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Extended Kurchatov-type methods for solving nonlinear equations

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Анотація. Широке коло задач із різних галузей може бути розв'язано шляхом зведення їх до нелінійних рівнянь у відповідних просторах. Ці рівняння зазвичай розв'язуються ітераційними методами. Розглянуто трикрокові ітераційні методи типу Курчатова для розв'язування нелінійних операторних рівнянь, використовуючи апроксимацію похідної Фреше оператора нелінійного рівняння поділеними різницями. Проведено аналіз локальної та напівлокальної збіжності, використовуючи лише умови для операторів, та з'ясовано умови та швидкість збіжності цих методів. Крім того, знайдено область єдиності розв'язку. Результати чисельних експериментів підтверджують теоретичні результати. Нова ідея може бути використана в інших ітераційних методах, що потребують знаходження оберненого оператора до поділених різниць першого порядку.

Abstract. A plethora of applications from diverse disciplines can be solved if reduced to nonlinear equations in suitable abstract spaces. Such equations are solved mostly iteratively. That is why, three-step iterative methods of the Kurchatov-type for solving nonlinear operator equations are investigated using approximation by the Fréchet derivative of an operator of a nonlinear equation by divided differences. We study the local and the semi-local convergence using conditions only on the operators on the methods. The conditions and speed of convergence of these methods are determined. Moreover, the domain of uniqueness is found for the solution. The results of numerical experiments validate the theoretical results. The new idea can be used on other iterative methods utilizing inverses of divided differences of order one.

1 Introduction

Let C_1 and C_2 stand for Banach spaces and Ω be a convex and nonempty subset of C_1. A plethora of applications from different areas can be solved if reduced to a nonlinear equation of the form

$$Q(x) = 0, \tag{1.1}$$

where $Q : \Omega \to C_2$ is a continuous operator. This reduction takes place using Mathematical Modelling [1, 2]. Then, a solution denoted by $s^* \in \Omega$ is to be found that answers the application. The solution may be a number or a vector or a matrix or a function. This task is very challenging in general. Obviously, the solution s^* is desired in closed form. However, in practice, this is achievable only in rare cases. That is why researchers mostly develop iterative methods convergent to s^* under some conditions on the initial data. Newton's method is an effective method for solving the equation (1.2). However, it requires an analytical definition of the operator Q'. If the operator Q is nondifferentiable, difference methods can be used. The simplest difference method is the Secant method [1, 2]. Many works are devoted to the study of this method. It uses the first-order

Key words: Banach spaces, Fréchet derivative, divided difference, Kurchatov's method, local and semi-local convergence.

divided difference instead of the Fréchet derivative. The order of convergence of the Secant method is $\frac{1+\sqrt{5}}{2}$. The quadratic order of convergence has a linear interpolation method first proposed by V.A. Kurchatov [3] in the one-dimensional case. In Banach spaces, the Kurchatov method was first investigated by S.M. Shakhno [4, 5]. The works of I.K.Argyros, H.Ren, J.A.Ezquerro, M.A.Hernández [6–8] are devoted to the study of this method. Two-step variants of the Kurchatov method were studied by I.K. Argyros, S. George, H. Kumar, P.K. Parida, and S.M. Shakhno [9–11]. Wang et al. [12,13] and Cordero et al. [14,15] presented some Kurchatov-type methods with memory by using Kurchatov's divided difference operator. Some multi-step methods are explored in Ahmad et. al., Behl et. al. [16,17].

Three-step methods of the Kurchatov-type were proposed by X. Wang, Jin, Y.; Zhao, Y and X. Chen [12, 13]. The convergence is developed for two three-step Kurchatov-like methods defined by

$$\begin{aligned} &x_0, y_{-1} \in \Omega, \quad y_n = x_n - [2x_n - y_{n-1}, y_{n-1}; Q]^{-1} Q(x_n),\\ &z_n = x_n - [y_n, x_n; Q]^{-1} Q(x_n),\\ &x_{n+1} = z_n - [y_n, z_n; Q]^{-1} Q(z_n) \end{aligned} \tag{1.2}$$

and

$$\begin{aligned} &x_0, z_{-1} \in \Omega, \quad y_n = x_n - [2x_n - z_{n-1}, z_{n-1}; Q]^{-1} Q(x_n),\\ &z_n = x_n - [y_n, x_n; Q]^{-1} Q(x_n),\\ &x_{n+1} = z_n - [y_n, z_n; Q]^{-1} Q(z_n), \end{aligned} \tag{1.3}$$

where $[\cdot, \cdot; Q] : C_1 \times C_2 \to \mathcal{L}(C_1, C_2)$ is divided difference of order one, and $\mathcal{L}(C_1, C_2)$ stands for the space of continuous operator from C_1 into C_2. There exist restriction with results using Taylor series to show convergence of iterative methods which constitute the motivation for this paper.

Motivation

(i) The local convergence order 4.56, and 5, is provided for method (1.2) and method (1.3), respectively in [13] for $C_1 = C_2 = \mathbb{R}^i$ (i is a natural number) using Taylor series, and conditions which are reaching up to the sixth derivative of the operator Q not in these methods. Let us consider a simple function

$$Q(t) = \begin{cases} \delta_1 \log t^2 + \delta_2 t^5 + \delta_3 t^4, & \text{if} \quad t \neq 0,\\ 0, \quad \text{if} \quad t = 0. \end{cases}$$

Moreover, restrict the function Q on the inteval $\Omega = [-\frac{3}{2}, 2]$ and suppose that the real parametrs δ_k, $k = 1, 2, 3$ satisfy $\delta_1 \neq 0$ and $\delta_2 + \delta_3 = 0$. Then, $s^* = 1 \in \Omega$ solves the equation $Q(t) = 0$. However, the function $Q^{(3)}(t)$ is not continuous at $t = 0 \in \Omega$. Thus, the results in [13] cannot assure that $\lim_{n\to\infty} x_n = s^*$ for either method. But these methods converge, if e.g. $x_0 = 1.2$. This is an indication that the conditions in [13] can be replaced by weaker ones.

