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Increased and extended convergence of a family of three-step methods

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Анотація. У різних наукових та інженерних дисциплінах широке коло застосувань можна звести до задач розв'язування нелінійних рівнянь або систем рівнянь у ретельно вибраному абстрактному просторі. Через значні труднощі або навіть неможливість знаходження аналітичних розв'язків, для отримання наближених розв'язків зазвичай використовують ітераційні методи. Ця стаття сфокусована на представленні ефективного сімейства трикрокових ітераційних методів, які демонструють високий порядок збіжності. Враховуючи умови φ-неперервності, накладені на використовувані оператори, проаналізовано властивості локальної та напівлокальної збіжності. Нова методологія, представлена в цій статті, не обмежується конкретними методами і може бути застосована до ширшого діапазону підходів, які передбачають використання обернених лінійних операторів або матриць.

Abstract. In various scientific and engineering disciplines, a wide range of applications can be simplified to the task of solving equations or systems of equations within a carefully selected abstract space. Due to the inherent difficulty or even impossibility of finding analytical solutions, iterative methods are commonly employed to obtain the desired solutions. This article focuses on the presentation of efficient family of three-step iterative methods that exhibit high convergence order. The analysis delves into the local and semi-local convergence properties, considering φ-continuity conditions imposed on the operators utilized. The novel methodology introduced in this article is not limited to specific methods but can be applied to a broader range of approaches that involve the use of inverses of linear operators or matrices.

1 Introduction

In the realm of applied science and technology, numerous challenges can be addressed by reformulating them as non-linear equations in the following format:

$$G(x) = 0, \tag{1.1}$$

where $G : Q \subset Q_1 \to Q_2$ represents a differentiable function in the Fréchet sense. Here, Q_1, Q_2 signifies a complete normed linear space, while Q corresponds to a non-empty, open and convex set.

Typically, closed-form solutions for these nonlinear equations are elusive. Hence, iterative methods are commonly employed to seek their solutions. Among these methods, Newton's method is frequently employed due to its quadratic convergence and a widely used iterative technique for solving equation (1.1). In recent times, significant progress has been made in the field of science and mathematics, leading to the discovery and application of numerous higher-order iterative methods for solving nonlinear equations [2–12, 14–16, 18–23]. However, these methods often suffer from a major drawback, namely the need for computing second and higher-order derivatives, which renders them impractical for real-world applications. The computation cost associated with evaluating G'' in each iteration makes classical cubic convergent schemes less suitable. It is worth noting that many of these methods rely on Taylor expansions to establish convergence results, necessitating derivatives of order higher than the method itself.

Key words: non-linear equations, Fréchet derivative, convergence, Banach space.

The analysis of local and semi-local behavior of iterative methods provides valuable insights into convergence properties, error bounds, and the region of uniqueness for solutions. Several studies [1–5, 18] have focused on investigating the local and semi-local convergence of efficient iterative techniques, yielding significant results such as convergence radii, error estimates, and extended applicability of these methods. These findings are particularly valuable as they shed light on the intricacies involved in selecting appropriate initial points for the iterative process.

In the present article, we introduce and investigate a particular class of methods composed of three sequential steps. The main focus of this study is to establish convergence theorems for these methods, building upon the groundwork laid out in a previous work [23]. The three-step family of methods (TSFM) is defined for $x_0 \in Q$ and each $m = 0, 1, 2, \dots$ by

$$\begin{aligned} y_m &= x_m - bG'(x_m)^{-1}G(x_m),\\ z_m &= T(x_m, y_m),\\ x_{m+1} &= z_m - \left[\frac{1}{b}G'(y_m)^{-1} + \left(1 - \frac{1}{b}\right)G'(x_m)^{-1}\right]G(z_m), \end{aligned} \tag{1.2}$$

where $b \in \mathbb{R} - \{0\}$ is a parameter and T is any iteration operator of convergence order $q \geq 2$.

The local convergence order $q + 2$ is determined in [23] by means of the Taylor expansion series approach when $Q_1 = Q_2 = \mathbb{R}^k$. Moreover, assumptions on the existence and boundedness of $G^{(4)}$ are required limiting the applicability of TSFM to solve equations with operators at least four times differentiable. But TSFM may converge if only G' appearing on it exists. To illustrate the concept, let us consider a motivational example where G is defined on the interval $Q = [-0.5, 1.5]$ as follows:

$$G(x) = \begin{cases} \frac{1}{3}x^3\ln(x) + 8x^5 - 8x^4, & \text{if} \quad x \neq 0,\\ 0, & \text{if} \quad x = 0. \end{cases} \tag{1.3}$$

We can observe that the solution $\bar{x} = 1 \in Q$ and the third derivative is given by

$$G'''(x) = \frac{11}{3} - 192x + 480x^2 + 2\ln(x).$$

It is obvious that G''' is unbounded on Q. Thus, utilising the findings in [23], convergence isn't always guaranteed. That is why the local analysis should be based on information only in TSFM (i.e., on G and G'). Such an analysis is presented in Section 2 for Banach space valued operators using generalized continuity conditions on G'. Moreover, the more important semi-local analysis of TSFM not given in [23] is studied in Section 3 by means of majorizing sequences. The methodology applied on (1.2) can be used to extend the applicability of other methods analogously [1–22].

