2024, Vol. 2, pp. 56–65
doi: 10.30970/ana.2024.2.56

Iterative methods for the infiltration advance problem

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Анотація. Ми розглядаємо проблему просування інфільтрації в іригаційній гідравліці, змодельовану за допомогою рівняння Льюїса-Мілна. Ми пропонуємо два ітераційних методи, які дають прийнятні результати порівняно з існуючими формулами для різних добре відомих функцій інфільтрації. Метод послідовних наближень, який розглядається як метод фіксованої точки, і метод варіаційної ітерації (на основі методу множників Лагранжа) дають формули після декількох ітерацій і простих обчислень порівняно з існуючими методами, де має бути відоме перетворення Лапласа функції інфільтрації.

Abstract. We examine the infiltration advance problem in irrigation hydraulics modeled through the Lewis-Milne equation. We propose two iterative methods that give reasonable results compared with existing formulas for various well-known infiltration functions. The method of successive approximations seen as a fixed-point method and the variational iteration method (based on Lagrange multipliers method) produce formulas after few iterations and simple calculations compared to existing methods where the Laplace transform of the infiltration function has to be known.

1 Introduction

The problem of simultaneous advance and infiltration of water on the soil surface observed in surface irrigations is of central importance. The equation of Lewis and Milne (1938) [15] forms the basis for all models attempting to address the problem (Cook et al., 2013 [4]). However, since it ignores water ponding at the soil surface it is valid only for shallow flow situations. Philip and Farrell (1964) [23] showed that using Laplace transformation, a general analytical solution of the Lewis-Milne equation can be achieved when there is a constant inflow at the system's entrance. Furthermore, applying various infiltration functions to the general analytical solution, such as (Horton, 1940 [12]; Kostiakov, 1932 [13]; Philip, 1957 [21]) resulted in specific analytical solutions for the advance $x(t)$ (where x is the distance of advance of the surface water and t is the advance time) for each infiltration equation when the water surface quantity is considered negligible. However, in most cases, the used infiltration equations are empirical, and hence, their parameters lack physical meaning.

A important feature of Philip and Farrell (1964) [23] solution is that no assumption is made about the relationship between advance and time. They also introduced an implicit general solution to the inverse problem of this equation, which can be used to study the infiltration process of border irrigation (Mao et al., 2011 [16]).

Or and Silva (1996) [19] presented a relatively simple numerical prediction method for the advance problem using a differential form of the Lewis-Milne equation, in which the volume of surface water is again considered negligible. Valiantzas (2000) [26] proposed an advance prediction equation derived by removing the surface volume term from the volume balance equation. However, it assumes that advance is described by an exponential equation and that infiltration follows the extended Kostiakov-Lewis equation.

2020 *Mathematics Subject Classification*: 45D05, 45K05, 65R20.
Key words: irrigation, infiltration, Lewis-Milne equation, Volterra integral equation, integro-differential equation, successive approximations.

Cook et al. (2013) [4] used a two-parameter infiltration equation given by Philip (1969) [22] in the Lewis-Milne equation, with physical soil parameters being sorptivity S and saturated hydraulic conductivity K_s, providing a solution for all times using the Laplace transform. For short advance times, their solution is the same as that by Philip and Farrell (1964) [23] when the infiltration equation used is the Philip (1957) [21] equation, while for long times, the solution was the same as that of Collis-George (1974) [3].

From the above, it generally emerges that the analytical solution of the Lewis-Milne equation usually relies on Laplace transformation, and different assumptions appear among the methods. Such assumptions concern the volume of surface water, the shape of the advance curve, and the form of the infiltration equation, where equations with empirical parameters are usually used instead of equations containing physically meaningful parameters such as sorptivity and saturated hydraulic conductivity.

The purpose of our work is to present two new approximate methods when different infiltration equations (two-term, empirical or implicit) are incorporated into the Lewis-Milne equation to produce an advance function. We focus on border irrigation but the problem remains in general the same also in furrow irrigation (Philip and Farrell, 1964 [23]).

The paper is organized as follows: in Section 2 we formulate mathematically the problem and we present the five different infiltration functions to be considered in this work. The two iterative methods are presented in Section 3. The approximate solutions as the output of the iterative schemes after few steps are derived in Section 4.

2 Problem formulation

The physical problem is described by the Lewis-Milne equation

$$qt = cx(t) + \int_0^t y(t-\tau)x'(\tau)d\tau, \tag{2.1}$$

where $x(t)$ is the position of the advance front as a function of time t (start of the flow), q is the constant flow rate at the system's entrance and c is the water depth on the soil surface. Of great interest is the infiltration equation $y(t)$ since its form affects the solution of (2.1). In this study, we explore different distinct forms; however, it is worth noting that the iterative method presented herein have broader applicability and can be applied to additional cases.