(ii) There are no estimates on the norms $\|s^* - x_n\|$, say upper bounds that can tell us in advance the number of iterates to be carried out to arrive at an error tolerance $\varepsilon > 0$.

(iii) No results are discussing the isolution of the solution s^*.

(iv) The results in [13] are restricted on $\mathbb{R}^i$.

(v) The more interesting semilocal analysis of convergence for both methods is not studied in previous works.

The restrictions (i) – (v) are dealt with positively in this paper. In particular,

Novelty

(i)' The new sufficient conditions involve only the operator on these methods.

(ii)' The number of iterates to reach error tolerance $\varepsilon > 0$ is known in advance, since a priori error estimates are given in this paper.

(iii)' Isolation of the solution is also discussed.

(iv)' The new results hold for Banach space-valued operators.

(v)' The semi-local analysis of convergence relies on majorizing sequences.

Both the local as well as semi-local analyses are given under conditions on generalized continuity for the divided difference $[\cdot,\cdot;Q]$ [2,6,7,9,11]. The same approach is applicable to similar methods utilizing Taylor series and inverses of linear operators [1,3–5,8,10,12,14–20].

We provide the local as well as the semi-local convergence analysis for these methods under generalized conditions. The local convergence is given in Section 2. The semi-local convergence is presented in Section 3, followed by the examples and the concluding remarks in Section 4 and Section 5, respectively.

2 Local convergence analysis

The symbols $V(s^*,\delta), V[s^*,\delta]$ are used to denote the open and close ball, respectively in C_1 with center s^* and radius $\delta > 0$. Let also A stand for the interval $[0,+\infty)$. Consider functions $\omega_0 : A_0 \times A_0 \to \mathbb{R}_+$ and $\omega : A_0 \times A_0 \to \mathbb{R}_+$ for $A_0 \subset A$. These functions are assumed to be continuous and nondecreasing. The local convergence conditions relating the real functions ω_0 and ω to the operators on both methods (1.2) and (1.3) provided that there exist $s^* \in \Omega$ solving the equation $Q(x) = 0$.

Suppose:

(H_1) There exists a minimal positive solution denoted by ρ_0 of the equation

$$\omega_0(3t,t) - 1 = 0.$$

Set $A_0 = [0,\rho_0]$.

(H_2) There exists a function w with domain $A_0 \times A_0$ and values in $\mathbb{R}_+$ such that for $g_1 : A_0 \to \mathbb{R}_+, g_2 : A_0 \to \mathbb{R}_+$ and $g_3 : A_0 \to \mathbb{R}_+$ designed by

$$g_1(t) = \frac{w(2t,t)}{1 - w_0(3t,t)},$$

$$g_2(t) = \frac{w((1+g_1(t))t,t)}{1 - w_0(g_1(t)t,t)}$$

and

$$g_3(t) = \frac{w((g_1(t)+g_2(t))t, g_2(t)t)g_2(t)}{1 - w_0(g_1(t)t, g_2(t)t)},$$

the equations $g_k(t) - 1 = 0$, $k = 1,2,3$ have minimal positive solutions. These solutions are denoted by r_k, respectively.

Set

$$r = \min\{r_k\} \quad \text{and} \quad A_1 = [0,r). \tag{2.1}$$

These definitions imply that for each $t \in A_1$

$$0 \le w_0(3t,t) < 1, \tag{2.2}$$

$$0 \le w_0(g_1(t)t,t) < 1, \tag{2.3}$$

$$0 \le w_0(g_1(t)t, g_2(t)t) < 1 \tag{2.4}$$

and

$$0 \leq g_k(t) < 1. \tag{2.5}$$

The parameter r is shown to be a radius of convergence for the method (1.2), see Theorem 2.1. The parameter r and the functions w_0 and w are associated to the divided difference on the method (1.2), and the method (1.3).

(H_3) There exists an invertible operator $L \in \mathcal{L}(C_1, C_2)$.

(H_4) $\|L^{-1}([x, y; Q] - L)\| \leq \omega_0(\|x - s^*\|, \|y - s^*\|)$ for each $x, y \in \Omega$.

Set $\Omega_1 = \Omega \cap V(s^*, \varrho_0)$.

(H_5) $\|L^{-1}([x, y; Q] - [z, s^*; Q])\| \leq \omega(\|x - z\|, \|y - s^*\|)$ for each $x, y, z \in \Omega_1$ and $\omega : A_0 \times A_0 \to \mathbb{R}$. and

(H_6) $V[s^*, \bar{r}] \subset \Omega$, where $\bar{r} = 3r$.

$r > 0$ is determined later based on which method is used.

