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The boundary value problem for the two-dimensional Laplace equation with transmission conditions given on open Lipschitz curve

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Анотація. Розглянуто крайову задачу для рівняння Лапласа в обмеженій двовимірній ліпшицевій області з крайовою умовою спряження, заданою на незамкненій кривій. Ця умова включає в себе стрибок розв'язку і значення його нормальної похідної. Доведено еквівалентність розглянутої крайової задачі та отриманої варіаційної задачі. Доведено існування і єдиність розв'язку поставлених задач у відповідних функціональних просторах. На основі інтегрального подання розв'язку крайова задача зведена до системи граничних інтегральних рівнянь, яка має єдиний розв'язок.

Abstract. We consider boundary value problem for Laplace equation in bounded two-dimensional Lipschitz domain with transmission boundary condition given upon open curve. This conditions includes itself the jump of solution of boundary value problem and the meaning of boundary value of its normal derivative. We prove the equivalence of considered boundary value problem and obtained variational problem. As a result we prove existence and uniqueness of solution of the posed problems in appropriate functional spaces. Based on the integral representation of the solution the considered boundary value problem is reduced to the system of boundary integral equation which has unique solution.

1 Introduction

In their work [6] G.S. Kit and Ya.S. Podstrigach proposed mathematical model for determination of the stationary temperature field in two dimensional infinite body with a linear slit having definite heat resistance. This mathematical model was reduced to the boundary value problem for the two-dimensional Laplace equation in plane with boundary conditions of transmission type and using representation of the solution via potential of double layer it was obtained Prandtl's integro-differential equation.

Problems with boundary conditions of transmission type for the second order elliptic equations and systems in Lipschitz domain were considered in [2, 3, 9, 11].

The aim of the present paper was to prove existence and uniqueness of solution of the boundary value problem for the two-dimensional Laplace equation in bounded Lipschitz domain with transmission boundary condition of special type given upon open Lipschitz curve. We introduce some functional spaces and trace operators in domain with open Lipschitz curve and show that this problem is equivalent to some variational problem which has unique solution according to the Lax-Milgram Lemma. We also obtained system of the boundary integral and integro-differential equations which is equivalent to the considered boundary value problem.

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2 Functional spaces and trace operators

Let $\Omega_+ \subset \mathbb{R}^2$ be a bounded connected Lipschitz domain. This means that its boundary curve Σ is locally the graph of a Lipschitz function [4, 5]. The points of $\mathbb{R}^2$ we denote $x = (x_1, x_2)$, $y = (y_1, y_2)$. Let us note that Σ can be piecewise smooth and have corner points. $\overline{\Omega}_+ = \Omega_+ \cup \Sigma$. We suppose that S is an open Lipschitz curve with the end points a and b, $\overline{S} = S \cup \{a, b\}$ and $\overline{S} \subset \Omega_+$. We denote $\Omega = \Omega_+ \setminus \overline{S}$ and consider S as a part of a some closed bounded Lipschitz curve $\Sigma_0 = \overline{S} \cup S_0$, $\Sigma_0 \subset \Omega_+$.

Since Σ and Σ_0 are Lipschitz almost everywhere we can define outward pointing unit vector of the normal $\vec{n}_x$, $x \in \Sigma$ or $x \in \Sigma_0$. Depend on the direction of $\vec{n}_x$, $x \in S$, we consider S as a double sided curve with sides S^+ and S^-.

In Ω_+ we consider the Laplace operator $Lu = -\Delta u = -\sum_{i=1}^{2} \frac{\partial^2 u}{\partial x_i^2}$ and the Hilbert spaces $H^1(\Omega_+)$ and $H^1(\Omega_+, L)$ of real functions with norms and inner products

$$\|u\|^2_{H^1(\Omega_+)} = \|\nabla u\|^2_{L_2(\Omega_+)} + \|u\|^2_{L_2(\Omega_+)}, \|u\|^2_{H^1(\Omega_+,L)} = \|u\|^2_{H^1(\Omega_+)} + \|Lu\|^2_{L_2(\Omega_+)},$$

$$(u, v)_{H^1(\Omega_+)} = (\nabla u, \nabla v)_{L_2(\Omega_+)} + (u, v)_{L_2(\Omega_+)},$$

$$(u, v)_{H^1(\Omega_+,L)} = (u, v)_{H^1(\Omega_+)} + (Lu, Lv)_{L_2(\Omega_+)}.$$

The trace operators $\gamma^+_{0,\Sigma} : H^1(\Omega_+) \to H^{1/2}(\Sigma)$ and $\gamma^+_{1,\Sigma} : H^1(\Omega_+, L) \to H^{-1/2}(\Sigma)$ are continuous and surjective [4, 5]. Here $\gamma^+_{1,\Sigma} u \in H^{-1/2}(\Sigma) = (H^{1/2}(\Sigma))'$ and for $u \in C^1(\overline{\Omega}_+)$ coincides with boundary value of normal derivative $\frac{\partial u}{\partial n_x}$ on Σ.

