} = g^{ij} \omega_i \frac{\partial}{\partial x^j}$, where (g^{ij}) is the inverse matrix of the matrix (g_{ij}).

For each $x \in M$, the scalar product $g^{-1} = (g^{ij})$ is defined on the cotangent space T_x^*M by $g^{-1}(\omega,\theta) = g(\widetilde{\omega},\widetilde{\theta}) = g^{ij}\omega_i\theta_j$. In this case, we have $\widetilde{\omega} = g^{-1}\circ\omega$.

If ∇ is the Levi-Civita connection of (M,g), then we have

$$\nabla_X\widetilde{\omega} = \widetilde{\nabla_X\omega},$$
$$Xg^{-1}(\omega,\theta) = g^{-1}(\nabla_X\omega,\theta) + g^{-1}(\omega,\nabla_X\theta)$$

for all $X \in \Im_0^1(M)$ and $\omega,\theta \in \Im_1^0(M)$.

3. Vertical rescaled Cheeger–Gromoll metric

Definition 3.1. Let (M,g) be a Riemannian manifold and $f : M \to]0,+\infty[$ be a strictly positive smooth function on M. On the cotangent bundle T^*M, we define a vertical rescaled Cheeger–Gromoll metric denoted by g^f:

$$g^f(X^H,Y^H) = g(X,Y)^V = g(X,Y)\circ\pi, \tag{3.1}$$
$$g^f(X^H,\theta^V) = 0, \tag{3.2}$$
$$^f(\omega^V,\theta^V) = \frac{f}{\alpha}(g^{-1}(\omega,\theta) + g^{-1}(\omega,p)g^{-1}(\theta,p)) \tag{3.3}$$

for all $X,Y \in \Im_0^1(M)$, $\omega,\theta \in \Im_1^0(M)$, where $\alpha = 1 + \|p\|^2$ and $\|p\| = \sqrt{g^{-1}(p,p)}$ is the norm of p with respect to the metric g.

Note that if $f = 1$, then g^f is the Cheeger–Gromoll metric [19].

Lemma 3.2. *Let (M,g) be a Riemannian manifold and $\rho : \mathbb{R} \to \mathbb{R}$ be a smooth function. Then we have the following:*

1. $X^H(\rho(r^2))_\xi = 0,$
2. $\omega^V(\rho(r^2))_\xi = 2\rho'(r^2)g^{-1}(\omega,p)_x,$
3. $X^H(g^{-1}(\theta,p))_\xi = g^{-1}(\nabla_X\theta,p)_x,$
4. $\omega^V(g^{-1}(\theta,p))_\xi = g^{-1}(\omega,\theta)_x,$
5. $X^H(g(Y,Z))_\xi = Xg(X,Y)_x = g(\nabla_XY,Z)_x + g(Y,\nabla_XZ)_x,$
6. $X^H(g^{-1}(\theta,\eta))_\xi = Xg^{-1}(\theta,\eta)_x = g^{-1}(\nabla_X\theta,\eta)_x + g^{-1}(\theta,\nabla_X\eta)_x,$
7. $\omega^V(g(Y,Z))_\xi = 0,$
8. $\omega^V(g^{-1}(\theta,\eta))_\xi = 0$

for all $\xi = (x,p) \in T^*M$, $X,Y,Z \in \Im_0^1(M)$ and $\omega,\theta,\eta \in \Im_1^0(M)$, $r^2 = g^{-1}(p,p)$.

Proof. Locally, Lemma 3.2 follows from formulas (2.2) and (2.3). □

Lemma 3.3. *Let (M,g) be a Riemannian manifold and (T^*M,g^f) be its cotangent bundle equipped with the vertical rescaled Cheeger–Gromoll metric. Then we have the following:*

(1) $X^H g^f(\theta^V,\eta^V)=\frac{1}{f}X(f)g^f(\theta^V,\eta^V)+g^f((\nabla_X\theta)^V,\eta^V)+g^f(\theta^V,(\nabla_X\eta)^V)$,

(2) $$\omega^V g^f(\theta^V,\eta^V)=\frac{-2}{\alpha}g^{-1}(\omega,p)g^f(\theta^V,\eta^V)+\frac{1}{\alpha}g^{-1}(\omega,\theta)g^f(\eta^V,\mathcal{P}^V)+\frac{1}{\alpha}g^{-1}(\omega,\eta)g^f(\theta^V,\mathcal{P}^V)$$

for all $X\in\Im^1_0(M)$ *and* $\omega,\theta,\mathcal{P}\in\Im^0_1(M)$ *such that* $\mathcal{P}_x=p\in T^*_xM$, *(*$\mathcal{P}^V$ *is the canonical vertical or Liouville vector field on* T^*M*).*