Hence, we arrived at:

Theorem 2.1. *Suppose that the conditions* (H_1)–(H_6) *hold, and* $x_0, y_{-1} \in V(s^*, r) - \{s^*\}$. *Then, the sequence* $\{x_n\}$ *provided by the method (1.2) exists, stays in the ball* $V(s^*, r)$ *and is convergent to* s^* *so that*

$$\|y_n - s^*\| \leq g_1(r)\|x_n - s^*\| \leq \|x_n - s^*\| < r, \tag{2.6}$$

$$\|z_n - s^*\| \leq g_2(r)\|x_n - s^*\| \leq \|x_n - s^*\| \tag{2.7}$$

and

$$\|x_{n+1} - s^*\| \leq g_3(r)\|x_n - s^*\| \leq \|x_n - s^*\|. \tag{2.8}$$

Proof. The choice of x_0 and $y_{-1} \in V(s^*, r) - \{s^*\}$, (2.1), (2.2), and (H_3) - (H_5) imply that

$$\|2x_0 - y_{-1} - s^*\| \leq 2\|x_0 - s^*\| + \|y_{-1} - s^*\| \leq 2r + r = 3r = \bar{r}$$

and

$$\|L^{-1}([2x_0 - y_{-1}, y_{-1}; Q] - L)\| \leq w_0(2\|x_0 - s^*\| + \|y_{-1} - s^*\|, \|y_{-1} - s^*\|) < 1.$$

Then, the Banach Lemma on invertible operators assures the existence of the inverse for the linear operator $[2x_0 - y_{-1}, y_{-1}; Q]$ [1, 2, 6, 7, 9, 11, 18] and

$$\|[2x_0 - y_{-1}, y_{-1}; Q]^{-1}L\| \leq \frac{1}{1 - w_0(2\|x_0 - s^*\| + \|y_{-1} - s^*\|, \|y_{-1} - s^*\|)}. \tag{2.9}$$

Thus, the iterate y_0 exists by the first substep of the method (1.2) if $n = 0$. We can also write

$$\begin{aligned} y_0 - x^* &= x_0 - x^* - [2x_0 - y_{-1}, y_{-1}; Q]^{-1}Q(x_0) \\ &= [2x_0 - y_{-1}, y_{-1}; Q]^{-1}([2x_0 - y_{-1}, y_{-1}; Q] - [x_0, s^*; Q])(x_0 - s^*). \end{aligned} \tag{2.10}$$

The application of (H_5), (2.7) (for $k = 1$), (2.9) in (2.10) and give

$$\begin{aligned} \|y_0 - x^*\| \leq h_0^{(1)} &:= \frac{w(\|x_0 - y_{-1}\|, \|y_{-1} - s^*\|)\|x_0 - s^*\|}{1 - w_0(2\|x_0 - s^*\| + \|y_{-1} - s^*\|, \|y_{-1} - s^*\|)} \\ &\leq g_1(r)\|x_0 - s^*\| \leq \|x_0 - s^*\| < r, \end{aligned} \tag{2.11}$$

since $\|x_0 - y_{-1}\| \leq \|x_0 - s^*\| + \|y_{-1} - s^*\| \leq r + r = 2r$. Hence, the item (2.6) holds if $n = 0$ and the iterate $y_0 \in V(s^*, r)$. It also follows from (2.1), (2.3), (H_4) and (2.11) that

$$\|L^{-1}([y_0, x_0; Q] - L)\| \le w_0(\|y_0 - s^*\|, \|x_0 - s^*\|) < 1.$$

Thus, the operator $[y_0, x_0; Q]$ is invertible, and

$$\|[y_0, x_0; Q]^{-1}L\| \le \frac{1}{1 - w_0(\|y_0 - s^*\|, \|x_0 - s^*\|)}. \tag{2.12}$$

Moreover, the iterate z_0 exists by the second substep of the method (1.2), and we can write

$$\begin{aligned} z_0 - s^* &= x_0 - s^* - [y_0, x_0; Q]^{-1}Q(x_0) \\ &= [y_0, x_0; Q]^{-1}([y_0, x_0; Q] - [x_0, s^*; Q])(x_0 - s^*). \end{aligned} \tag{2.13}$$

The application of (H_5), and the usage of (2.1), (2.5), (2.12) in (2.13) give in turn

$$\begin{aligned} \|z_0 - x^*\| \le h_0^{(2)} &:= \frac{w(\|y_0 - x_0\|, \|x_0 - s^*\|)\|x_0 - s^*\|}{1 - w_0(\|y_0 - s^*\|, \|x_0 - s^*\|)} \\ &\le g_2(r)\|x_0 - s^*\| \le \|x_0 - s^*\|. \end{aligned}$$

So, the item (2.7) holds if $n = 0$, and the iterate $z_0 \in V(s^*, r)$. Then, the application of (2.1), (2.4), and (H_5) give

$$\|L^{-1}([y_0, z_0; Q] - L)\| \le w_0(\|y_0 - s^*\|, \|z_0 - x^*\|) < 1.$$

Hence, the linear operator $[y_0, z_0; Q]$ is invertible, and

$$\|[y_0, z_0; Q]^{-1}L\| \le \frac{1}{1 - w_0(\|y_0 - s^*\|, \|z_0 - s^*\|)}. \tag{2.14}$$

Furthermore, the iterate x_1 exists, and we can write

$$\begin{aligned} x_1 - s^* &= z_0 - s^* - [y_0, z_0; Q]^{-1}F(z_0) \\ &= [y_0, z_0; Q]^{-1}([y_0, z_0; Q] - [z_0, s^*; Q])(z_0 - s^*). \end{aligned} \tag{2.15}$$

Using (2.1), (2.5) (for $k = 3$), (2.14), in (2.15) we get in turn that

$$\begin{aligned} \|x_1 - s^*\| \le h_0^{(3)} &:= \frac{w(\|y_0 - z_0\|, \|z_0 - s^*\|)\|z_0 - s^*\|}{1 - w_0(\|y_0 - s^*\|, \|z_0 - s^*\|)} \\ &\le g_3(r)\|x_0 - s^*\| \le \|x_0 - s^*\|. \end{aligned}$$

Thus, the item (2.8) holds and the iterate $x_1 \in V(s^*, r)$. The calculations can be repeated provided x_0, y_0, x_1 are switched by x_m, y_m, x_{m+1} (m a nature number). This way the induction for items (2.6) – (2.8) is completed, and the iterates $x_m, y_m, x_{m+1} \in V(s^*, r)$. Then, from the estimation

$$\|x_{m+1} - s^*\| \le c\|x_m - s^*\| \le c^{m+1}\|x_0 - s^*\| < r,$$

where $c = g_3(\|x_0 - s^*\|) \in [0, 1)$, we conclude that $\lim_{m\to\infty} x_m = s^*$. □

Next, the local analysis of convergence for the method (1.3) follows analogously.