Let us denote by $C_0^\infty(\Omega)$ the class of infinitely differentiable functions with compact support in Ω. If $\varphi \in C_0^\infty(\Omega)$ then $\varphi(x) = 0$, $x \in S$.

We introduce the Hilbert spaces $H^1(\Omega)$ and $H^1(\Omega, L)$ of real functions with norms and inner products

$$\|u\|^2_{H^1(\Omega)} = \|\nabla u\|^2_{L_2(\Omega)} + \|u\|^2_{L_2(\Omega)}, \quad \|u\|^2_{H^1(\Omega,L)} = \|u\|^2_{H^1(\Omega)} + \|Lu\|^2_{L_2(\Omega)},$$

$$(u, v)_{H^1(\Omega)} = (\nabla u, \nabla v)_{L_2(\Omega)} + (u, v)_{L_2(\Omega)},$$

$$(u, v)_{H^1(\Omega,L)} = (u, v)_{H^1(\Omega)} + (Lu, Lv)_{L_2(\Omega)},$$

where derivatives $\frac{\partial u}{\partial x_i} \in L_2(\Omega)$ are defined as

$$\left(\frac{\partial u}{\partial x_i}, \varphi\right)_{L_2(\Omega)} = -\int_\Omega u(x)\frac{\partial \varphi(x)}{\partial x_i} dx = -\left(u, \frac{\partial \varphi}{\partial x_i}\right)_{L_2(\Omega)}$$

for all $\varphi \in C_0^\infty(\Omega)$.

We consider some trace operators in Ω. We denote $\gamma^\pm_{0,S}$ and $\gamma^\pm_{1,S}$ the restrictions of trace operators $\gamma^\pm_{0,\Sigma_0}$ and $\gamma^\pm_{1,\Sigma_0}$ on S respectively [1, 10]. Then we have $\gamma^\pm_{0,S} : H^1(\Omega) \to H^{1/2}(S)$ and $\gamma^\pm_{1,S} : H^1(\Omega, L) \to H^{-1/2}(S)$.

We introduce the space $H_0^1(\Omega) = \{u \in H^1(\Omega) : \gamma^\pm_{0,S} u = 0, \gamma^+_{0,\Sigma} u = 0\}$ and denote dual space $H^{-1}(\Omega) = (H_0^1(\Omega))'$. Then $L : H^1(\Omega) \to H^{-1}(\Omega)$. We also have that $H_0^1(\Omega)$ is a closure of $C_0^\infty(\Omega)$ in the norm $\|\cdot\|_{H^1(\Omega)}$.

We use the next trace operators : $[\gamma_{0,S}] = \gamma^+_{0,S} - \gamma^-_{0,S}$, $[\gamma_{1,S}] = \gamma^+_{1,S} - \gamma^-_{1,S}$. Let $\overline{\Omega}_1 \subset \Omega_+$ be a Lipschitz domain bounded by the closed curve Σ_0. $\overline{\Omega}_1 = \Omega_1 \cup \Sigma_0$, $\Omega_2 = \Omega_+ \setminus \overline{\Omega}_1$. We denote by $u_i = r_{\Omega_i} u$ the restriction of $u \in H^1(\Omega)$ to Ω_i, $i = 1, 2$. It's obviously that $u_i \in H^1(\Omega_i)$, $i = 1, 2$.

Lemma 2.1. *If $u \in H^1(\Omega_+)$ then $\gamma^+_{0,S} u = \gamma^-_{0,S} u$.*

Proof. For arbitrary $\varphi \in C_0^1(\overline{\Omega}_+) = \{\varphi \in C^1(\overline{\Omega}_+) : \varphi(x) = 0, x \in \Sigma\}$ and for $i = 1, 2$ we have

$$\int_{\Omega_1} \frac{\partial u_1(x)}{\partial x_i}\varphi(x)dx = -\int_{\Omega_1} u_1(x)\frac{\partial \varphi(x)}{\partial x_i}dx + \int_{\Sigma_0} \gamma^+_{0,\Sigma_0} u_1(y)\varphi(y)\cos(\vec{n}_y,\vec{y}_i)ds_y$$

$$\int_{\Omega_2} \frac{\partial u_2(x)}{\partial x_i}\varphi(x)dx = -\int_{\Omega_1} u_2(x)\frac{\partial \varphi(x)}{\partial x_i}dx - \int_{\Sigma_0} \gamma^-_{0,\Sigma_0} u_2(y)\varphi(y)\cos(\vec{n}_y,\vec{y}_i)ds_y$$