Proof. The proof of Lemma 3.3 follows directly from Lemma 3.2. □

Theorem 3.4. *Let* (M,g) *be a Riemannian manifold and* (T^*M,g^f) *be its cotangent bundle equipped with the vertical rescaled Cheeger–Gromoll metric. If* ∇ *(respectively,* ∇^f *) denotes the Levi-Civita connection of* (M,g) *(respectively,* (T^*M,g^f) *), we have*

(1) $(\nabla^f_{X^H}Y^H)_\xi=(\nabla_XY)^H_\xi+\frac{1}{2}(pR_x(X,Y))^V$,

(2) $(\nabla^f_{X^H}\theta^V)_\xi=(\nabla_X\theta)^V_\xi+\frac{1}{2f(x)}X_x(f)\theta^V_\xi+\frac{f(x)}{2\alpha}(R_x(\widetilde{p},\widetilde{\theta})X)^H$,

(3) $(\nabla^f_{\omega^V}Y^H)_\xi=\frac{1}{2f(x)}Y_x(f)\omega^V_x+\frac{f(x)}{2\alpha}(R_x(\widetilde{p},\widetilde{\omega})Y)^H$,

(4) $$(\nabla^f_{\omega^V}\theta^V)_\xi=-\frac{1}{2f(x)}g^f_\xi(\omega^V,\theta^V)(\mathrm{grad}\,f)^H_\xi-\frac{1}{\alpha f(x)}[g^f_\xi(\omega^V,\mathcal{P}^V)\theta^V_\xi+g^f_\xi(\theta^V,\mathcal{P}^V)\omega^V_\xi]+[\frac{\alpha+1}{\alpha f(x)}g^f_\xi(\omega^V,\theta^V)-\frac{1}{\alpha f^2(x)}g^f_\xi(\omega^V,\mathcal{P}^V)g^f_\xi(\theta^V,\mathcal{P}^V)]\mathcal{P}^V_\xi$$

for all $\xi=(x,p)\in T^*M$, $X,Y\in\Im^1_0(M)$ *and* $\omega,\theta,\mathcal{P}\in\Im^0_1(M)$ *such that* $\mathcal{P}_x=p\in T^*_xM$, *(* $\mathcal{P}^V$ *is the canonical vertical or Liouville vector field on* T^*M *) and* $pR(X,Y)=p_aR^a_{ijk}X^iY^j\,dx^k$, *where* R^a_{ijk} *are local components of the curvature tensor* R *on* (M,g).

Proof. The proof of Theorem 3.4 follows from the Kozul formula and Lemma 3.3.

(1) Direct calculations give us

$$\begin{aligned}2g^f(\nabla^f_{X^H}Y^H,Z^H)&=X^Hg^f(Y^H,Z^H)+Y^Hg^f(Z^H,X^H)-Z^Hg^f(X^H,Y^H)\\&\quad+g^f(Z^H,[X^H,Y^H])+g^f(Y^H,[Z^H,X^H])\\&\quad-g^f(X^H,[Y^H,Z^H])\\&=Xg(Y,Z)+Yg(Z,X)-Zg(X,Y)+g(Z,[X,Y])\\&\quad+g(Y,[Z,X])-g(X,[Y,Z])\\&=2g(\nabla_XY,Z)=2g^f((\nabla_XY)^H,Z^H)\end{aligned}$$

and

$$2g^f(\nabla^f_{X^H}Y^H,\eta^V)=X^Hg^f(Y^H,\eta^V)+Y^Hg^f(\eta^V,X^H)-\eta^Vg^f(X^H,Y^H)$$

$$\begin{aligned}&+g^f(\eta^V,[X^H,Y^H])+g^f(Y^H,[\eta^V,X^H])\\&-g^f(X^H,[Y^H,\eta^V])\\&=g^f(\eta^V,[X^H,Y^H])=g^f((pR(X,Y))^V,\eta^V).\end{aligned}$$

Thus we have

$$\nabla^f_{X^H}Y^H=(\nabla_XY)^H+\frac{1}{2}(pR(X,Y))^V.$$

(2) By straightforward calculations, we obtain

$$\begin{aligned}2g^f(\nabla^f_{X^H}\theta^V,Z^H)&=X^Hg^f(\theta^V,Z^H)+\theta^Vg^f(Z^H,X^H)-Z^Hg^f(X^H,\theta^V)\\&\quad+g^f(Z^H,[X^H,\theta^V])+g^f(\theta^V,[Z^H,X^H])\\&\quad-g^f(X^H,[\theta^V,Z^H])\\&=g^f(\theta^V,[Z^H,X^H])=g^f((pR(Z,X))^V,\theta^V)\\&=\frac{f}{\alpha}\big(g^{-1}(pR(Z,X),\theta)+g^{-1}(pR(Z,X),p)g^{-1}(\theta,p)\big)\\&=\frac{f}{\alpha}g^f((R(\widetilde{p},\widetilde{\theta})X)^H,Z^H),\end{aligned}$$