Theorem 2.2. *Suppose that the conditions $(H_1) - (H_6)$ hold, and $x_0, y_{-1} \in V(s^*, r) - \{s^*\}$. Then, the sequence $\{x_n\}$ given by the method (1.3) exists, stays in the ball $V(s^*, r)$, and is convergent to s^* such that the items (2.6)-(2.8) hold.*

Proof. We follow the proof of Theorem 2.1. But we have instead the estimates

$$\begin{aligned} \|y_m - x^*\| &\le \frac{w(\|x_m - z_{m-1}\|, \|z_{m-1} - s^*\|)\|x_m - s^*\|}{1 - w_0(2\|x_m - s^*\| + \|z_{m-1} - s^*\|, \|z_{m-1} - s^*\|)} \\ &\le g_1(r)\|x_m - s^*\| \le \|x_m - s^*\|, \end{aligned}$$

$$\|z_m - s^*\| \leq \frac{w(\|y_m - x_m\|, \|x_m - s^*\|)\|x_m - s^*\|}{1 - w_0(\|y_m - x^*\|, \|x_m - s^*\|)}$$
$$\leq g_2(r)\|x_m - s^*\| \leq \|x_m - s^*\|$$

and

$$\|x_{m+1} - s^*\| \leq \frac{w(\|z_m - y_m\|, \|z_m - s^*\|)\|z_m - s^*\|}{1 - w_0(\|y_m - s^*\|, \|z_m - s^*\|)}$$
$$\leq g_3(r)\|x_m - s^*\| \leq \|x_m - s^*\|.$$

□

The uniqueness of the solution is determined in the next result.

Proposition 2.1. *(a) The conditions (H_3) and (H_4) hold in the ball $V(s^*, \varrho)$ for some $\varrho > 0$. and*
(b) There exists $\varrho_1 > \varrho$ such that

$$\omega_0(0, \varrho_1) < 1.$$

Set $\Omega_2 = \Omega \cap V[s^*, \varrho_1]$.

Then, the equation $Q(x) = 0$ is uniquely solvable by s^* in the domain Ω_2.

Proof. Assume the existence of the divided difference $[s^*, v; Q]$ for $v \neq s^*$. Then, it follows by (a)-(b) that

$$\|L^{-1}([s^*, v; Q] - L)\| \leq \omega_0(0, \|s^* - v\|) \leq \omega_0(0, \varrho_1) < 1.$$

Thus, the Banach Lemma asserts $[s^*, v; Q]^{-1} \in \mathcal{L}(C_1, C_2)$. Consequently, $v = s^*$ follows from the identity

$$s^* - v = [s^*, v; Q]^{-1}(Q(s^*) - Q(v)) = [s^*, v; Q]^{-1}(0).$$

□

Remark 2.1. *A possible choice for $\varrho = r$.*

3 Semi-local convergence

The functions v_0 and v correspond to ω_0, ω of the previous section and have the same properties. Moreover, the conditions relating them to the methods (1.2) and (1.3) are:
(S_1) There exists a minimum positive solution denoted by ρ_2 of the equation $v_0(3t, t) - 1 = 0$. Set $A_1 = [0, \rho_2)$.
(S_2) There exists function $v : A_1 \times A_1 \to \mathbb{R}_+$ which is continuous as well as non-decreasing.

Define sequences for $b_{-1}, a_0, b_0 \geq 0$ with $a_0 \leq b_{-1}$:

$$c_n = b_n + \frac{v(b_n - b_{n-1}, b_{n-1} - a_n)(b_n - a_n)}{1 - v_0(b_n - a_0, a_n - a_0)},$$

$$\begin{aligned} \gamma_n &= (1 + v_0(c_n - a_0, a_n - a_0))(c_n - a_n) \\ &+ (1 + v_0(2(a_n - a_0) + (b_{n-1} - a_0), b_{n-1} - a_0))(b_n - a_n), \\ a_{n+1} &= c_n + \frac{\gamma_n}{1 + v(b_n - a_0, c_n - a_0)}, \\ \delta_{n+1} &= (1 + v_0(a_{n+1} - a_0, a_n - a_0))(a_{n+1} - a_n) \\ &+ (1 + v_0(2(a_n - a_0) + b_{n-1} - a_0, b_{n-1} - a_0))(b_n - a_n) \end{aligned} \tag{3.1}$$

and

$$b_{n+1} = a_{n+1} + \frac{\delta_{n+1}}{1 - v_0(2(a_{n+1} - a_0) + b_n - a_0, b_n - a_0)}.$$

These sequences are shown to be majoriting for the method (1.2) in the Theorem 3.1. But first, a convergence condition is required.
(S_3) There exists $\tau \in [0, \rho_2)$ such that for each $n = 0, 1, 2, ...$

$$v_0(b_n - a_0, a_n - a_0) < 1,$$
$$v_0(2(a_{n+1} - a_0) + b_n - a_0, b_n - a_0) < 1$$

and $a_n < \tau$.

It follows by this condition and the definition of the scalar sequences that

$$a_n \leq b_n \leq c_n \leq a_{n+1} \leq \tau$$

and there exists $a \in [0, \tau]$ such that

$$\lim_{n \to +\infty} a_n = a.$$

The functions v_0 and v relate to the operators on the method (1.2).
(S_4) There exists an invertible operator $L \in \mathcal{L}(C_1, C_2)$.
(S_5) $\|L^{-1}([x, y; Q] - L)\| \leq v_0(\|x - x_0\|, \|y - x_0\|)$ for each $x, y \in \Omega$, and some $x_0 \in \Omega$.