Since $\frac{\partial u(x)}{\partial x_i} \in L_2(\Omega_+)$ then

$$\int_{\Omega_1} \frac{\partial u_1(x)}{\partial x_i}\varphi(x)dx + \int_{\Omega_2} \frac{\partial u_2(x)}{\partial x_i}\varphi(x)dx = \int_{\Omega_+} \frac{\partial u(x)}{\partial x_i}\varphi(x)dx$$

and $\int_{\Sigma_0}[\gamma_{0,\Sigma_0}]u(y)\varphi(y)\cos(\vec{n}_y,\vec{y}_i)ds_y = 0$ for all $\varphi \in C^1(\Sigma_0)$. It means that $\gamma^+_{0,\Sigma_0}u = \gamma^-_{0,\Sigma_0}u$ or $\gamma^+_{0,S}u = \gamma^-_{0,S}u$. □

Corollary 2.2. *If $u \in H^1(\Omega)$ then $[\gamma_{0,S_0}]u = 0$. If $u \in H^1(\Omega)$ and $\gamma^+_{0,S}u = \gamma^-_{0,S}u$ then $u \in H^1(\Omega_+)$.*

From corollary 1 it follows that we have trace operator $[\gamma_{0,S}] : H^1(\Omega) \to H^{1/2}_{00}(S)$ where $H^{1/2}_{00}(S) = \{g \in H^{1/2}(S) : p_0 g \in H^{1/2}(\Sigma_0)\}$. Here $p_0 g$ is extension of the function g on S_0 by zero, i.e. $p_0 g(x) = 0$ if $x \in S_0$. The norm in $H^{1/2}_{00}(S)$ is given as [7] $\|g\|_{H^{1/2}_{00}(S)} = \|p_0 g\|_{H^{1/2}(\Sigma_0)}$.

As a consequence we obtain that for $u \in H^1(\Omega)$ the norm $\|\cdot\|_{H^1(\Omega)}$ we can present as

$$\|u\|^2_{H^1(\Omega)} = \|u_1\|^2_{H^1(\Omega_1)} + \|u_2\|^2_{H^1(\Omega_2)},$$

where $u_i = r_{\Omega_i}u$, $i = 1,2$.

Lemma 2.3. *The trace maps $\gamma^\pm_{0,S} : H^1(\Omega) \to H^{1/2}(S)$ and $[\gamma_{0,S}] : H^1(\Omega) \to H^{1/2}_{00}(S)$ are continuous and surjective.*

Proof. Let $g_+ \in H^{1/2}(S)$ be an arbitrary function. We denote by $pg_+ \in H^{1/2}(\Sigma_0)$ the extension of g_+ on Σ_0. The trace map $\gamma^+_{0,\Sigma_0} : H^1(\Omega_1) \to H^{1/2}(\Sigma_0)$ is continuous and surjective. Thus there exists function $u_1 \in H^1(\Omega_1)$ with trace $\gamma^+_{0,\Sigma_0}u_1 = pg_+$ and

$$\|pg\|_{H^{1/2}(\Sigma_0)} \le c\|u_1\|_{H^1(\Omega_1)}. \tag{2.1}$$

Let $g_0 \in H^{1/2}_{00}(S)$, $p_0 g_0 \in H^{1/2}(\Sigma_0)$ is the extension of g_0 on Σ_0 by zero. Then for functions $pg_- = pg_+ + pg_0 \in H^{1/2}(\Sigma_0)$ and $g \in H^{1/2}(\Sigma)$ there exist function $u_2 \in H^1(\Omega_2)$ with traces $\gamma^-_{0,\Sigma_0}u_1 = pg_-$ and $\gamma^+_{0,\Sigma}u_1 = g$.

Thus we have function u where u_i are the restrictions of u on Ω_i, $i = 1,2$. Since $u_i \in H^1(\Omega_i)$ and $[\gamma_{0,S_0}]u = 0$ then $u \in H^1(\Omega)$ and $\gamma^+_{0,S}u = g_+$.