The remaining sections of this paper are organized as follows: Section 4 focuses on the investigation of convergence properties of specific cases of equation (1.2). Section 5 includes numerical applications that utilize the convergence results derived in the previous sections. Finally, Section 6 concludes this paper with closing remarks.

2 Local Analysis

Throughout this Section we assume the existence of a solution $\bar{x} \in Q$ for the equation $G(x) = 0$. Let $M = [0, +\infty)$. Moreover, assume:

(C_1) There exists a function $\varphi_0 : M_0 \to M$ which is continuous and non-decreasing (FCN) so that $\varphi_0(t) - 1 = 0$ admits a smallest solution (SS). Denote such solution by ρ_0 and let $M_0 = [0, \rho_0)$.

(C_2) There exist FCN $\varphi : M_0 \to M$, $g_2 : M_1 \subset M_0 \to M$, for some subset of M_0, such that $g_1(t) - 1 = 0$ and $g_2(t) - 1 = 0$ admit SS, where

$$g_1(t) = \frac{\int_0^1 \varphi((1-\theta)t)d\theta + |1-b|(1+\int_0^1 \varphi_0(\theta t)d\theta)}{1-\varphi_0(t)}.$$

Denote such solutions by r_1, r_2, respectively.

(C_3) The equations $\varphi_0(g_1(t)t) - 1 = 0$ and $\varphi_0(g_2(t)t) - 1 = 0$ admit SS in $M_1 - \{0\}$. Denote such solutions by ρ_1, ρ_2, respectively. Let $\rho = min\{\rho_1, \rho_2\}$ and $M_2 = [0, \rho)$.

(C_4) The equation $g_3(t) - 1 = 0$ admits SS in $M_2 - \{0\}$, denoted as r_3, where $g_3 : M_2 \to M$ is given as

$$g_3(t) = \Bigg[\frac{\int_0^1 \varphi((1-\theta)g_2(t)t)d\theta}{1-\varphi_0(g_2(t)t)} + \Bigg(\frac{\bar{\bar{\varphi}}(t)}{(1-\varphi_0(t))(1-\varphi_0(g_2(t)t))}$$
$$+ \left|1-\frac{1}{b}\right| \frac{\bar{\varphi}(t)}{(1-\varphi_0(t))(1-\varphi_0(g_1(t)t))}\Bigg)\left(1+\int_0^1 \varphi_0(\theta g_2(t)t)d\theta\right)\Bigg] g_2(t),$$

where

$$\bar{\varphi}(t) = \begin{cases} \varphi((1+g_1(t))t), \\ \varphi_0(t) + \varphi_0(g_1(t)t) \end{cases}$$

and

$$\bar{\bar{\varphi}}(t) = \begin{cases} \varphi((g_1(t)+g_2(t))t), \\ \varphi_0(g_1(t)t) + \varphi_0(g_2(t)t). \end{cases}$$

In practice the smallest of the two versions of functions $\bar{\varphi}$ and $\bar{\bar{\varphi}}$ are chosen.

(C_5) There exists a linear operator $\mathscr{A} : Q_1 \to Q_2$ such that $\mathscr{A}^{-1} \in \mathscr{L}(Q_2, Q_1)$, which is the space of linear continuous operators mapping Q_2 into Q_1.

(C_6) $\|\mathscr{A}^{-1}(G'(x) - \mathscr{A})\| \le \varphi_0(\|x - \bar{x}\|)$ for all $x \in Q$.

(C_7)

$$\|T(x, x - bG'(x)^{-1}G(x)) - \bar{x}\| \le g_2(\|x - \bar{x}\|)\|x - \bar{x}\|$$

and

$$\|\mathscr{A}^{-1}(G'(y) - G'(x))\| \le \varphi(\|y - x\|)$$

for all $x \in S_0 = S(\bar{x}, \rho_0) \cap Q$.
Notice that by the assumptions (C_1) and (C_6), $\varphi_0(\|x - \bar{x}\|) < 1$. Thus, $G'(x)^{-1} \in \mathscr{L}(Q_2, Q_1)$ by the standard Banach perturbation Lemma [1,15,17] involving linear operators. Hence, the second variable term in T of assumption (C_7) is well defined.
and

(C_8) $S[\bar{x}, r] \subset Q$, where $r = \min\{r_i\}$, $i = 1, 2, 3$.

Notice that the usual assumption in such studies is that $\bar{x}$ is simple, i.e., $G'(\bar{x})^{-1} \in \mathscr{L}(Q_2, Q_1)$ [23]. However, such assumption is not made here. Therefore, the results can be used to approximate a solution $\bar{x}$ of multiplicity greater than one.