where

$$\begin{aligned}g^{-1}(pR(Z,X),\theta)&=g^{kl}(pR(Z,X))_k\theta_l=p_sR^s_{ijk}Z^iX^j\widetilde{\theta}^k\\&=g_{st}\widetilde{p}^tR^s_{ijk}Z^iX^j\widetilde{\theta}^k=R_{ijkt}Z^iX^j\widetilde{\theta}^k\widetilde{p}^t\\&=g(R(Z,X)\widetilde{\theta},\widetilde{p})=g(R(\widetilde{p},\widetilde{\theta})X,Z)\\&=g^f((R(\widetilde{p},\widetilde{\theta})X)^H,Z^H)\end{aligned}$$

and

$$\begin{aligned}g^{-1}(pR(Z,X),p)&=g^{kl}(pR(Z,X))_kp_l=(pR(Z,X))_k\widetilde{p}^k,\\&=p_sR^s_{ijk}Z^iX^j\widetilde{p}^k=g_{st}\widetilde{p}^tR^s_{ijk}Z^iX^j\widetilde{p}^k\\&=R_{ijkt}Z^iX^j\widetilde{p}^t\widetilde{p}^k=g(R(Z,X)\widetilde{p},\widetilde{p})=0.\end{aligned}$$

Then it follows that

$$\begin{aligned}2g^f(\nabla^f_{X^H}\theta^V,\eta^V)&=X^Hg^f(\theta^V,\eta^V)+\theta^Vg^f(\eta^V,X^H)-\eta^Vg^f(X^H,\theta^V)\\&\quad+g^f(\eta^V,[X^H,\theta^V])+g^f(\theta^V,[\eta^V,X^H])\\&\quad-g^f(X^H,[\theta^V,\eta^V])\\&=X^Hg^f(\theta^V,\eta^V)+g^f(\eta^V,[X^H,\theta^V])+g^f(\theta^V,[\eta^V,X^H]).\end{aligned}$$

Using the first formula of Lemma 4.10, we have

$$2g^f(\nabla^f_{X^H}\theta^V,\eta^V)=\frac{1}{f}X(f)g^f(\theta^V,\eta^V)+g^f((\nabla_X\theta)^V,\eta^V)+g^f(\theta^V,(\nabla_X\eta)^V)$$

$$+ g^f(\eta^V, (\nabla_X \theta)^V) - g^f(\theta^V, (\nabla_X \eta)^V)$$
$$= 2g^f((\nabla_X \theta)^V, \eta^V) + \frac{1}{f} X(f) g^f(\theta^V, \eta^V),$$

and thus

$$\nabla^f_{X^H} \theta^V = (\nabla_X \theta)^V + \frac{1}{2f} X(f) \theta^V + \frac{f}{2} (R(\widetilde{p}, \widetilde{\theta})X)^H.$$

The other formulas are obtained by a similar calculation. □

4. Para-Kähler–Norden Structures

An almost product structure φ on a manifold M is a $(1,1)$ tensor field on M such that $\varphi^2 = id_M$, $\varphi \neq \pm id_M$ (id_M is the identity tensor field of type $(1,1)$ on M). The pair (M, φ) is called an almost product manifold.

A linear connection ∇ on (M, φ) such that $\nabla \varphi = 0$ is said to be an almost product connection. There exists an almost product connection on every almost product manifold [11].

An almost para-complex manifold is an almost product manifold (M, φ) such that the two eigenbundles TM^+ and TM^- associated to the two eigenvalues $+1$ and -1 of φ, respectively, have the same rank. Note that the dimension of an almost para-complex manifold is necessarily even [5].

An almost para-complex Norden manifold (M^{2m}, φ, g) is a real $2m$-dimensional differentiable manifold M^{2m} with an almost para-complex structure φ and a Riemannian metric g such that

$$g(\varphi X, Y) = g(X, \varphi Y) \tag{4.1}$$

for all $X, Y \in \Im^1_0(M)$. In this case, g is called a pure metric with respect to φ or para-Norden metric (B-metric) [22].

A para-complex Norden manifold (para-Kähler–Norden) is an almost para-complex Norden manifold (M^{2m}, φ, g) such that φ is integrable, i.e, $\nabla \varphi = 0$ (B-manifold), where ∇ is the Levi-Civita connection of g [20, 22].

A Tachibana operator ϕ_φ applied to the pure metric g is given by

$$\begin{aligned}(\phi_\varphi g)(X, Y, Z) = (\varphi X)(g(Y, Z)) - X(g(\varphi Y, Z)) + g((L_Y \varphi)X, Z) \\ + g((L_Z \varphi)X, Y)\end{aligned} \tag{4.2}$$

for all $X, Y, Z \in \Im^1_0(M)$ [23].