The surjectivity of $\gamma^-_{0,S} : H^1(\Omega) \to H^{1/2}(S)$ we can show analogously. Then from (2.1) we obtain

$$\|g\|_{H^{1/2}(S)} = \inf_{pg \in H^{1/2}(\Sigma_0)} \|pg\|_{H^{1/2}(\Sigma_0)} \le c\|u_1\|_{H^1(\Omega_1)} \le c\|u\|_{H^1(\Omega_+)}.$$

Here c – some positive constant. □

Lemma 2.4. *If $u \in H^1(\Omega_+, L)$ then $\gamma^+_{0,S}u = \gamma^-_{0,S}u$ and $\gamma^+_{1,S}u = \gamma^-_{1,S}u$.*

Proof. Since $u \in H^1(\Omega_+)$ then from Lemma 1 we have $\gamma^+_{0,S}u = \gamma^-_{0,S}u$. For arbitrary $u \in H^1(\Omega_+, L)$ and $v \in H^1(\Omega_+)$ we have the first Green formula

$$(\nabla u, \nabla v)_{L_2(\Omega_+)} = (Lu, v)_{L_2(\Omega_+)} + \langle \gamma^+_{1,\Sigma}u, \gamma^+_{0,\Sigma}v\rangle$$

Also for the restrictions of u and v on Ω_i, $i = 1,2$, we have

$$(\nabla u, \nabla v)_{L_2(\Omega_1)} = (Lu, v)_{L_2(\Omega_1)} + \langle \gamma^+_{1,\Sigma_0}u, \gamma^+_{0,\Sigma_0}v\rangle,$$

$$(\nabla u, \nabla v)_{L_2(\Omega_2)} = (Lu, v)_{L_2(\Omega_2)} - \langle \gamma^-_{1,\Sigma_0} u, \gamma^-_{0,\Sigma_0} v\rangle + \langle \gamma^+_{1,\Sigma} u, \gamma^+_{0,\Sigma} v\rangle,$$

Since $(Lu, v)_{L_2(\Omega_+)} = (Lu, v)_{L_2(\Omega_1)} + (Lu, v)_{L_2(\Omega_2)}$ then we obtain $\langle \gamma^+_{1,\Sigma_0} u, \gamma^+_{0,\Sigma_0} v\rangle - \langle \gamma^-_{1,\Sigma_0} u, \gamma^-_{0,\Sigma_0} v\rangle = 0$. From Lemma 1 we have $\gamma^+_{0,\Sigma_0} v = \gamma^-_{0,\Sigma_0} v$ thus $\langle [\gamma_{1,\Sigma_0}]u, \gamma^+_{0,\Sigma_0} v\rangle = 0$. Trace operator $\gamma^+_{0,\Sigma_0} : H^1(\Omega_1) \to H^{1/2}(\Sigma_0)$ is surjective and as a consequence we have $[\gamma_{1,\Sigma_0}]u = 0$. For arbitrary function $g_0 \in H^{1/2}_{00}(S)$ there exists function $v \in H^1(\Omega_1)$ that $\gamma^+_{0,S} v = g_0$ and $\gamma^+_{0,S_0} v = 0$. Thus from the equality $\langle [\gamma_{1,\Sigma_0}]u, \gamma^+_{0,\Sigma_0} v\rangle = 0$ we obtain $\langle [\gamma_{1,\Sigma_0}]u, g_0\rangle = 0$ for arbitrary $g_0 \in H^{1/2}_{00}(S)$. It give us $[\gamma_{1,\Sigma_0}]u = 0$ or $\gamma^+_{1,S} u = \gamma^-_{1,S} u$. □

Corollary 2.5. *If $u \in H^1(\Omega, L)$ then $[\gamma_{0,S_0}]u = 0$ and $[\gamma_{1,S_0}]u = 0$. If $u \in H^1(\Omega, L)$ and $\gamma^+_{0,S} u = \gamma^-_{0,S} u$, $\gamma^+_{1,S} u = \gamma^-_{1,S} u$ then $u \in H^1(\Omega_+, L)$.*

From Corollary 2 it follows that we have trace operator $[\gamma_{1,S}] : H^1(\Omega, L) \to H^{-1/2}_{00}(S)$ where $H^{-1/2}_{00}(S) = \{f \in H^{1/2}(S) : p_0 f \in H^{-1/2}(\Sigma_0)\}$. Here $p_0 g$ is extension by zero of functional f on S_0 and $H^{-1/2}_{00}(S) = (H^{1/2}(S))'$, $H^{-1/2}(S) = (H^{1/2}_{00}(S))'$ [8, 10].