- The two-term infiltration equation

$$y(t) = S\sqrt{t} + At, \tag{2.2}$$

where S is the sorptivity and A is proportional to the saturated hydraulic conductivity K_s. This equation was proposed by Philip (1957) [21] and it is a truncation of a series solution and it is valid only for infiltration at short to medium times.

- The linear equation

$$y(t) = y_0 + K_s t, \tag{2.3}$$

for some initial infiltration value y_0 before $y'(t) = K_s$. This equation was suggested by Philip and Farrell (1964) [23] and it holds for large times.

A power law relation of the form

$$y(t) = \kappa t^{\alpha}, \tag{2.4}$$

where $\kappa > 0$ and $\alpha \in (0, 1]$ are empirical constants, an equation proposed by Kostiakov (1932) [13].

- The two-parameter implicit infiltration equations

$$t = \frac{y}{K_s} - \frac{S^2}{2K_s^2} \ln\left(1 + \frac{2K_s y}{S^2}\right) \tag{2.5}$$

and

$$t = \frac{y}{K_s} + \frac{S^2}{2K_s^2} \left(e^{-\frac{2K_s y}{S^2}} - 1\right), \tag{2.6}$$

proposed by Green and Ampt (1911) [7] and Talsma and Parlange (1972) [25], respectively. We consider these two equations together since they share the same characteristic that t is given as a function of y and thus an inversion is needed before proceeding further.

The equation (2.1) is an integro-differential equation of Volterra type and an equivalent form can be obtained by taking the time derivative and apply the Leibniz rule:

$$q = cx'(t) + \int_0^t \frac{\partial y}{\partial t}(t-\tau)x'(\tau)d\tau + y(0)x'(t). \tag{2.7}$$

In this work we consider the direct problem to compute $x(t)$, from (2.1) or (2.7) given the infiltration function $y(t)$ and the constants q and c. The majority of the previous works solve (2.1) by applying the Laplace transform, a method that is useful only if the transform of the infiltration function is known, see (Cook et al., 2013 [4]; Philip and Farrell, 1964 [23]) and the references therein.

3 Methods

In this section we present the fundamentals of two iterative methods: the successive approximation method (SAM) and the variational iterative method (VIM). Our objectives are to:

(a) show that the two methods coincide for the particular problem,

(b) test the effectiveness in solving the infiltration advance problem and

(c) compare the approximate solutions with established ones.

It is worth noting, however, that a comprehensive theoretical exploration of convergence and error analysis lies beyond the scope of this paper.

3.1 The successive approximation method (SAM)

The SAM or the method of successive approximations is an iterative method considered by several authors, already decades ago, to solve differential equations, integral equations (both of Fredholm and Volterra type) and optimal control problems. We refer the interested reader to the early works (Bückner, 1948 [1]; Chen, 1981 [2]; Mitter, 1966 [17]; Wiggins, 1978 [27]) and the book by Kress (2014) [14] for the fundamentals of the method. To our knowledge this is the first time that this method is applied to (2.1).

The SAM is based on a fixed-point iteration, where the solution of a general equation $x(t) = F(t, x)$, is given in a form of a sequence $x_0(t), x_1(t), \ldots$ under the formula

$$x_{n+1}(t) = F(t, x_n), \quad n = 0, 1, \ldots .$$

In our case, using (2.1) we get

$$x_{n+1}(t) = \frac{qt}{c} - \frac{1}{c}\int_0^t y(t-\tau)x_n'(\tau)d\tau. \tag{3.1}$$

This method requires an initial guess $x_0(t)$. Theoretically, the solution is given by $x(t) = \lim_{n\to\infty} x_n(t)$, but in practice the first two or three terms result already in a good approximation of the exact solution.

3.2 The variational iteration method (VIM)

The main idea behind the VIM is the method of Lagrange multipliers. It is a new method, compared to SAM, proposed by He in 1997 (He, 1997 [9], 1998 [10]), see also the review paper by He and Wu (2007) [11]. It is widely used since it solves a large class of linear and non-linear problems easily and accurately, modelled both by differential and integral equations (Dehghan and Tatari, 2006 [6]; Hamoud et al., 2018 [8]; Momani and Abuasad, 2006 [18]; Ramos, 2008 [24]).