In a para-complex Norden manifold, a para-Norden metric g is called para-holomorphic if

$$(\phi_\varphi g)(X, Y, Z) = 0 \tag{4.3}$$

for all $X, Y, Z \in \Im^1_0(M)$ [22].

A para-holomorphic Norden manifold is an almost para-complex Norden manifold (M^{2m}, φ, g) such that g is a para-holomorphic, i.e., $\phi_\varphi g = 0$.

It is well known that the almost para-holomorphic Norden manifold (M^{2m}, φ, g) is para-Kähler–Norden if and only if g is paraholomorphic, i.e., $\phi_\varphi g = 0$ is equivalent to $\nabla\varphi = 0$, which was proven in [22]. By virtue of this point of view, para-holomorphic Norden manifolds are similar to para-Kähler–Norden manifolds [20].

4.1. Let (M, g) be a Riemannian manifold. We consider an almost para-complex structure J on T^*M defined by

$$\begin{cases} JX^H = -X^H \\ J\omega^V = \omega^V \end{cases} \tag{4.4}$$

for all $X \in \Im_0^1(M)$ and $\omega \in \Im_1^0(M)$ [4].

Theorem 4.1. *Let (M, g) be a Riemannian manifold, (T^*M, g^f) be its cotangent bundle equipped with the vertical rescaled Cheeger–Gromoll metric and the almost para-complex structure J defined by (4.4). The triple (T^*M, J, g^f) is an almost para-complex Norden manifold.*

Proof. For all $X, Y \in \Im_0^1(M)$ and $\omega, \theta \in \Im_1^0(M)$, from (4.4) we have

1. $g^f(JX^H, Y^H) = g^f(-X^H, Y^H) = g^f(X^H, -Y^H) = g^f(X^H, JY^H)$;
2. $g^f(JX^H, \theta^V) = g^f(-X^H, \theta^V) = 0 = g^f(X^H, \theta^V) = g^f(X^H, J\theta^V)$;
3. $g^f(J\omega^V, Y^H) = g^f(\omega^V, Y^H) = 0 = g^f(\omega^V, -Y^H) = g^f(\omega^V, JY^H)$;
4. $g^f(J\omega^V, \theta^V) = g^f(\omega^V, \theta^V) = g^f(\omega^V, J\theta^V)$,

i.e., g^f is pure with respect to J. Hence, (T^*M, J, g^f) is an almost para-complex Norden manifold. □

Proposition 4.2. *Let (M, g) be a Riemannian manifold, (T^*M, g^f) be its cotangent bundle equipped with the vertical rescaled Cheeger–Gromoll metric and the almost para-complex structure J defined by (4.4). Then we get*

1. $(\phi_J g^f)(X^H, Y^H, Z^H) = 0$;
2. $(\phi_J g^f)(\omega^V, Y^H, Z^H) = 0$,
3. $(\phi_J g^f)(X^H, \theta^V, Z^H) = 2g^f\big((pR(X, Z))^V, \theta^V\big)$;
4. $(\phi_J g^f)(X^H, Y^H, \eta^V) = 2g^f\big((pR(X, Y))^V, \eta^V\big)$;
5. $(\phi_J g^f)(\omega^V, \theta^V, Z^H) = 0$,
6. $(\phi_J g^f)(\omega^V, Y^H, \eta^V) = 0$;
7. $(\phi_J g^f)(X^H, \theta^V, \eta^V) = \dfrac{-2}{f} X(f) g^f(\theta^V, \eta^V)$;
8. $(\phi_J g^f)(\omega^V, \theta^V, \eta^V) = 0$

for all $X, Y, Z \in \Im_0^1(M)$ *and* $\omega, \theta, \eta \in \Im_1^0(M)$, *where* R *denotes the curvature tensor of* (M, g).

Proof. We calculate the Tachibana operator ϕ_J applied to the pure metric g^f. This operator is characterized by (4.2).

1. From Lemma 3.3, we have

$$\begin{aligned}(\phi_J g^f)(X^H, Y^H, Z^H) &= (JX^H)g^f(Y^H, Z^H) - X^H g^f(JY^H, Z^H)\\ &\quad + g^f\left((L_{Y^H}J)X^H, Z^H\right) + g^f\left(Y^H, (L_{Z^H}J)X^H\right)\\ &= -X^H g^f(Y^H, Z^H) + X^H g^f(Y^H, Z^H)\\ &\quad + g^f\left(L_{Y^H}JX^H - J(L_{Y^H}X^H), Z^H\right)\\ &\quad + g^f\left(Y^H, L_{Z^H}JX^H - J(L_{Z^H}X^H)\right)\\ &= -g^f\left([Y^H, X^H], Z^H\right) - g^f\left(J[Y^H, X^H], Z^H\right)\\ &\quad - g^f\left(Y^H, [Z^H, X^H]\right) - g^f\left(Y^H, J[Z^H, X^H]\right) = 0.\end{aligned}$$