It follows by this condition and the definition of ρ_2 that for $y_{-1} \in \Omega_3 = V(x_0, \rho)$

$$\|2x_0 - y_{-1} - x_0\| = \|x_0 - y_{-1}\| < \rho,$$

so

$$\|L^{-1}([2x_0 - y_{-1}, y_{-1}; Q] - L)\| \leq v_0(\|x_0 - y_{-1}\|, \|y_{-1} - x_0\|) < 1.$$

Thus, the linear operator $[2x_0 - y_{-1}, y_{-1}; Q]$ is invertible, and we can take

$$b_0 \geq \|[2x_0 - y_{-1}, y_{-1}; Q]^{-1} Q(x_0)\|$$

and $\|y_{-1} - x_0\| \leq b_{-1} - a_0$, provided that $b_{-1} \leq a_0 \leq b_0$.

Set $\Omega_4 = V(x_0, \rho_2) \cap \Omega$.
(S_6) $\|L^{-1}([x, y; Q] - [\bar{x}, \bar{y}; Q])\| \leq v(\|x - \bar{x}\|, \|y - \bar{y}\|)$ for each $x, y, \bar{x}, \bar{y}, \in \Omega_4$. and
(S_7) $V(x_0, 3a) \subset \Omega$.

Next, the semi-local analysis of convergence for the method (1.2) follows.

Theorem 3.1. *Suppose that the conditions $(S_1) - -(S_7)$ hold. Then, the sequence $\{x_n\}$ given by the method (1.2) exists, stays in $V[x_0, a]$ and is convergent to a solution $s^* \in V[x_0, a]$ so that*

$$\|y_n - x_n\| \leq b_n - a_n, \tag{3.2}$$
$$\|z_n - y_n\| \leq c_n - b_n, \tag{3.3}$$
$$\|x_{n+1} - z_n\| \leq a_{n+1} - c_n \tag{3.4}$$

and

$$\|s^* - x_n\| \leq a - a_n.$$

Proof. Items (3.2)–(3.4) are shown using induction.

It follows by the initial conditions that (3.2) holds if $n = 0$. The rest of the proof exchanges the functions w_0, w, s^* and conditions (H) by v_0, v, x_0, s^* and conditions (S), respectively.

The motivation for the scalar sequences is

$$z_m - y_m = ([2x_m - y_{m-1}, y_{m-1}; Q]^{-1} - [y_m, x_m; Q]^{-1})Q(x_m)$$
$$-[y_m, x_m; Q]^{-1}([2x_m - y_{m-1}, y_{m-1}; Q] - [y_m, x_m; Q])(y_m - x_m),$$

since

$$y_m - x_m = [2x_m - y_{m-1}, y_{m-1}; Q]^{-1} Q(x_m),$$

$$\|z_m - y_m\| \le \frac{\upsilon(\|2x_m - y_{m-1} - y_m\|, \|y_{m-1} - x_m)\|y_m - x_m\|}{1 - \upsilon_0(\|y_m - x_0\|, \|x_m - x_0\|)}$$
$$\le \frac{\upsilon(b_m - a_m + a_m - b_{m-1}, b_{m-1} - a_m)(b_m - a_m)}{1 - \upsilon_0(\|b_m - a_0\|, \|a_m - a_0\|)} = c_m - b_m,$$

$$\|z_m - x_0\| \le \|z_m - y_m\| + \|y_m - x_0\|$$
$$\le c_m - b_m + b_m - a_0 = c_m - a_0 < a,$$

$$Q(z_m) = Q(z_m) - Q(x_m) - [2x_m - y_{m-1}, y_{m-1}; Q](y_m - x_m)$$
$$= [z_m, x_m; Q](z_m - x_m) - [2x_m - y_{m-1}, y_{m-1}; Q](y_m - x_m),$$

$$\|L^{-1}Q(z_m)\| \le (1 + \upsilon_0(\|z_m - x_0\|, \|x_m - x_0\|))\|z_m - x_m\|$$
$$+(1 + \upsilon_0(\|2x_m - y_{m-1} - x_0\|, \|y_{m-1} - x_0\|))\|y_m - x_m\|$$
$$\le (1 + \upsilon_0(c_m - a_0, a_m - a_0))(c_m - a_m)$$
$$+(1 + \upsilon_0(2(a_m - a_0) + b_{m-1} - a_0, b_{m-1} - a_0))(b_m - a_m) = \gamma_m,$$

$$x_{m+1} - z_m = -[z_m, x_m; Q]^{-1}(Q(z_m) - Q(x_m) + Q(x_m)),$$

$$\|x_{m+1} - z_m\| \le \frac{\gamma_m}{1 - \upsilon_0(b_m - a_0, a_m - a_0)} = a_{m+1} - c_m,$$

$$\|x_{m+1} - x_0\| \le \|x_{m+1} - z_m\| + \|z_m - x_0\| \le a_{m+1} - c_m + c_m - a_0 = a_{m+1} < a,$$

$$Q(z_{m+1}) = Q(x_{m+1}) - Q(x_m) - [2x_m - y_{m-1}, y_{m-1}; Q](y_m - x_m),$$

$$\|L^{-1}Q(x_{m+1})\| \le (1 + \upsilon_0(a_{m+1} - a_0, a_m - a_0))(a_{m+1} - a_m)$$
$$+(1 + \upsilon_0(2(a_m - a_0) + b_{m+1} - a_0))(b_m - a_m) = \delta_{m+1}.$$