The iteration formula for an equation of the form $F(t,x) = g(t)$, reads

$$x_{n+1}(t) = x_n(t) + \int_0^t \lambda(s) \left(F\left(s, x_n\right) - g(s)\right) ds,$$

where the value of the multiplier λ depends on the form of linear part of F. In this work, we have a linear first-order differential operator, see (2.7), resulting in $\lambda = -1$. Thus, the initial equation can be rewritten as $x'(t) + F\left(t, x, x'\right) = 0$, and the correctional functional takes the form

$$x_{n+1}(t) = x_n(t) - \int_0^t \left(x_n'(s) + F\left(t, x_n, x_n'\right)\right) ds,$$

where again an initial guess is needed and the approximate solution is the function obtained at the nth iteration step.

To apply the VIM, we consider the variant (2.7) and we obtain the following iterative scheme

$$x_{n+1}(t) = x_n(t) - \int_0^t \left(x_n'(s) - \frac{q}{c+y_0} + \frac{1}{c+y_0}\int_0^s \frac{\partial y}{\partial s}(s-\tau)x_n'(\tau)d\tau\right) ds.$$

If the infiltration function y is given by (2.2) or (2.4) then $y_0 \equiv 0$, and the above formula after performing the outer integration coincides with (3.1). Thus, both iterative methods for the particular infiltration functions result in the same iterative scheme.

In the following section, we will apply (3.1) considering the different infiltration functions y, given by (2.2)–(2.6).

4 Exact and approximate solutions

This section is divided into four parts with respect to the different infiltration functions. For the linear one, see (2.3), we obtain an exact solution and for the other cases we derive approximate solutions. The first steps of the iterative scheme are presented and we compare them with solutions previously employed.

4.1 The two-term equation (2.2)

We apply the SAM, see (3.1), with initial guess $x_0(t) = 0$. We obtain immediately, for $n = 0$, that $x_1(t) = \frac{qt}{c}$. In the next step, we get

$$\begin{aligned} x_2(t) &= \frac{qt}{c} - \frac{1}{c}\int_0^t y(t-\tau)x_1'(\tau)d\tau \\ &= \frac{qt}{c} - \frac{q}{c^2}\int_0^t (S\sqrt{t-\tau} + A(t-\tau))d\tau \\ &= \frac{qt}{c} - \frac{q}{c^2}\left(\frac{2S}{3}t^{3/2} + \frac{A}{2}t^2\right). \end{aligned}$$

For $n = 2$, we derive

$$\begin{aligned} x_3(t) &= \frac{qt}{c} - \frac{1}{c}\int_0^t y(t-\tau)x_2'(\tau)d\tau \\ &= \frac{qt}{c} - \frac{1}{c}\int_0^t \left(S\sqrt{t-\tau} + A(t-\tau)\right)\left(\frac{q}{c} - \frac{qS}{c^2}\sqrt{\tau} - \frac{qA}{c^2}\tau\right)d\tau \\ &= \frac{qt}{c} - \frac{2qS}{3c^2}t^{3/2} + \frac{q\left(\pi S^2 - 4Ac\right)}{8c^3}t^2 + \frac{8AqS}{15c^3}t^{5/2} + \frac{A^2q}{6c^3}t^3. \end{aligned} \tag{4.1}$$

In the next step, new terms involving powers of t with exponents 3, $\frac{7}{2}$ and 4 will be introduced and the last term in the above expression will be updated. In general, the nth iterative step will add three new terms with t in the power of n, $n - \frac{1}{2}$ and $n - 1$. Thus, if the nth step is chosen as approximate solution, then the last term has to be dropped since it does not contain all information.

The formula (4.1) aligns with (Philip and Farrell, 1964, Eq. (35) [23]), proving the validity of the proposed scheme. The authors derived this expression by applying the Laplace transform in (2.1), resulting in a solution that incorporates the complementary error function. Subsequently, an asymptotic expansion was employed for small values of t. It is evident that our approach, based on the straightforward integration of elementary functions, is a more preferable methodology.

In Section 5 we examine how the solution changes with the addition of extra terms.

4.2 The linear equation (2.3)

The simple linear form (2.3) of the infiltration function simplifies a lot the following calculations and results in an exact solution. This is in accordance with the theory of VIM where for linear problems one iteration step is enough to obtain an exact solution (Momani and Abuasad, 2006 [18]).