2. We also have

$$\begin{aligned}(\phi_J g^f)(\omega^V, Y^H, Z^H) &= (J\omega^V)g^f(Y^H, Z^H) - \omega^V g^f(JY^H, Z^H))\\ &\quad + g^f\left((L_{Y^H}J)\omega^V, Z^H\right) + g^f\left(Y^H, (L_{Z^H}J)\omega^V\right)\\ &= +g^f\left([Y^H, \omega^V], Z^H\right) - g^f\left(J[Y^H, \omega^V], Z^H\right)\\ &\quad + g^f\left(Y^H, [Z^H, \omega^V]\right) - g^f\left(Y^H, J[Z^H, \omega^V]\right)\\ &= 2g^f\left([Y^H, \omega^V], Z^H\right) + 2g^f\left(Y^H, [Z^H, \omega^V]\right)\\ &= 2g^f\left((\nabla_Y \omega)^V, Z^H\right) + 2g^f\left(Y^H, (\nabla_Z \omega)^V\right) = 0.\end{aligned}$$

3. We obtain

$$\begin{aligned}(\phi_J g^f)(X^H, \theta^V, Z^H) &= (JX^H)g^f(\theta^V, Z^H) - X^H g^f(J\theta^V, Z^H)\\ &\quad + g^f\left((L_{\theta^V}J)X^H, Z^H\right) + g^f\left(\theta^V, (L_{Z^H}J)X^H\right)\\ &= -g^f\left([\theta^V, X^H], Z^H\right) - g^f\left(J[\theta^V, X^H], Z^H\right)\\ &\quad - g^f\left(\theta^V, [Z^H, X^H]\right) - g^f\left(\theta^V, J[Z^H, X^H]\right)\\ &= -2g^f\left(\theta^V, [Z^H, X^H]\right) = -2g^f\left(\theta^V, (pR(Z, X))^V\right)\\ &= 2g^f\left((pR(X, Z))^V, \theta^V\right).\end{aligned}$$

4. Finally, we get

$$\begin{aligned}(\phi_J g^f)(X^H, Y^H, \eta^V) &= (JX^H)g^f(Y^H, \eta^V) - X^H g^f(JY^H, \eta^V)\\ &\quad + g^f\left((L_{Y^H}J)X^H, \eta^V\right) + g^f\left(Y^H, (L_{\eta^V}J)X^H\right)\\ &= -g^f\left([Y^H, X^H], \eta^V\right) - g^f\left(J[Y^H, X^H], \eta^V\right)\\ &\quad - g^f\left(Y^H, [\eta^V, X^H]\right) - g^f\left(Y^H, J[\eta^V, X^H]\right)\\ &= -2g^f\left([Y^H, X^H], \eta^V\right) = 2g^f\left((pR(X, Y))^V, \eta^V\right).\end{aligned}$$

The other formulas are obtained by a similar calculation. □

Theorem 4.3. *Let (M, g) be a Riemannian manifold, (T^*M, g^f) be its cotangent bundle equipped with the vertical rescaled Cheeger–Gromoll metric and the almost para-complex structure J defined by (4.4). The triple (T^*M, J, g^f) is a para-Kähler–Norden manifold if and only if M is flat and f is constant.*

Proof. For all $\overline{X}, \overline{Y}, \overline{Z} \in \Im_0^1(T^*M)$ such as $\overline{X} = X^H, \omega^V$, $\overline{Y} = Y^H, \theta^V$ and $\overline{Z} = Z^H, \eta^V$, by virtue of Proposition 4.2, we have.

$$(\phi_J g^f))(\overline{X}, \overline{Y}, \overline{Z}) = 0 \Leftrightarrow \begin{cases} 2g^f\left((pR(X,Z))^V, \theta^V\right) = 0 \\ 2g^f\left((pR(X,Y))^V, \eta^V\right) = 0 \\ \dfrac{-2}{f} X(f) g^f(\theta^V, \eta^V) = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} pR(X,Z) = 0 \\ pR(X,Y) = 0 \\ X(f) = 0 \end{cases} \Leftrightarrow R = 0 \,\text{and}\, f = \text{const.} \qquad \square$$

4.2. Now we study a quasi-para-Kähler–Norden manifold. The basic class of non-integrable almost paracomplex manifolds with para–Norden metric is the class of quasi-para-Kähler manifolds. An almost para-complex Norden manifold (M, φ, g) is a quasi-para-Kähler–Norden manifold if

$$\underset{X,Y,Z}{\sigma} g((\nabla_X \varphi)Y, Z) = 0$$

for all $X, Y, Z \in \Im_0^1(M)$, where σ is the cyclic sum by three arguments [7, 13]. It is well known that

$$\underset{X,Y,Z}{\sigma} g((\nabla_X \varphi)Y, Z) = 0$$

is equivalent to

$$(\phi_\varphi g)(X, Y, Z) + (\phi_\varphi g)(Y, Z, X) + (\phi_\varphi g)(Z, X, Y) = 0,$$

which was proven in [21].