Based on the above propositions as in [10] we obtain the first Green formula in domain Ω for $u \in H^1(\Omega, L)$ and $v \in H^1(\Omega)$:

$$(\nabla u, \nabla v)_{L_2(\Omega_+)} = (Lu, v)_{L_2(\Omega)} + \langle \gamma^+_{1,S} u, [\gamma_{0,S}]v\rangle +$$

$$+ \langle [\gamma_{1,S}]u, \gamma^-_{0,S} v\rangle + \langle \gamma^+_{1,\Sigma} u, \gamma^+_{0,\Sigma} v\rangle. \tag{2.2}$$

Here $\langle \cdot, \cdot \rangle$ are relations of duality between $H^{1/2}_{00}(S)$ and $H^{-1/2}(S)$, $H^{1/2}(S)$ and $H^{-1/2}_{00}(S)$, $H^{1/2}(\Sigma)$ and $H^{-1/2}(\Sigma)$ respectively.

3 Boundary value problem with transmission boundary condition and it's variational formulation

We consider the following boundary value problem in domain Ω.

Problem R. Find function $u \in H^1(\Omega, L)$ that satisfies

$$Lu = -\Delta u = 0 \quad \text{in} \quad \Omega, \tag{3.1}$$

$$[\gamma_{1,S}]u = 0, \quad \lambda[\gamma_{0,S}]u + \gamma^+_{1,S} u = f, \quad \gamma^+_{0,\Sigma} u = 0. \tag{3.2}$$

Here $f \in H^{-1/2}(S)$ and $\lambda \in C(\bar{S})$ are given.

We can connect with problem R the next variational problem.

Problem VR. Find function $u \in \widetilde{H}^1(\Omega) = \{u \in H^1(\Omega) : \gamma^+_{0,\Sigma} u = 0\}$ that satisfies

$$a(u, v) = l(v)$$

for every $v \in \widetilde{H}^1(\Omega)$.

Here

$$a(u, v) = (\nabla u, \nabla v)_{L_2(\Omega_+)} + (\lambda[\gamma_{0,S}]u, [\gamma_{0,S}]v)_{L_2(S)},$$

$$l(v) = \langle f, [\gamma_{0,S}]v\rangle.$$

Let us note that the space $H^1_0(\Omega_+) = \{u \in H^1(\Omega_+) : \gamma^+_{0,\Sigma} u = 0\}$ is a subspace of the space $\widetilde{H}^1(\Omega)$ and if function $u \in H^1_0(\Omega_+)$ then $[\gamma_{0,S}]u = 0$.

In the space $\widetilde{H}^1(\Omega)$ we introduce the following norm:

$$\|u\|^2_S = \|\nabla u\|^2_{L_2(\Omega)} + \int_S ([\gamma_{0,S}]u(y))^2 ds_y.$$

Lemma 3.1. *The norms $\|\cdot\|_{H^1(\Omega)}$ and $\|\cdot\|_S$ are equivalent, i.e. there exist constants $\alpha > 0$ and $\beta > 0$ that for all $u \in \widetilde{H}^1(\Omega)$ we have*

$$\alpha\|u\|_S \leq \|u\|_{H^1(\Omega)} \leq \beta\|u\|_S.$$

Proof. Let us show that there exists some constant β that for all $u \in \widetilde{H}^1(\Omega)$ we have $\|u\|_{H^1(\Omega)} \leq \beta\|u\|_S$. We suppose contrary. It means that there exists some sequence $\{v_n\} \in \widetilde{H}^1(\Omega)$ and $\geq n\|u\|_S$ for all $n \in \mathbb{N}$. Let $u_n = v_n/\|u\|_{H^1(\Omega)}$. Then $\|u_n\|_{H^1(\Omega)} = 1$ and

$$\|u_n\|_S^2 = \|\nabla u_n\|^2_{L_2(\Omega)} + \int_S ([\gamma_{0,S}]u_n(y))^2 ds_y \leq \frac{1}{n^2}. \tag{3.3}$$