We substitute (2.3) in (2.7), where now $y(0) = y_0$, to get

$$q = cx'(t) + K_s\int_0^t x'(\tau)d\tau + y_0x'(t).$$

Under the natural initial condition $x(0) = 0$, we get the first order linear ordinary differential equation

$$(c + y_0)\,x'(t) + K_sx(t) = q$$

with solution

$$x(t) = \frac{q}{K_s}\left(1 - e^{-\frac{K_s}{c+y_0}t}\right). \tag{4.2}$$

The form (4.2) is suitable for large times, since it is based on (2.3), and it first appeared by Collis-George (1974) [3] again with the use of the Laplace transform.

4.3 The power law equation (2.4)

As in the Subsection 4.1 for $x_0(t) = 0$, we get $x_1(t) = \frac{qt}{c}$, and the next term takes the form

$$\begin{aligned} x_2(t) &= \frac{qt}{c} - \frac{1}{c}\int_0^t y(t-\tau)x_1'(\tau)d\tau \\ &= \frac{qt}{c} - \frac{q\kappa}{c^2}\int_0^t (t-\tau)^\alpha d\tau \\ &= \frac{qt}{c} - \frac{\kappa q}{c(c+\alpha c)}t^{1+\alpha}. \end{aligned}$$

For $n = 2$, we get

$$\begin{aligned} x_3(t) &= \frac{qt}{c} - \frac{1}{c}\int_0^t y(t-\tau)x_2'(\tau)d\tau \\ &= \frac{qt}{c} - \frac{q\kappa}{c^2}\int_0^t (t-\tau)^\alpha \left(1 - \frac{\kappa(1+\alpha)}{c+\alpha c}\tau^\alpha\right) d\tau \\ &= \frac{qt}{c} - \frac{\kappa q}{c(c+\alpha c)}t^{1+\alpha} + \frac{\kappa^2 q\Gamma(1+\alpha)^2}{c^3\Gamma(2+2\alpha)}t^{1+2\alpha}, \end{aligned} \tag{4.3}$$

where $\Gamma(x)$ is the Gamma function. In the next step, a term proportional to $t^{1+3\alpha}$ will be added, thus all previously obtained terms will not change in contrast to what we have seen in Section 4.1.

In (Philip and Farrell, 1964, Eq. (20) [23]) the solution is constructed using the Laplace transform in a series representation

$$x(t) = \frac{qt}{c}\sum_{n=0}^{\infty}\frac{(-\beta t^\alpha)^n}{\Gamma(2+n\alpha)}, \qquad \text{for} \qquad \beta = \frac{\kappa}{c}\Gamma(1+\alpha).$$

The first three terms read

$$x(t) = \frac{qt}{c}\left(\frac{1}{\Gamma(2)} - \frac{\beta t^\alpha}{\Gamma(2+\alpha)} + \frac{\beta^2 t^{2\alpha}}{\Gamma(2+2\alpha)}\right)$$

and using that $\Gamma(2) = 1$, and $\Gamma(2+\alpha) = (1+\alpha)\Gamma(1) = 1+\alpha$, we get (4.3).

4.4 The two-parameter implicit infiltration equations

Starting with (2.5), we define $\gamma = \frac{2K_s}{S^2}$, and we rewrite it in the form

$$y - \frac{1}{\gamma}\ln(1+\gamma y) = K_s t.$$

We set $u = -(1+\gamma y)$, and we obtain

$$ue^u = -e^{-(\gamma K_s t+1)},$$

which can be solved using the Lambert W function with real argument (Corless et al., 1996 [5]). Recall the definitions: $W_0 = W$, if $W \geq -1$ and $W_{-1} = W$, if $W \leq -1$. Since $-e^{-(\gamma K_s t+1)} \geq -e^{-1}$ and $u(=W) \leq -1$, we get

$$y(t) = -\frac{S^2}{2K_s} - \frac{S^2}{2K_s}W_{-1}\left(-e^{-2\frac{K_s^2}{S^2}t-1}\right). \tag{4.4}$$

This equation was derived by Parlange et al. (2002) [20] for the dimensionless form of (2.5). It prevents us from an analytical representation of the advance position x but we can still derive an expression that can be numerically evaluated.