Theorem 4.4. *Let (M, g) be a Riemannian manifold, (T^*M, g^f) be its cotangent bundle equipped with the vertical rescaled Cheeger–Gromoll metric and the almost para-complex structure J defined by (4.4). The triple (T^*M, J, g^f) is a quasi-para-Kähler–Norden manifold if and only if f is constant.*

Proof. We put, for all $\overline{X}, \overline{Y}, \overline{Z} \in \Im_0^1(T^*M)$,

$$A(\overline{X}, \overline{Y}, \overline{Z}) = (\phi_J g^f)(\overline{X}, \overline{Y}, \overline{Z}) + (\phi_J g^f)(\overline{Y}, \overline{Z}, \overline{X}) + (\phi_J g^f)(\overline{Z}, \overline{X}, \overline{Y}).$$

By virtue of Proposition 4.2, we have

$$A(X^H, Y^H, Z^H) = 0, \qquad A(\omega^V, Y^H, Z^H) = 0,$$

$$A(\omega^V, \theta^V, Z^H) = -\frac{2}{f} Z(f) g^f(\omega^V, \theta^V), \qquad (\omega^V, \theta^V, \eta^V) = 0.$$

Then, for (T^*M, J, g^f) to be a quasi-para-Kähler–Norden manifold, it suffices that $Z(f) = 0$, for any $Z \in \Im_0^1(M)$, i.e., f is constant. $\square$

4.3. Now we study a generalization of the almost para-complex structure defined by (4.4).

Lemma 4.5. *Let (M,φ) be an almost para-complex manifold and define a tensor field $J_\varphi \in \Im^1_1(T^*M)$ by*

$$\begin{cases} J_\varphi X^H = -(\varphi X)^H \\ J_\varphi \omega^V = \omega^V \end{cases} \tag{4.5}$$

for all $X \in \Im^1_0(M)$ and $\omega \in \Im^0_1(M)$. Then the couple (T^*M, J_φ) is an almost para-complex manifold .

Proof. By virtue of (4.5), we have

$$\begin{cases} J^2_\varphi X^H = J_\varphi(J_\varphi X^H) = J_\varphi(-(\varphi X)^H) = (\varphi(\varphi X))^H = (\varphi^2 X)^H, \\ J^2_\varphi \omega^V = J_\varphi(J_\varphi \omega^V) = J_\varphi \omega^V = \omega^V \end{cases}$$

for any $X \in \Im^1_0(M)$ and $\omega \in \Im^0_1(M)$. Since $\varphi^2 = id_M$, then $J^2_\varphi = id_{T^*M}$. □

Theorem 4.6. *Let (M,φ,g) be an almost para-complex Norden manifold, (T^*M, g^f) be its cotangent bundle equipped with the vertical rescaled Cheeger–Gromoll metric and the almost para-complex structure J_φ defined by (4.5). The triple (T^*M, J_φ, g^f) is an almost para-complex Norden manifold.*

Proof. For all $X, Y \in \Im^1_0(M)$ and $\omega, \theta \in \Im^0_1(M)$, from (4.5) we have

$$\begin{aligned} g^f(J_\varphi X^H, Y^H) &= g^f(-(\varphi X)^H, Y^H) = -g(\varphi X, Y) = -g(X, \varphi Y) \\ &= g^f(X^H, -(\varphi Y)^H) = g^f(X^H, J_\varphi Y^H), \\ g^f(J_\varphi X^H, \theta^V) &= g^f(-(\varphi X)^H, \theta^V) = 0 = g^f(X^H, \theta^V) = g^f(X^H, J_\varphi \theta^V), \\ g^f(J_\varphi \omega^V, \theta^V) &= g^f(\omega^V, \theta^V) = g^f(\omega^V, J_\varphi \theta^V). \end{aligned}$$

Since g is pure with respect to φ, then g^f is pure with respect to J_φ. □

Proposition 4.7. *Let (M,φ,g) be an almost para-complex Norden manifold, (T^*M, g^f) be its cotangent bundle equipped with the vertical rescaled Cheeger–Gromoll metric and the almost para-complex structure J_φ defined by (4.5). Then we get*

1. $(\phi_{J_\varphi} g^f)(X^H, Y^H, Z^H) = -(\phi_\varphi g)(X, Y, Z)$;
2. $(\phi_{J_\varphi} g^f)(\omega^V, Y^H, Z^H) = 0$;
3. $(\phi_{J_\varphi} g^f)(X^H, \theta^V, Z^H) = g^f\big((pR(\varphi X, Z) + pR(X, Z))^V, \theta^V\big)$;
4. $(\phi_{J_\varphi} g^f)(X^H, Y^H, \eta^V) = g^f\big((pR(\varphi X, Y) + pR(X, Y))^V, \eta^V\big)$,
5. $(\phi_{J_\varphi} g^f)(\omega^V, \theta^V, Z^H) = 0$;