We consider $u_n^{(i)}$ restriction of u_n on Ω_i, $i = 1, 2$. Inasmuch as $\|u_n^{(i)}\|_{H^1(\Omega_i)} \leq 1$ for all n and there is compact imbedding $H^1(\Omega_i) \subset L_2(\Omega_i)$, $i = 1, 2$, then in $L_2(\Omega_i)$ there exists subsequence $u_{nk}^{(i)}$ and functions $u_i \in L_2(\Omega_i)$ that $\|u_{nk}^{(i)} - u_i\|_{L_2(\Omega_i)} = 0$, $n_k \to \infty$. Since $\nabla u_{nk}^{(i)} \in L_2(\Omega_i)$ then $\nabla u_i \in L_2(\Omega_i)$. From (3.3) we have $\|\nabla u_{nk}^{(i)}\|_{L_2(\Omega_i)} \leq 1/n_k$, $i = 1, 2$. Thus $\lim_{n_k\to\infty} \|\nabla u_{nk}^{(i)}\|_{L_2(\Omega_i)} = \|\nabla u_i\|_{L_2(\Omega_i)} = 0$ or $u^{(i)} = c_i = \text{const}$ in Ω_i. From the Corollary 1 it follows that $c_1 = c_2$. Since $\gamma^+_{0,\Sigma}u_2 = 0$ then $u_i(x) = 0$, $x \in \Omega_i$, $i = 1, 2$ or $u(x) = 0$, $x \in \Omega$ where $u_i = r_{\Omega_i}u$ are the restrictions of u on Ω_i. But according our supposition $\|u\|_{H^1(\Omega)} = 1$ and we obtained contradiction. Thus there exists some constant $\beta > 0$ that for all $x \in H^1(\Omega)$ we have $\|u\|_{H^1(\Omega)} \leq \beta\|u\|_S$.

Now we consider the inequality $\alpha\|u\|_S \leq \|u\|_{H^1(\Omega)}$. From the Lemma 2 we have that trace operator $[\gamma_{0,S}] : H^1(\Omega) \to H^{1/2}_{00}(S)$ is bounded. Thus

$$\|[\gamma_{0,S}]u\|_{L_2(S)} \leq \|[\gamma_{0,S}]u\|_{H^{1/2}_{00}(S)} \leq c\|u\|_{H^1(\Omega)},$$

where $c > 0$ – some constant. So there exists some constant $\alpha > 0$ that for all $x \in H^1(\Omega)$ we have $\alpha\|u\|_S \leq \|u\|_{H^1(\Omega)}$. □

Theorem 3.2. *Problems R and VR are equivalent.*

Proof. Let u be a solution of the problem R. It means that $u \in \widetilde{H}^1(\Omega)$ and $[\gamma_{1,S}]u = 0$. Using the first Green formula (2.2) and boundary condition (3.2) we have $a(u, v) = l(v)$ for every $v \in \widetilde{H}^1(\Omega)$. Thus u is a solution of the problem VR.

Let now $u \in \widetilde{H}^1(\Omega)$ be a solution of the problem VR. Then for every $v \in C_0^\infty(\Omega)$ we get $(\nabla u, \nabla v)_{L_2(\Omega_+)} = \langle Lu, v\rangle = 0$ or $Lu = 0$ in Ω. It means that $u \in H^1(\Omega, L)$.

From the first Green formula (2.2) for every $v \in \widetilde{H}^1(\Omega)$ we have

$$\langle \lambda[\gamma_{0,S}]u + \gamma^+_{1,S}u - f, [\gamma_{0,S}]v\rangle + \langle [\gamma_{1,S}]u, \gamma^-_{0,S}v\rangle = 0.$$

If $v \in H^1_0(\Omega_+) \subset \widetilde{H}^1(\Omega)$ or $[\gamma_{0,S}]v = 0$ we obtain $\langle [\gamma_{1,S}]u, \gamma^-_{0,S}v\rangle = 0$. Since trace operator $\gamma^-_{0,S} : H^1_0(\Omega_+) \to H^{1/2}(S)$ is surjective then for arbitrary function $h \in H^{1/2}(S)$ we have $\langle [\gamma_{1,S}]u, h\rangle = 0$ or $[\gamma_{1,S}]u = 0$. On the another side trace operator $[\gamma^-_{0,S}] : \widetilde{H}^1_0(\Omega) \to H^{1/2}_{00}(S)$ is also surjective. From the equality $\langle \lambda[\gamma_{0,S}]u + \gamma^+_{1,S}u - f, h\rangle$ which is valid for arbitrary function $h \in H^{1/2}(S)$ we obtain boundary condition $\lambda[\gamma_{0,S}]u + \gamma^+_{1,S}u = f$. Thus function u is a solution of the problem R. □

Theorem 3.3. *If $\lambda \in C(\bar{S})$, $\lambda(x) \geq 0$, $x \in \bar{S}$, then problem VR has a unique solution $u \in \widetilde{H}^1(\Omega)$ for arbitrary $f \in H^{-1/2}(S)$.*