The calculations requiring these assumptions for $x_0 \in S(\bar{x}, r) - \{\bar{x}\}$ and induction are in turn:

$$\|\mathscr{A}^{-1}(G'(x) - \mathscr{A})\| \leq \varphi_0(\|x - \bar{x}\|) < \varphi_0(r) < 1 \quad \text{for} \quad x \in S(\bar{x}, r) - \{\bar{x}\},$$

$$\text{so} \quad \|G'(x)^{-1}\mathscr{A}\| \leq \frac{1}{1 - \varphi_0(\|x - \bar{x}\|)},$$

$$y_j - \bar{x} = x_j - \bar{x} - G'(x_j)^{-1}G(x_j) + (1-b)G'(x_j)^{-1}G(x_j),$$

$$\|y_j - \bar{x}\| \leq \frac{\left[\int_0^1 \varphi((1-\theta)\|x_j - \bar{x}\|)d\theta + |(1-b)| \left(1 + \int_0^1 \varphi_0(\theta\|x_j - \bar{x}\|)d\theta\right)\right] \|x_j - \bar{x}\|}{1 - \varphi_0(\|x_j - \bar{x}\|)}$$

$$\leq g_1(\|x_j - \bar{x}\|)\|x_j - \bar{x}\| \leq \|x_j - \bar{x}\| < r,$$

$$\|z_j - \bar{x}\| = \|T(x_j, y_j) - \bar{x}\| \leq g_2(\|x_j - \bar{x}\|)\|x_j - \bar{x}\|$$

$$\leq \|x_j - \bar{x}\|$$

and

$$x_{j+1} - \bar{x} = z_j - \bar{x} - G'(z_j)^{-1}G(z_j) + \left[G'(z_j)^{-1} - \frac{1}{b}G'(y_j)^{-1} - (1-b)G'(x_j)^{-1}\right] G(z_j)$$

$$= z_j - \bar{x} - G'(z_j)^{-1}G(z_j) + \left[(G'(z_j)^{-1} - G'(y_j)^{-1}) + \left(1 - \frac{1}{b}\right)(G'(y_j)^{-1} - G'(x_j)^{-1})\right] G(z_j),$$

$$\|x_{j+1} - \bar{x}\| \leq \left[\frac{\int_0^1 \varphi((1-\theta)\|z_j - \bar{x}\|)d\theta}{1 - \varphi_0(\|z_j - \bar{x}\|)} + \left(\frac{\bar{\bar{\varphi}}(\|x_j - \bar{x}\|)}{(1 - \varphi_0(\|x_j - \bar{x}\|))(1 - \varphi_0(\|z_j - \bar{x}\|))}\right.\right.$$

$$\left.\left. + \left|1 - \frac{1}{b}\right| \frac{\bar{\varphi}(\|x_j - \bar{x}\|)}{(1 - \varphi_0(\|x_j - \bar{x}\|))(1 - \varphi_0(\|y_j - \bar{x}\|))}\right)(1 + \int_0^1 \varphi_0(\theta\|z_j - \bar{x}\|)d\theta)\right] \|z_j - \bar{x}\|$$

$$\leq g_3(\|x_j - \bar{x}\|)\|x_j - \bar{x}\| \leq \|x_j - \bar{x}\|,$$

where

$$\bar{\varphi}(\|x_j - \bar{x}\|) = \begin{cases} \varphi(\|x_j - \bar{x}\| + \|y_j - \bar{x}\|) \\ \varphi_0(\|x_j - \bar{x}\|) + \varphi_0(\|y_j - \bar{x}\|) \end{cases}$$

and

$$\bar{\bar{\varphi}}(\|x_j - \bar{x}\|) = \begin{cases} \varphi(\|y_j - \bar{x}\| + \|z_j - \bar{x}\|) \\ \varphi_0(\|y_j - \bar{x}\|) + \varphi_0(\|z_j - \bar{x}\|). \end{cases}$$

Hence, we showed by induction:

Theorem 2.1. *Given the assumptions (C_1)-(C_8), it is established that $\{x_m\} \subset S(\bar{x}, r)$ and converges to $\bar{x}$ as m tends to positive infinity, provided that the initial value x_0 lies in the set $S(\bar{x}, r) - \{\bar{x}\}$.* We now provide a result that establishes the uniqueness of the solution in the context of local convergence.

Proposition 2.1. *Assume that there exists a solution $x^* \in S(\bar{x}, \rho_3)$ of the equation $G(x) = 0$, where $\rho_3 > 0$.*

Furthermore, assume the condition in (C_6) is satisfied within the ball $S(\bar{x}, \rho_3)$, and there exists a larger radius $\rho_4 \geq \rho_3$ such that

$$\int_0^1 \varphi_0(\theta\rho_4)d\theta < 1. \tag{2.1}$$

Let $S_1 = Q \cap S[\bar{x}, \rho_4]$. Then, $\bar{x}$ is the unique solution of the equation $G(x) = 0$ within the set S_1.