6. $(\phi_{J_\varphi} g^f)(\omega^V, Y^H, \eta^V) = 0$;

7. $(\phi_{J_\varphi} g^f)(X^H, \theta^V, \eta^V) = \dfrac{-1}{f}(\varphi X(f) + X(f))g^f(\theta^V, \eta^V)$;

8. $(\phi_{J_\varphi} g^f)(\omega^V, \theta^V, \eta^V) = 0$

for all $X, Y, Z \in \Im^1_0(M)$ and $\omega, \theta, \eta \in \Im^0_1(M)$, where R denotes the curvature tensor of (M, g).

Proof. We calculate the Tachibana operator ϕ_{J_φ} applied to the pure metric g^f. With the same steps as in the proof of Proposition 4.2, we get the results. □

Theorem 4.8. *Let (M, φ, g) be an almost para-complex Norden manifold, (T^*M, g^f) be its cotangent bundle equipped with the vertical rescaled Cheeger–Gromoll metric and the almost para-complex structure J_φ defined by (4.5). The triple (T^*M, J_φ, g^f) is a para-Kähler–Norden manifold if and only if M is a flat para-Kähler–Norden manifold and f is constant.*

Proof. For all $\overline{X}, \overline{Y}, \overline{Z} \in \Im^1_0(T^*M)$ such as $\overline{X} = X^H, \omega^V$, $\overline{Y} = Y^H, \theta^V$ and $\overline{Z} = Z^H, \eta^V$, by virtue of Proposition 4.7, we have

$$(\phi_{J_\varphi} g^f))(\overline{X}, \overline{Y}, \overline{Z}) = 0 \Leftrightarrow \begin{cases} (\phi_\varphi g)(X, Y, Z) = 0 \\ g^f\left((pR(\varphi X, Z) + pR(X, Z))^V, \theta^V\right) = 0 \\ g^f\left((pR(\varphi X, Y) + pR(X, Y))^V, \eta^V\right) = 0 \\ \dfrac{-1}{f}(\varphi X(f) + X(f))g^f(\theta^V, \eta^V) = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} (\phi_\varphi g)(X, Y, Z) = 0 \\ pR(\varphi X + X, Z) = 0 \\ pR(\varphi X + X, Y) = 0 \\ (\varphi X + X)(f) = 0 \end{cases}.$$

Since $\varphi \neq \pm id_M$, then

$$(\phi_{J_\varphi} g^f))(\overline{X}, \overline{Y}, \overline{Z}) = 0 \Leftrightarrow \begin{cases} \phi_\varphi g = 0 \\ R = 0 \\ f = \text{const} \end{cases}.$$ □

Theorem 4.9. *Let (M, φ, g) be a para-Kähler–Norden manifold, (T^*M, g^f) be its cotangent bundle equipped with the vertical rescaled Cheeger–Gromoll metric and the almost para-complex structure J_φ defined by (4.5). The triple (T^*M, J_φ, g^f) is a quasi-para-Kähler–Norden manifold if and only if f is constant.*

Proof. Since (M, φ, g) is a para-Kähler–Norden manifold, then for all $X, Y, Z \in \Im^1_0(M)$ $(\phi_\varphi g)(X, Y, Z) = 0$, and $R(\varphi Y, Z) = R(Y, \varphi Z)$. With the same steps as in the proof of Theorem 4.4, we get the results. □

4.4. Now consider the almost product structure J defined by (4.4) and the Levi-Civita connection ∇^f of (T^*M, g^f) given by Theorem 3.4. We define a tensor field S of type $(1,2)$ and a linear connection $\widehat{\nabla}$ on T^*M ,

$$S(\overline{X},\overline{Y}) = \frac{1}{2}\big[(\nabla^f_{J\overline{Y}}J)\overline{X} + J\big((\nabla^f_{\overline{Y}}J)\overline{X}\big) - J\big((\nabla^f_{\overline{X}}J)\overline{Y}\big)\big], \tag{4.6}$$

$$\widehat{\nabla}_{\overline{X}}\overline{Y} = \nabla^f_{\overline{X}}\overline{Y} - S(\overline{X},\overline{Y}) \tag{4.7}$$

for all $\overline{X},\overline{Y}\Im^1_0(T^*M)$, is an almost product connection on T^*M (see [11, p.151] for more details).