Proof. Let us show that for $\lambda \in C(\bar{S})$, $\lambda(x) \geq 0$, $x \in \bar{S}$, the bilinear form $a(u, v) : \widetilde{H}^1(\Omega) \times \widetilde{H}^1(\Omega) \to \mathbb{R}$ is continuous and $\widetilde{H}^1(\Omega)$-elliptic and functional $l : \widetilde{H}^1(\Omega) \to \mathbb{R}$ is continuous. By using Lemma 2 we have the following inequality:

$$|a(u, v)| \leq |(\nabla u, \nabla v)_{L_2(\Omega)}| + M \int_S |[\gamma_{0,S}]u(y)| \cdot |[\gamma_{0,S}]v(y)| ds_y \leq$$

$$\leq (1 + M \cdot c)\|u\|_{\widetilde{H}^1(\Omega)}\|v\|_{\widetilde{H}^1(\Omega)},$$

where $M = \max_{x\in S}\lambda(x)$, $c > 0$ – constant. Thus $a(u,v)$ is continuous.

If $\lambda(x) \geq 0$, $x \in \bar{S}$ then

$$a(u,u) = \|\nabla u\|^2_{L_2(\Omega)} + \int_S \lambda([\gamma_{0,S}]u(y))^2 ds_y \geq \|\nabla u\|^2_{L_2(\Omega)} + m\|[\gamma_{0,S}]u\|^2_{L_2(\Omega)},$$

where $m = \min_{x\in S}\lambda(x)$.

If $m \geq 1$ then $a(u,u) \geq \|u\|^2_S$. If $m < 1$ then $a(u,u) \geq m\|u\|^2_S$. From Lemma 4 we obtain existence of some constant $c > 0$ that for all $u \in \widetilde{H}^1(\Omega)$ we have $a(u,u) \geq c\|u\|^2_{\widetilde{H}^1(\Omega)}$, i.e. the bilinear form $a(u,v)$ is $\widetilde{H}^1(\Omega)$-elliptic.

Since the trace operator $[\gamma_{0,S}] : \widetilde{H}^1(\Omega) \to H^{1/2}_{00}(S)$ is continuous we can get

$$|l(v)| = |\langle f, [\gamma_{0,S}]v\rangle| \leq \|f\|_{H^{-1/2}(S)}\|[\gamma_{0,S}]v\|_{H^{1/2}_{00}(S)} \leq c\|f\|_{H^{-1/2}(S)}\|v\|_{\widetilde{H}^1(\Omega)},$$

where $c > 0$ – come constant which does not depend on v. Thus functional $l : \widetilde{H}^1(\Omega) \to \mathbb{R}$ is continuous. Then by the Lax-Milgram Lemma we obtain that problem VR has a unique solution $u \in \widetilde{H}^1(\Omega)$ for arbitrary $f \in H^{-1/2}(S)$. □

Corollary 3.4. *Boundary value problem R has a unique solution $u \in H^1(\Omega, L)$ for arbitrary $f \in H^{-1/2}(S)$ and $\lambda \in C(\bar{S})$, $\lambda(x) \geq 0$, $x \in \bar{S}$.*

4 Integral representation of solution and system of boundary equations

Let $Q(x,y) = \frac{1}{2\pi}\ln\frac{1}{|x-y|}$ be the fundamental solution of operator $L = -\Delta$. For function $u \in H^1(\Omega)$ which satisfies Laplace equation (3.1) we have the following integral representation:

$$u(x) = V[\gamma_{1,S}]u(x) - W[\gamma_{0,S}]u(x) + V_\Sigma[\gamma_{1,\Sigma}]u(x) - W_\Sigma[\gamma_{0,\Sigma}]u(x),$$

where

$$V\tau(x) = \int_S Q(x,y)\tau(y)ds_y, \qquad W\mu(x) = \int_S \frac{\partial Q(x,y}{\partial n_y}\mu(y)ds_y,$$

Analogously for Σ.

If function $u \in H^1(\Omega, L)$ is a solution of the problem R then in Ω we have the next integral representation:

$$u(x) = -W\mu(x) + V_\Sigma\sigma(x), \tag{4.1}$$

where $\mu = [\gamma_{0,S}]u$, $\sigma = [\gamma_{1,\Sigma}]u$,

$$W\mu(x) = \int_S \frac{\partial Q(x,y}{\partial n_y}\mu(y)ds_y, \qquad V_\Sigma\sigma(x) = \int_\Sigma Q(x,y)\sigma(y)ds_y.$$

Let us note that we may use in (4.1) potential of simple or double layer on Σ but on S we have only potential of double layer.