Proof. Define the linear operator $\mathscr{W} = \int_0^1 G'(\bar{x} + \theta(x^* - \bar{x}))d\theta$. Utilizing condition in (C_6) and (2.1), we can deduce the following:

$$\begin{aligned}\|G'(\bar{x})^{-1}(\mathscr{W} - G'(\bar{x}))\| &\le \int_0^1 \varphi_0(\theta\|x^* - \bar{x}\|)d\theta \\ &\le \int_0^1 \varphi_0(\theta\rho_4)d\theta \\ &< 1.\end{aligned}$$

Therefore, $\mathscr{W}^{-1} \in \mathscr{L}(Q_2, Q_1)$, and based on the approximation

$$x^* - \bar{x} = \mathscr{W}^{-1}(G(x^*) - G(\bar{x})) = \mathscr{W}^{-1}(0) = 0,$$

we conclude that $x^* = \bar{x}$. □

3 Semi-Local Analysis

Analogous to the local analysis but with the role of $\bar{x}$, "φ" is exchanged by x_0, "ψ" functions which are developed below.

Assume:

(H_1) There exists FCN $\psi_0 : M \to M$ such that $\psi_0(t) - 1 = 0$ admits SS. Denote such solution by R. Let $M_3 = [0, R]$ and $S_3 = S(x_0, R) \cap Q$.

(H_2) Same as (C_5).

(H_3) $\|\mathscr{A}^{-1}(G'(x) - \mathscr{A})\| \le \psi_0(\|x - x_0\|)$ for all $x \in Q$.

(H_4) There exists FCN $\psi : M_3 \to M$ so that

$$\|\mathscr{A}^{-1}(G'(y) - G'(x))\| \le \psi(\|y - x\|) \quad \text{for all} \quad x, y \in S_3.$$

(H_5) $\|T(x, x - bG'(x)^{-1}G(x)) - (x - bG'(x)^{-1}G(x))\| \le \gamma(x) - \beta(x)$ for some FCN functions $\gamma, \beta : M_3 \to M$ and all $x \in S_3$.

Define the real sequence $\{\alpha_m\}$ for $\alpha_0 = 0, \beta_0 \ge |b|\|G'(x_0)^{-1}G(x_0)\|$ and all $m = 0, 1, 2, \cdots$ for some FCN $\bar{\beta} : M_3 \to M$,

$$\begin{aligned}\gamma_m &= \beta_m + \bar{\beta}_m, \\ \lambda_m &= \left(1 + \int_0^1 \psi_0(\alpha_m + \theta(\gamma_m - \alpha_m))d\theta\right)(\gamma_m - \alpha_m) + \frac{1}{|b|}(1 + \psi_0(\alpha_m))(\beta_m - \alpha_m), \\ \bar{\psi}_m &= \begin{cases} \psi(\beta_m - \alpha_m) \\ \psi_0(\alpha_m) + \psi_0(\beta_m) \end{cases},\end{aligned}$$

$$\alpha_{m+1} = \gamma_m + \frac{1}{1 - \psi_0(\alpha_m)}\left(1 + \frac{\bar{\psi}_m}{|b|(1 - \psi_0(\beta_m))}\right)\lambda_m, \tag{3.1}$$

$$\delta_{m+1} = \left(1 + \int_0^1 \psi_0(\alpha_m + \theta(\alpha_{m+1} - \alpha_m))d\theta\right)(\alpha_{m+1} - \alpha_m) + \frac{1}{|b|}(1 + \psi_0(\alpha_m))(\beta_m - \alpha_m)$$

and

$$\beta_{m+1} = \alpha_{m+1} + |b|\frac{\delta_{m+1}}{1 - \psi_0(\alpha_{m+1})}.$$

As in the local case, $\psi_0(\|x_0 - x_0\|) = \psi_0(0) < \psi_0(R) \le 1$. Thus, $G'(x_0)^{-1} \in \mathscr{L}(Q_2, Q_1)$ and the iterate β_0 is well defined. Notice also that by $\gamma(x_m) = \gamma_m$ and $\beta(x_m) = \beta_m$. These functions are further specialized if the operator T is precised (see the Numerical Section).

(H_6) There exists $R_0 \in [0, R)$ such that for all $m = 0, 1, 2, \dots$

$$\psi_0(\alpha_m) < 1 \quad \text{and} \quad \alpha_m \le R_0.$$

It follows by this assumption and (3.1) that

$$0 \le \alpha_m \le \beta_m \le \gamma_m \le \alpha_{m+1} < R_0$$

and the sequence $\{\alpha_m\}$ is convergent to its least upper bound $R^* \in [0, R_0]$. This limit is unique.

And

(H_7) $S[x_0, R^*] \subset Q$.