Consequently, we obtain

$$\|y_{m+1} - x_{m+1}\| \le \frac{\delta_{m+1}}{1 - \upsilon_0(2(a_{m+1} - a_0) + b_m - a_0, b_m - a_0)} = b_{m+1} - a_{m+1}$$

and

$$\|y_{m+1} - x_0\| \le \|y_{m+1} - x_{m+1}\| + \|x_{m+1} - x_0\|$$
$$\le b_{m+1} - a_{m+1} + a_{m+1} - a_0 = b_{m+1} < a.$$

The induction is complete. It follows that the sequence $\{x_m\}$ is Cauchy establishing the existence of its limit. By letting $m \to +\infty$ in (3.2) we obtain $Q(s^*) = 0$. Then, from the estimate $\|x_{m+i} - x_m\| \le a_{m+i} - a_m$, and by letting $i \to +\infty$, we complete the proof. □

Method (1.3)

The majorizing sequences are slightly different from the ones used in the method (1.2)

$$c_n = b_n + \frac{\upsilon(b_n - a_n + c_{n-1} - a_n, c_{n-1} - a_n)(b_n - a_n)}{1 - \upsilon_0(b_n - a_n, a_n - a_0)},$$

$$\bar{\gamma}_n = (1 + \upsilon_0(c_n - a_0, a_n - a_0 - \upsilon_0(b_n - a_n, a_n - a_0))(c_n - a_n)$$
$$+(1 + \upsilon_0(2(a_n - a_0) + c_{n-1} - a_0, c_{n-1} - a_0)(b_n - a_n),$$

$$a_{n+1} = c_n + \frac{\bar{\gamma}_n}{1 - v(b_n - a_0, c_n - a_0)}, \tag{3.5}$$

$$\begin{aligned}\bar{\delta}_{n+1} &= (1 + v_0(a_{n+1} - a_0, a_n - a_0))(a_{n+1} - a_n) \\ &+ (1 + v_0(2(a_n - a_0) + c_{n-1} - a_0, c_{n-1} - a_0))(b_n - a_n)\end{aligned}$$

and

$$b_{n+1} = a_{n+1} + \frac{\bar{\delta}_{n+1}}{1 - v_0(2(a_n - a_0) + b_{n-1} - a_0, b_n - a_0)}.$$

The corresponding assumptions are also made:

$$\|[2x_0 - z_{-1}, z_{-1}; Q]^{-1} Q(x_0)\| \leq b_0 - a_0$$

and

$$\|z_{-1} - x_0\| \leq c_{-1} - a_0$$

provided that $a_0 \leq b_0$ and $a_0 \leq c_{-1}$.

Then, the corresponding semi-local analysis of convergence for the method (1.3) is analogous to the one given for method (1.2) in Theorem 3.1.

Theorem 3.2. *Suppose that the conditions $(S_1) - (S_7)$ hold but with (3.5), replacing 3.1 and for $\|z_{-1} - x_0\| \leq c_{-1} - a_0$ replacing $\|y_{-1} - x_0\| \leq b_{-1} - a_0$. Then, the conclusions of Theorem 3.1 hold for the method (1.3).*

Proof. Simply exchange (1.2), (3.1), by (1.3), (3.3), respectively. □

Note that the limit of the sequence (3.3) is not necessarily the same with the one given by (3.1), however we use the same symbol.

The uniqueness of a solution of the equation (1.1) is given in the next result.

Proposition 3.1. *Suppose: there exists a solution $\bar{s} \in V(x_0, \rho_4)$ of the equation $Q(x) = 0$ for some $\rho_4 > 0$; The condition (S_4) holds; The condition (S_5) holds in the ball $V(x_0, \rho_4)$ and there exists $\rho_5 \geq \rho_4$ such that $v_0(\rho_4, \rho_5) < 1$. Set $\Omega_4 = V[x_0, \rho_5] \cap \Omega$. Then the equation $Q(x) = 0$ is uniquely solvable by $\bar{s}$ in the set Ω_4.*

Proof. Let $\bar{x} \in \Omega_4$ with $Q(\bar{x}) = 0$ and $\bar{x} \neq \bar{s}$. Then, the divided difference $[\bar{x}, \bar{s}; Q]$ is well defined. It then follows that

$$\begin{aligned}\|L^{-1}([\bar{x}, \bar{s}; Q] - L)\| &\leq v_0(\|\bar{x} - x_0\|, \|\bar{s} - x_0\|) \\ &\leq v_0(\rho_4, \rho_5) < 1.\end{aligned}$$

Therefore, the linear operator $[\bar{x}, \bar{s}; Q]$ is invertible. So, $\bar{s} = \bar{x}$ follows from the identity

$$\bar{x} - \bar{s} = [\bar{x}, \bar{s}; Q]^{-1}(Q(\bar{x}) - Q(\bar{s})) = [\bar{x}, \bar{s}; Q]^{-1}(0 - 0) = 0.$$

□

Remark 3.1.

(i) Possible choices for the linear operator L can be $L = I$ or $L = Q'(s^*)$ (local case) or $L = Q'(x_0)$ or $L = [2x_0 - y_{-1}, y_{-1}; Q]$ (semi-local case) provided that these operators are invertible. Other choices are possible as long as the conditions (H) (local case) or the conditions (S) semi-local case hold.

(ii) The point a can be replaced by ρ_2 in the condition (S_7).

(iii) Under all the conditions $(S_1) - (S_7)$, take $\bar{s} = s^*$ and $\rho_4 = a$ in Proposition.

4 Numerical experiments

In this section, the considered methods are tested on a wide range of matrices to give an idea about the different situations that optimization algorithms have to face when coping with these kinds of problems. Details about the test functions can be found in [19].