For $\tau \in H^{-1/2}_{00}(S)$ and $\mu \in H^{1/2}_{00}(S)$ we have wellknown jump relation [4, 10]:

$$\gamma_0^+ V\tau = \gamma_0^- V\tau, \quad \gamma_1^\pm V\tau = \pm\frac{1}{2}\tau + N\tau,$$

$$\gamma_1^+ W\mu = \gamma_1^- W\mu, \quad \gamma_0^\pm W\mu = \mp\frac{1}{2}\mu + M\mu,$$

where

$$N\tau = \frac{1}{2}(\gamma_1^+ V\tau + \gamma_1^- V\tau), \; M\mu = \frac{1}{2}(\gamma_0^+ W\mu + \gamma_0^- W\mu).$$

The same relations we have for Σ.

If $\tau \in L_2(S)$ then

$$N\tau(x) = \int_S \frac{\partial Q(x,y}{\partial n_x}\tau(y)ds_y, \qquad M\mu(x) = \int_S \frac{\partial Q(x,y}{\partial n_y}\mu(y)ds_y,$$

If we use boundary conditions of the problem R and the jump relations then for searching of unknown τ and μ we obtain the following system of boundary equations:

$$\begin{cases} \lambda\mu + H\mu + \gamma^+_{1,S}V_\Sigma\sigma = f, \\ -\gamma^+_{0,\Sigma}W\mu + K_\Sigma\sigma = 0. \end{cases} \tag{4.2}$$

Here $H\mu = -\gamma^+_{1,S}W\mu$ is singular integro-differential operator defined on S, and $K_\Sigma\sigma = \gamma^+_{0,\Sigma}V_\Sigma\sigma$ is integral operator with weak singularity given on Σ.

Let us note that operators H for arbitrary Σ and K_Σ if $\mathrm{Cap}_\Sigma < 1$ are positive definite [8], i.e.

$$\langle H\mu, \mu\rangle \geq c_1\|\mu\|^2_{H^{1/2}_{00}(S)}, \quad \langle\sigma, K_\Sigma\sigma\rangle \geq c_2\|\sigma\|^2_{H^{-1/2}(\Sigma)}.$$

where $c_1 > 0$, $c_2 > 0$ – some constants.

Theorem 4.1. *Problem R is equivalent to the system of equations (4.2), i.e. the solution $u(x)$ of the problem R has presentation (4.1), where $\mu \in H^{1/2}_{00}(S)$ and $\sigma \in H^{-1/2}(\Sigma)$ are the solutions of the system (4.2) and vice versa function $u(x)$ given by (4.1) where μ and σ are solutions of the system (4.2) is solution of the problem R.*

Proof. Function $u(x)$ given by (4.1) satisfies Laplace equation (3.1) in Ω and $u \in H^1(\Omega, L)$. If $\mu \in H^{1/2}_{00}(S)$ and $\sigma \in H^{-1/2}(\Sigma)$ are the solutions of the system (4.2) then $u(x)$ satisfies boundary conditions (3.2).

Let now function $u(x) \in H^1(\Omega, L)$ is a solution of the problem R. Then it has integral representation (4.1), where $\mu \in H^{1/2}_{00}(S)$ and $\sigma \in H^{-1/2}(\Sigma)$. If we use boundary conditions (3.2) we can reduce this problem to the system (4.2). □

Corollary 4.2. *If $\lambda \in C(\bar{S}), \lambda(x) \geq 0, x \in \bar{S}$ then system (4.2) has unique solution for arbitrary $f \in H^{-1/2}(S)$ and we have the following estimates:*

$$\|\mu\|^2_{H^{1/2}_{00}(S)} + \|\sigma\|^2_{H^{-1/2}(\Sigma)} \leq c\|f\|^2_{H^{-1/2}(S)},$$

where $c > 0$ – some constant.

When f in the boundary condition (3.2) is a smooth function, for instance $f \in C^{1,\alpha}(\overline{S})$, the system of boundary equations (4.2) has the following form:

$$\begin{cases} \lambda(x)\mu(x) - \left(\int_S \frac{\partial Q(x,y)}{\partial n_x \partial n_y}\mu(y)ds_y\right)^{\pm} + \int_\Sigma \frac{\partial Q(x,y)}{\partial n_x}\sigma(y)ds_y = f(x), \quad x \in S \\ \qquad - \int_S \frac{\partial Q(x,y)}{\partial n_y}\mu(y)ds_y + \int_\Sigma Q(x,y)\sigma(y)ds_y = 0, \quad x \in \Sigma. \end{cases}$$

Let us note that $\mu(y)$ must satisfies conditions $\mu(a) = \mu(b) = 0$, where a and b are the ends of S.

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