The sequence $\{\alpha_m\}$ is motivated by a series of calculations as in the local case which in turn are:

$$\begin{aligned}
\|z_j - y_j\| &= \|T(x_j, y_j) - y_j\| \le \gamma_j - \beta_j,\\
\|z_j - x_0\| &\le \|z_j - y_j\| + \|y_j - x_0\| \le \gamma_j - \beta_j + \beta_j - \alpha_j = \gamma_j < R^*,\\
G(z_j) &= G(z_j) - G(x_j) - \frac{1}{b}G'(x_j)(y_j - x_j),\\
\|\mathscr{A}^{-1}G(z_j)\| &\le \left(1 + \int_0^1 \psi_0(\|x_j - x_0\| + \theta\|z_j - x_j\|)d\theta\right)\|z_j - x_j\|\\
&\quad + \frac{1}{|b|}(1 + \psi_0(\|x_j - x_0\|))\|y_j - x_j\|\\
&\le \left(1 + \int_0^1 \psi_0(\alpha_j + \theta(\gamma_j - \alpha_j))d\theta\right)(\gamma_j - \alpha_j) + \frac{1}{|b|}(1 + \psi_0(\alpha_j))(\beta_j - \alpha_j) = \lambda_j,\\
\|x_{j+1} - z_j\| &= \left\|\left[G'(x_j)^{-1} + \frac{1}{b}(G'(y_j) - G'(x_j)^{-1})\right]G(z_j)\right\|\\
&\le \frac{1}{1 - \psi_0(\|x_j - x_0\|)}\left[1 + \frac{\bar{\psi}_j}{|b|(1 - \psi_0(\|y_j - x_0\|))}\right]\lambda_j\\
&\le \alpha_{j+1} - \gamma_j,\\
\|x_{j+1} - x_0\| &\le \|x_{j+1} - z_j\| + \|z_j - x_0\| \le \alpha_{j+1} - \gamma_j + \gamma_j - \alpha_0 = \alpha_{j+1} < R^*, \qquad (3.2)\\
G(x_{j+1}) &= G(x_{j+1}) - G(x_j) - \frac{1}{b}G'(x_j)(y_j - x_j),\\
\|\mathscr{A}^{-1}G(x_{j+1})\| &\le \left(1 + \int_0^1 \psi_0(\alpha_j + \theta(\alpha_{j+1} - \alpha_j))d\theta\right)(\alpha_{j+1} - \alpha_j)\\
&\quad + \frac{1}{|b|}(1 + \psi_0(\alpha_j))(\beta_j - \alpha_j) = \delta_{j+1},\\
\|y_{j+1} - x_{j+1}\| &\le |b|\frac{\delta_{j+1}}{1 - \psi_0(\|x_{j+1} - x_0\|)}\\
&\le |b|\frac{\delta_{j+1}}{1 - \psi_0(\alpha_{j+1})} = \beta_{j+1} - \alpha_{j+1}
\end{aligned}$$

and

$$\begin{aligned}
\|y_{j+1} - x_0\| &\le \|y_{j+1} - x_{j+1}\| + \|x_{j+1} - x_0\|\\
&\le \beta_{j+1} - \alpha_{j+1} + \alpha_{j+1} - \alpha_0 = \beta_{j+1} < R^*.
\end{aligned}$$

Thus, the iterates $\{x_j\}, \{y_j\}, \{z_j\} \subset S(x_0, R^*)$ are Cauchy in Banach space Q_1. Hence, there exists $\bar{x} \in S[x_0, R^*]$ so that $\lim_{j\to+\infty} x_j = \bar{x}$. Moreover, by (3.2) $G(\bar{x}) = 0$. Furthermore, by the estimate

$$\|x_{j+k} - x_j\| \leq \alpha_{j+k} - \alpha_j,$$

the useful items

$$\|\bar{x} - x_j\| \leq R^* - \alpha_j$$

become available.

Hence, we arrive at.

Theorem 3.1. *Subject to the conditions (H_1)-(H_7), the sequence $\{x_m\}$ converges towards a solution $\bar{x} \in S[x_0, R^*]$ of the equation $G(x) = 0$.*

We establish the uniqueness of the solution domain in the following proposition.

Proposition 3.2. *Assume the following conditions:*

(i) There exists a solution $\bar{x}^*$ of the equation $G(x) = 0$ in $S(x_0, R_1)$ for some $R_1 > 0$.

(ii) Condition (H_3) holds on $S(x_0, R_1)$.

(iii) There exists $R_2 > R_1$ such that

$$\int_0^1 \psi_0((1-\theta)R_1 + \theta R_2)d\theta < 1.$$

Set $S_4 = Q \cap S[x_0, R_2]$.

Then, the only point in the domain S_4 that satisfies the equation $G(x) = 0$ is $\bar{x}^*$.