Example 4.1. Let us consider the cases:

1. Freudenstein & Roth

$$\begin{aligned} &n = 2,\\ &f_1(x) = -13 + x_1 + ((5 - x_2)x_2 - 2)x_2,\\ &f_2(x) = -29 + x_1 + ((x_2 + 1)x_2 - 14)x_2,\\ &x_0 = (0.5, -2). \end{aligned}$$

2. Powell badly scaled

$$\begin{aligned} &n = 2,\\ &f_1(x) = 10^4 x_1 x_2 - 1,\\ &f_2(x) = \exp(-x_1) + \exp(-x_2) - 1.0001,\\ &x_0 = (0, 1). \end{aligned}$$

3. Broyden tridiagonal problem

$$\begin{aligned} &n = 3,\\ &f_1(x) = (3 - 2x_1)x_1 - 2x_2 + 1,\\ &f_2(x) = (3 - 2x_2)x_2 - x_1 - 2x_3 + 1,\\ &f_3(x) = (3 - 2x_3)x_3 - x_2 + 1,\\ &x_0 = (0, 0, 0). \end{aligned}$$

4. Discrete boundary value problem

$$\begin{aligned} &n = 3,\\ &h = 0.25,\\ &f_1(x) = 2x_1 + h^2\frac{(x_1 + 1 + h)^3}{2} - x_2,\\ &f_2(x) = 2x_2 + h^2\frac{(x_2 + 1 + 2h)^3}{2} - x_1 - x_3,\\ &f_3(x) = 2x_3 + h^2\frac{(x_3 + 1 + 3h)^3}{2} - x_2,\\ &x_0 = (0.25, 0.5, 0.75). \end{aligned}$$

5. A trigonometric - exponential system

$$\begin{aligned} &f_k(x) = 3x_k^3 + 2x_{k+1} - 5,\\ &+ \sin(x_k - x_{k+1})\sin(x_k + x_{k+1}), \quad if \quad k = 1,\\ &f_k(x) = 3x_k^3 + 2x_{k+1} - 5 + \sin(x_k - x_{k+1})\sin(x_k + x_{k+1}),\\ &+ 4x_k - x_{k-1}\exp(x_{k-1} - x_k) - 3, \quad if \quad 1 < k < n,\\ &f_k(x) = 4x_k - x_{k-1}\exp(x_{k-1} - x_k) - 3, \quad k = n. \end{aligned}$$

6. Five-diagonal system

$$
\begin{aligned}
f_1(x) &= 4(x_1 - x_2^2) + x_2 - x_3^2,\\
f_2(x) &= 8x_2(x_2^2 - x_1) - 2(1 - x_2) + 4(x_2 - x_3^2) + x_3 - x_4^2,\\
f_3(x) &= 8x_3(x_3^2 - x_2) - 2(1 - x_3) + 4(x_3 - x_4^2) + x_2^2 - x_1,\\
f_4(x) &= 8x_4(x_4^2 - x_3) - 2(1 - x_4) + 4(x_4 - x_5^2) + x_3^2 - x_2,\\
f_5(x) &= 8x_5(x_5^2 - x_4) - 2(1 - x_5) + x_4^2 - x_3.
\end{aligned}
$$

7. Extended power singular function

$$
\begin{aligned}
f_k(x) &= x_k + 10x_{k+1}, && \text{mod } (k,4) = 1,\\
f_k(x) &= \sqrt{5}(x_{k+1} - x_{k+2}), && \text{mod } (k,4) = 2,\\
f_k(x) &= (x_{k-1} - 2x_k)^2, && \text{mod } (k,4) = 3,\\
f_k((x) &= \sqrt{10}(x_{k-3} - x_k)^2, && \text{mod } (k,4) = 0.
\end{aligned}
$$

Table 4.1. The number of iterations to obtain an approximation to the solutions of the test problems using both common and three-step methods

Problem	Method					
	(1.2)	(1.3)	(4.3)	(4.4)	(4.1)	(4.2)
Freudenstein & Roth	25	8	19	41	29	14
Powell badly scaled	6	5	16	13	10	-
Broyden tridiagonal problem	2	2	6	4	3	2
Discrete boundary value problem	2	2	5	4	3	2
A trigonometric - exponential system $n = 2,\quad x_0 = (2, 0)$	4	3	7	6	5	4
A trigonometric - exponential system $n = 3,\quad x_0 = (5, 5, 5)$	4	4	10	9	7	5
A trigonometric - exponential system $n = 5,\quad x_0 = (-9, -9, -9, -9, -9)$	7	30	15	-	15	13
Five-diagonal system $n = 5,\quad x_0 = (-5, 5, -5, 5, -5)$	16	-	-	-	19	-
Five-diagonal system $n = 5,\quad x_0 = (3, 3, 3, 3, 3)$	4	4	-	-	11	-
Extended power singular function	5	5	9	8	20	13

We use the condition $\|Q(x_n)\| \le \varepsilon$, where $\varepsilon = 10^{-8}$, for stopping the computational process. The obtained results are confronted with those from the Secant method, expressed as:

$$x_{n+1} = x_n - [x_{n-1}, x_n; Q]^{-1} Q(x_n), \quad n = 0, 1, 2, ... \tag{4.1}$$

and the Kurchatov method, defined by:

$$x_{n+1} = x_n - [x_{n-1}, 2x_n - x_{n-1}; Q]^{-1} Q(x_n), \quad n = 0, 1, 2, ... \tag{4.2}$$

Here, x_{-1}, x_0 represents the given initial approximations.