Lemma 4.10. *Let (M,g) be a Riemannian manifold, T^*M be its cotangent bundle equipped with the vertical rescaled Cheeger–Gromoll metric g^f and the almost product structure J defined by (4.4). Then the tensor field S is as follows:*

(1) $S(X^H, Y^H) = \frac{1}{2}(pR(X,Y))^V$,

(2) $S(X^H, \theta^V) = -\frac{1}{f}X(f)\theta^V + \frac{f}{2\alpha}(R(\widetilde{p},\widetilde{\theta})X)^H$,

(3) $S(\omega^V, Y^H) = -\frac{1}{f}Y(f)\omega^V$,

(4) $S(\omega^V, \theta^V) = -\frac{1}{2f}g^f(\omega^V,\theta^V)(grad\, f)^H$

for all $X,Y \in \Im^1_0(M)$ and $\omega,\theta \in \Im^0_1(M)$.

Proof. The proof of Lemma 4.10 follows directly from Theorem 3.4, formula (4.4) and formula (4.6). □

Theorem 4.11. *Let (M,g) be a Riemannian manifold, T^*M be its cotangent bundle equipped with the vertical rescaled Cheeger–Gromoll metric g^f and the almost product structure J defined by (4.4). Then the almost product connection $\widehat{\nabla}$ defined by (4.7) is as follows:*

(1) $\widehat{\nabla}_{X^H}Y^H = (\nabla_X Y)^H$,

(2) $\widehat{\nabla}_{X^H}\theta^V = (\nabla_X\theta)^V + \frac{3}{2f}X(f)\theta^V$,

(3) $\widehat{\nabla}_{\omega^V}Y^H = \frac{f}{2\alpha}(R(\widetilde{p},\widetilde{\omega})Y)^H$,

(4) $\widehat{\nabla}_{\omega^V}\theta^V = -\frac{1}{\alpha f}\Big[g^f(\omega^V,\mathcal{P}^V)\theta^V + g^f(\theta^V,\mathcal{P}^V)\omega^V\Big] + \Big[\frac{\alpha+1}{\alpha f}g^f(\omega^V,\theta^V) -$
$\frac{1}{\alpha f^2}g^f(\omega^V,\mathcal{P}^V)g^f(\theta^V,\mathcal{P}^V)\Big]\mathcal{P}^V$

for all $X,Y \in \Im^1_0(M)$ and $\omega,\theta \in \Im^0_1(M)$.

Proof. The proof of Theorem 4.11 follows directly from Theorem 3.4, Lemma 4.10 and formula (4.7). □

Lemma 4.12. *Let (M, g) be a Riemannian manifold, T^*M be its cotangent bundle equipped with the vertical rescaled Cheeger–Gromoll metric g^f and the almost product structure J defined by (4.4), and let $\widehat{T}$ denote the torsion tensor of $\widehat{\nabla}$. Then we have*

1) $\widehat{T}(X^H, Y^H) = (pR(X,Y))^V$,

2) $\widehat{T}(X^H, \theta^V) = \dfrac{3}{2f}X(f)\theta^V - \dfrac{f}{2\alpha}(R(\widetilde{p}, \widetilde{\theta})X)^H$,

3) $\widehat{T}(\omega^V, Y^H) = -\dfrac{3}{2f}Y(f)\omega^V + \dfrac{f}{2\alpha}(R(\widetilde{p}, \widetilde{\omega})Y)^H$,

4) $\widehat{T}(\omega^V, \theta^V) = 0$

for all $X, Y \in \Im^1_0(M)$ and $\omega, \theta \in \Im^0_1(M)$.

Proof. The proof of Lemma 4.12 follows directly from Lemma 4.10 and the formula

$$\widehat{T}(\overline{X}, \overline{Y}) = \widehat{\nabla}_{\overline{X}}\overline{Y} - \widehat{\nabla}_{\overline{Y}}\overline{X} - [\overline{X}, \overline{Y}] = S(\overline{Y}, \overline{X}) - S(\overline{X}, \overline{Y})$$

for all $\overline{X}, \overline{Y} \in \Im^1_0(T^*M)$. □

From Lemma 4.12, we obtain.

Theorem 4.13. *Let (M, g) be a Riemannian manifold, T^*M be its cotangent bundle equipped with the vertical rescaled Cheeger–Gromoll metric g^f, and let $\widehat{\nabla}$ be the almost product connection defined by (4.7). Then $\widehat{\nabla}$ is symmetric if and only if f is a constant function and M is flat.*

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Пара-комплексні структури Нордена в кодотичному розшаруванні з вертикальною масштабованою метрикою Чігера-Громолла

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У статті розглядається деформація (у вертикальному розшаруванні) метрики Чігера–Громолла на кодотичному розшаруванні T^*M над m-вимірним рімановим многовидом (M, g), що називається вертикальною масштабованою метрикою Чігера–Громолла. Досліджено паранорденові властивості вертикальної масштабованої метрики Чігера–Громолла.

Ключові слова: кодотичні розшарування, горизонтальний ліфт, вертикальний ліфт, вертикальна масштабована метрика Чігера–Громолла, пара-комплексна структура, чиста метрика.