Proof. Let us assume that there exists $x^{'} \in S_4$ such that $G(x) = 0$. Conditions (ii) and (iii) allow us to obtain the following inequality:

$$\begin{aligned}\|G'(x_0)^{-1}(\mathscr{X} - G'(x_0))\| &\leq \int_0^1 \psi_0((1-\theta)\|\bar{x}^* - x_0\| + \theta\|x^{'} - x_0\|)d\theta \\ &\leq \int_0^1 \psi_0((1-\theta)R_1 + \theta R_2)d\theta \\ &< 1,\end{aligned}$$

where $\mathscr{X} = \int_0^1 G'(\bar{x}^* + \theta(x^{'} - \bar{x}^*))d\theta$. Hence, we conclude that $x^{'} = \bar{x}^*$. □

Remark 3.1.

(i) In condition (H_7), the limit point R^* can be replaced by R.

(ii) Under all the assumptions (H_1)–(H_7), let $\bar{x}^* = \bar{x}$ and $R_1 = R^*$ in Proposition 3.2.

4 Special choices and Applications

Let us specialize b and T, so that we can determine the function g_2 in the local case and the majorant sequence in the semi-local case. In particular, the fifth order methods studied respectively in [13, 22] have been reduced to

$$b = 1, \quad T(x_j, y_j) = x_j - \frac{1}{2}(G'(y_j)^{-1} + G'(x_j)^{-1})G(x_j)$$

and

$$b = \frac{1}{2}, \quad T(x_j, y_j) = x_j - G'(y_j)^{-1}G(x_j).$$

That is, method (1.2) specializes to

$$\begin{aligned} y_j &= x_j - G'(x_j)^{-1}G(x_j), \\ z_j &= x_j - \frac{1}{2}(G'(y_j)^{-1} + G'(x_j)^{-1})G(x_j), \\ x_{j+1} &= z_j - G'(y_j)^{-1}G(z_j) \end{aligned} \tag{4.1}$$

and

$$\begin{aligned} y_j &= x_j - \frac{1}{2}G'(x_j)^{-1}G(x_j), \\ z_j &= x_j - G'(y_j)^{-1}G(x_j) \\ x_{j+1} &= z_j - (2G'(y_j)^{-1} - G'(x_j)^{-1})G(z_j), \end{aligned} \tag{4.2}$$

respectively.

We shall also consider the sixth order method studied in [21], for

$$b = \frac{2}{3}, \quad T(x_j, y_j) = x_j - \frac{1}{2}\left(-I + \frac{9}{4}G'(y_j)^{-1}G'(x_j) + \frac{3}{4}G'(x_j)^{-1}G'(y_j)\right)G'(x_j)^{-1}G(x_j).$$

That is, we have the method

$$\begin{aligned} y_j &= x_j - \frac{2}{3}G'(x_j)^{-1}G(x_j), \\ z_j &= x_j - \frac{1}{2}\left(-I + \frac{9}{4}G'(y_j)^{-1}G'(x_j) + \frac{3}{4}G'(x_j)^{-1}G'(y_j)\right)G'(x_j)^{-1}G(x_j) \\ x_{j+1} &= z_j - \frac{1}{2}(3G'(y_j)^{-1} - G'(x_j)^{-1})G(z_j). \end{aligned} \tag{4.3}$$

Next, we determine the function g_2 for each of the above choices.

Method (4.1)

The motivational estimates are:

$$z_m - \bar{x} = x_m - \bar{x} - G'(x_m)^{-1}G(x_m) - \frac{1}{2}(G'(y_m)^{-1} - G'(x_m)^{-1})G(x_m),$$

$$\|z_m - \bar{x}\| \le \left[\frac{\int_0^1 \varphi((1-\theta)\|x_m - \bar{x}\|)d\theta}{1 - \varphi_0(\|x_m - \bar{x}\|)} + \frac{\bar{\varphi}_m\left(1 + \int_0^1 \varphi_0(\theta\|x_m - \bar{x}\|)d\theta\right)}{2(1 - \varphi_0(\|x_m - \bar{x}\|))(1 - \varphi_0(\|y_m - \bar{x}\|))}\right]\|x_m - \bar{x}\|,$$

thus, we can choose

$$g_2(t) = \frac{\int_0^1 \varphi((1-\theta)t)d\theta}{1 - \varphi_0(t)} + \frac{\bar{\varphi}(t)(1 + \int_0^1 \varphi_0(\theta t)d\theta)}{2(1 - \varphi_0(t))(1 - \varphi_0(g_1(t)t))}.$$

Method (4.2)

The calculations here yields

$$z_m - \bar{x} = x_m - \bar{x} - G'(x_m)^{-1}G(x_m) + (G'(x_m)^{-1} - G'(y_m)^{-1})G(x_m),$$

$$\|z_m - \bar{x}\| \le \left[\frac{\int_0^1 \varphi((1-\theta)\|x_m - \bar{x}\|)d\theta}{1 - \varphi_0(\|x_m - \bar{x}\|)} + \frac{\bar{\varphi}_m\left(1 + \int_0^1 \varphi_0(\theta\|x_m - \bar{x}\|)d\theta\right)}{(1 - \varphi_0(\|x_m - \bar{x}\|))(1 - \varphi_0(\|y_m - \bar{x}\|))}\right]\|x_m - \bar{x}\|.$$