Furthermore, to expand the comparison, we introduce two additional Kurchatov-like methods with additional processing steps:

$$\begin{aligned} A_n &= [x_{n-1}, 2x_n - x_{n-1}; Q], \\ y_n &= x_n - A_n^{-1} Q(x_n), \\ x_{n+1} &= y_n - A_n^{-1} Q(y_n), \quad n = 0, 1, 2, \dots \end{aligned} \tag{4.3}$$

and

$$\begin{aligned} A_n &= [x_{n-1}, 2x_n - x_{n-1}; Q], \\ y_n &= x_n - A_n^{-1} Q(x_n), \\ B_n &= [x_n, 2y_n - x_n; Q], \\ x_{n+1} &= y_n - B_n^{-1} Q(y_n), \quad n = 0, 1, 2, \dots \end{aligned} \tag{4.4}$$

Here A_n is divided difference of operator Q for points x_{n-1} and $2x_n - x_{n-1}$. Values x_0, x_{-1} are initial approximations.

Table 4.1 shows the number of iterations for finding the solution of the described functions by developed and well-known methods.

Example 4.2. We will compare the results of the methods for the subsequent problem

$$\begin{gathered} 3x^2 y + y^2 - 1 + |x - 1| = 0, \\ x^4 + xy^3 - 1 + |y| = 0. \end{gathered}$$

The solution of this problem is $s^* \approx (0.89465537, 0.32782652)$, $Q(s^*) = 0$. A bar chart was generated to illustrate the variation in results based on different initial approximations.

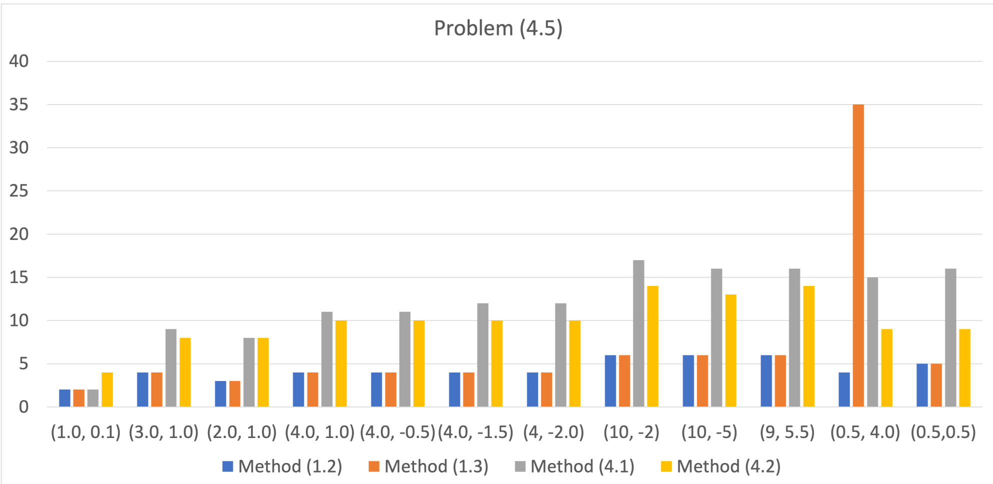

Fig. 4.1. This figure displays results for both standard and newly developed methods across different initial approximations

Example 4.3. To extend our investigation, we add a Three-Hump Camel function [20] to assess the methods' performance across a range of tolerance levels

$$f(x) = 2x_1^2 - 1.05x_1^4 + \frac{x_1^6}{6} + x_1 x_2 + x_2^2.$$

All methods successfully converged to the exact solution for the given problem. The proposed methods demonstrated superior efficiency by requiring fewer iterations, indicating their effectiveness

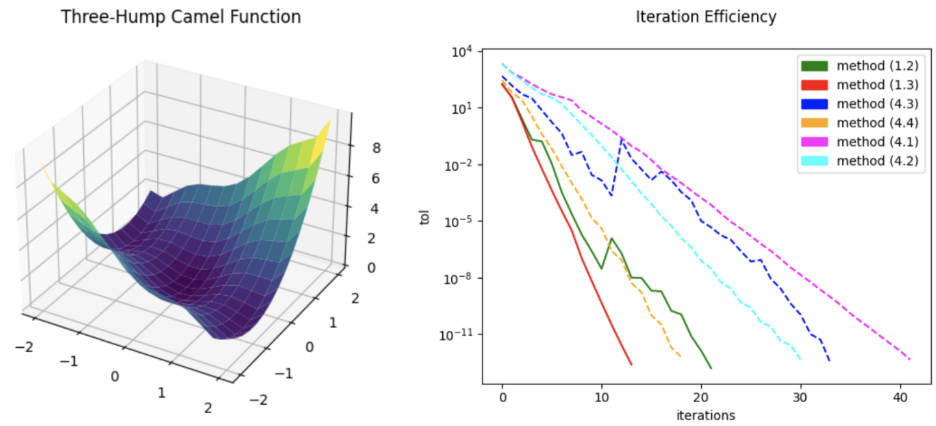

Fig. 4.2. This figure shows the tolerance history of Kurchatov-like methods for the three-hump camel function

in solving similar problems. However, it's noteworthy that while the proposed methods outperform others, the margin of difference in iteration counts was relatively modest and the solution depends on the features of the function.

5 Conclusions

Local and semi-local convergence analysis for two three-step Kurchatov-type methods is provided under the generalized Lipschitz conditions for only divided differences of order one. Regions of convergence and uniqueness of the solution are established. The results of the numerical experiment are given. There is no substantial advantage of one method over the others. The proposed methods demonstrate the ability to effectively discover solutions with a smaller number of required iterations or locate solutions within larger systems. The developed technique does not rely on the studied methods. That is why it can also be used on other methods that contain inverses of divided differences or inverses of linear operators in general. The future work involves the application of this process on other iterative methods [1, 3–5, 8, 10, 12, 14–20].

Declarations

Conflict of Interest.

The authors declare that there are no conflicts of interest.

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Additional information

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