Thus,

$$g_2(t) = \frac{\int_0^1 \varphi((1-\theta)t)d\theta}{1 - \varphi_0(t)} + \frac{\bar{\varphi}(t)\left(1 + \int_0^1 \varphi_0(\theta t)d\theta\right)}{(1 - \varphi_0(t))(1 - \varphi_0(g_1(t)t))}.$$

Method (4.3)

This time we get in turn

$$z_m - \bar{x} = x_m - \bar{x} - G(x_m)^{-1}G(x_m) - \frac{1}{2}\Big[-I - 2I + \frac{9}{4}(G'(y_m)^{-1}G'(x_m) - I) + \frac{9}{4}I$$
$$+ \frac{3}{4}(G'(x_m)^{-1}G'(y_m) - I) + \frac{3}{4}I\Big] G'(x_m)^{-1}G(x_m),$$

$$\|z_m - \bar{x}\| \leq \Bigg[\frac{\int_0^1 \varphi((1-\theta)\|x_m - \bar{x}\|)d\theta}{1 - \varphi_0(\|x_m - \bar{x}\|)} + \frac{3\bar{\varphi}_m}{8}\Bigg(\frac{1}{1-\varphi_0(\|x_m - \bar{x}\|)}$$
$$+ \frac{3}{1 - \varphi_0(\|y_m - \bar{x}\|)}\Bigg)\frac{\left(1 + \int_0^1 \varphi_0(\theta\|x_m - \bar{x}\|)d\theta\right)}{1 - \varphi_0(\|x_m - \bar{x}\|)}\Bigg]\|x_m - \bar{x}\|.$$

Hence,

$$g_2(t) = \frac{\int_0^1 \varphi((1-\theta)t)d\theta}{1-\varphi_0(t)} + \frac{3\bar{\varphi}(t)}{8}\left(\frac{1}{1-\varphi_0(t)} + \frac{3}{1-\varphi_0(g_1(t)t)}\right)\frac{\left(1 + \int_0^1 \varphi_0(\theta t)d\theta\right)}{1-\varphi_0(t)}.$$

The iterates for the semi-local case follow as given below:

Method (4.1)

It follows by eliminating the iterate x_m from the second sub-step by means of the first sub-step

$$z_m - y_m = \left(G'(x_m)^{-1} - \frac{1}{2}G'(y_m)^{-1} - \frac{1}{2}G'(x_m)^{-1}\right)G(x_m)$$
$$= -\frac{1}{2}G'(y_m)^{-1}(G'(x_m) - G'(y_m))G'(x_m)^{-1}G(x_m)$$
$$= \frac{1}{2}G'(y_m)^{-1}(G'(x_m) - G'(y_m))(y_m - x_m)$$

and

$$\|z_m - y_m\| \leq \frac{1}{2}\frac{\bar{\psi}_m(\beta_m - \alpha_m)}{1 - \psi_0(\beta_m)} = \gamma_m - \beta_m.$$

Method (4.2)

Similarly, we have for this method

$$z_m - y_m = \frac{1}{2}G'(x_m)^{-1}G(x_m) - G'(y_m)^{-1}G(x_m)$$
$$= G'(y_m)^{-1}\left[(G'(x_m) - G'(y_m)) + G'(x_m)\right](y_m - x_m)$$

and

$$\|z_m - y_m\| \leq \frac{\bar{\psi}_m + 1 + \psi_0(\alpha_m)}{1 - \psi_0(\beta_m)}(\beta_m - \alpha_m) = \gamma_m - \beta_m.$$

Method (4.3)

For this method, we can get

$$z_m - y_m = \left[\frac{2}{3}I + \frac{1}{2}I - \frac{9}{8}G'(y_m)^{-1}G'(x_m) - \frac{3}{8}G'(x_m)^{-1}G'(y_m)\right]\left(-\frac{3}{2}(y_m - x_m)\right)$$
$$= \frac{3}{16}\left[8I + 9(G'(y_m)^{-1}G'(x_m) - I) + 3(G'(x_m)^{-1}G'(y_m) - I)\right](y_m - x_m),$$
$$\|z_m - y_m\| \leq \frac{3}{16}\left(8 + \frac{9\bar{\psi}_m}{1-\psi_0(\beta_m)} + 3\frac{\bar{\psi}_m}{1-\psi_0(\alpha_m)}\right)(\beta_m - \alpha_m) = \gamma_m - \beta_m.$$

5 Numericals

The following examples demonstrate how to apply the local convergence criteria.

Example 5.1. *We study a system of differential equations of the form:*

$$G_1'(x_1) = e^{x_1}, \quad G_2'(x_2) = (e-1)x_2 + 1, \quad G_3'(x_3) = 1$$

with the starting point $G_1(0) = G_2(0) = G_3(0) = 0$. We let $G = (G_1, G_2, G_3)$ and $Q_1 = Q_2 = \mathbb{R}^3$ and $Q = S[0,1]$. The point $\bar{x} = (0,0,0)^T$ is a solution of the system. We define a function G on Q for any vector $x = (x_1, x_2, x_3)^T$ as

$$G(x) = (e^{x_1} - 1, \frac{e-1}{2}x_2^2 + x_2, x_3)^T.$$

The function G has this derivative matrix:

$$G'(x) = \begin{bmatrix} e^{x_1} & 0 & 0 \\ 0 & (e-1)x_2 + 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and we see that $G'(\bar{x}) = I$. To check the local convergence criteria, we need to meet the conditions $(C_1) - (C_8)$. We can do that by choosing $\varphi_0(t) = (e-1)t$, $\varphi(t) = e^{\frac{1}{e-1}}t$, $\rho_0 = 0.581977$ and $S_0 = S \cap S(\bar{x}, \rho_0)$. The radii for methods (4.1),(4.2),(4.3) are given in Table 5.1.

Table 5.1. Estimates for Example (5.1)

Radii	r_1	r_2	r_3	$r = min\{r_i\}$
Method (4.1)	0.382692	0.235533	0.21314	0.21314
Method (4.2)	0.164331	0.138126	0.121147	0.121147
Method (4.3)	0.229929	0.1244	0.123482	0.123482

Example 5.2. *Let $Q_1 = Q_2 = Q = \mathbb{R}$. We define a function G on Q by $G(x) = sinx$. The derivative of G is $G'(x) = cosx$. The fixed point is $x^* = 0$. To check the conditions $(C_1) - (C_8)$, we choose $\varphi_0(t) = \varphi(t) = t$, $\rho_0 = 1$ and $S_0 = S \cap S(\bar{x}, \rho_0)$. The radii of convergence are shown in Table 5.2.*

Example 5.3. *We analyze the equation system given by*

$$G(x) = \left(x_1^2 + \sin(x_1) - \exp(x_2), 3x_1 - \cos(x_1) - x_2\right)^T,$$

where the initial conditions are set as $x_0 = \{-1.5, -2\}^T$. We obtain the solution

$$\bar{x} = \{-0.907430217073695685..., -3.338063225186236275...\}^T.$$

Table 5.2. Estimates for Example (5.2)

Radii	r_1	r_2	r_3	$r = min\{r_i\}$
Method (4.1)	0.666667	0.409084	0.369883	0.369883
Method (4.2)	0.285714	0.238655	0.209218	0.209218
Method (4.3)	0.4	0.214922	0.213304	0.213304

Table 5.3. Estimates for Example (5.3)

Methods	$\|x_1 - \bar{x}\|$	$\|x_2 - \bar{x}\|$
Method (4.1)	0.00106372955621983	$2.968770575826642 \cdot 10^{-11}$
Method (4.2)	0.04276386733727288	$8.68882879660436 \cdot 10^{-9}$
Method (4.3)	0.01838299860784418	$2.841080214146303 \cdot 10^{-11}$

Methods	$\|x_3 - \bar{x}\|$	$\|x_4 - \bar{x}\|$
Method (4.1)	$2.968770575826642 \cdot 10^{-11}$	$2.968770575826642 \cdot 10^{-11}$
Method (4.2)	$2.968760738845313 \cdot 10^{-11}$	$2.968760738845313 \cdot 10^{-11}$
Method (4.3)	$2.968760738845313 \cdot 10^{-11}$	$2.968760738845313 \cdot 10^{-11}$

Table 5.3 provides error estimates for the methods under consideration.

Example 5.4. *Consider the function* $G(x) = (g_1(x), \dots, g_n(x))$, *where*

$$g_i(x) = x_{(i)} + 1 - 2\ln\left(1 + \sum_{j=1, j\neq i}^{n} x_{(j)}\right), \quad 1 \le i \le n,$$

with $n = 8$. *The initial conditions are given by* $x_0 = \{5.35, \dots, 5.35\}^T$, *and the solution is*

$$\bar{x} = \{6.7539323119353358594..., \dots, 6.7539323119353358594...\}^T.$$

By utilizing methods (4.1), (4.2), and (4.3), it is observed that the system achieves convergence to the solution $\bar{x}$ *within 6, 7, and 5 iterations, respectively.*

6 Conslusion

We have introduced a novel technique that allows for the demonstration of both local convergence analysis and semi-local convergence analysis of high convergence order methods, utilizing only the derivatives present within the method itself. Previous works have often assumed the existence of high-order derivatives that may not be inherent to the method, thereby limiting their applicability. In contrast, our technique overcomes this limitation and provides error bounds and uniqueness results that were previously unavailable. Moreover, this technique exhibits a high level of generality as it is independent of the specific method being employed. Consequently, it can be readily applied to extend the applicability of other higher order methods, including single step or multi-step methods [6–12, 14, 19, 20, 20, 23].

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