The iterative scheme (3.1) now reads: Set $x_0(t) = 0$, to get $x_1(t) = \frac{qt}{c}$. For $n = 1$, we obtain

(using again γ for the sake of presentation)

$$
\begin{aligned}
x_2(t) &= \frac{qt}{c} - \frac{1}{c}\int_0^t y(t-\tau)x_1'(\tau)d\tau \\
&= \frac{qt}{c} - \frac{q}{c^2}\int_0^t \left(-\frac{1}{\gamma} - \frac{1}{\gamma}W_{-1}\left(-e^{-(1+\gamma K_s(t-\tau))}\right)\right)d\tau \\
&= \frac{qt}{c} + \frac{q}{c^2\gamma}t + \frac{q}{c^2\gamma}\left[\frac{W_{-1}^2\left(-e^{-(1+\gamma K_s t)}e^{\gamma K_s \tau}\right)}{2\gamma K_s} + \frac{W_{-1}\left(-e^{-(1+\gamma K_s t)}e^{\gamma K_s \tau}\right)}{\gamma K_s}\right]_0^t \\
&= \frac{qt}{c} + \frac{q}{c^2\gamma}t + \frac{q}{c^2\gamma}\left(-\frac{1}{2\gamma K_s} - \frac{W_{-1}^2\left(-e^{-(1+\gamma K_s t)}\right)}{2\gamma K_s} - \frac{W_{-1}\left(-e^{-(1+\gamma K_s t)}\right)}{\gamma K_s}\right) \\
&= -\frac{q}{2c^2\gamma^2 K_s} + \frac{q}{c}\left(1+\frac{1}{c\gamma}\right)t - \frac{q}{c^2\gamma}\left(\frac{W_{-1}^2\left(-e^{-(1+\gamma K_s t)}\right)}{2\gamma K_s} + \frac{W_{-1}\left(-e^{-(1+\gamma K_s t)}\right)}{\gamma K_s}\right).
\end{aligned}
$$

To derive the above formula we have used the following properties of the Lambert W function: $W\left(-\frac{1}{e}\right) = -1$ and

$$
\int W\left(\alpha e^{\beta t}\right)dt = \frac{W^2\left(\alpha e^{\beta t}\right)}{2\beta} + \frac{W\left(\alpha e^{\beta t}\right)}{\beta} + \text{constant}.
$$

In the next step, products of Lambert functions will appear in the integrand. Then, there exist no formulas to compute the integral explicitly. One possibility to overcome this problem is by expanding the Lambert function around the branch point.

Using similar arguments and straightforward calculations, we can get an explicit representation of y from (2.6) which reads

$$
y(t) = \frac{S^2}{2K_s} + K_s t + \frac{S^2}{2K_s}W_0\left(-e^{-2\frac{K_s^2}{S^2}t-1}\right).
$$

The main difference compared to (4.4) is the appearance of the linear term $K_s t$.

At the second iteration step, we get

$$
\begin{aligned}
x_2(t) =& \frac{qt}{c} - \frac{1}{c}\int_0^t y(t-\tau)x_1'(\tau)d\tau \\
=& \frac{qt}{c} - \frac{q}{c^2}\int_0^t \left(\frac{1}{\gamma} + K_s(t-\tau) + \frac{1}{\gamma}W_0\left(-e^{-(1+\gamma K_s(t-\tau))}\right)\right)d\tau \\
=& \frac{qt}{c} - \frac{q}{c^2\gamma}t - \frac{qK_s}{2c^2}t^2 - \frac{q}{c^2\gamma}\left[\frac{W_0^2\left(-e^{-(1+\gamma K_s t)}e^{\gamma K_s \tau}\right)}{2\gamma K_s} + \frac{W_0\left(-e^{-(1+\gamma K_s t)}e^{\gamma K_s \tau}\right)}{\gamma K_s}\right]_0^t \\
=& \frac{qt}{c} - \frac{q}{c^2\gamma}t - \frac{qK_s}{2c^2}t^2 - \frac{q}{c^2\gamma}\left(-\frac{1}{2\gamma K_s} - \frac{W_0^2\left(-e^{-(1+\gamma K_s t)}\right)}{2\gamma K_s} - \frac{W_0\left(-e^{-(1+\gamma K_s t)}\right)}{\gamma K_s}\right) \\
=& \frac{q}{2c^2\gamma^2 K_s} + \frac{q}{c}\left(1-\frac{1}{c\gamma}\right)t - \frac{qK_s}{2c^2}t^2 \\
&+ \frac{q}{c^2\gamma}\left(\frac{W_0^2\left(-e^{-(1+\gamma K_s t)}\right)}{2\gamma K_s} + \frac{W_0\left(-e^{-(1+\gamma K_s t)}\right)}{\gamma K_s}\right).
\end{aligned}
$$

We note that since $y(0) = 0$ holds for both infiltration functions, the VIM coincides with SAM also here. Given that we consider only two terms in the series expansion we expect that the approximate solutions will be close to the exact ones only for short times.

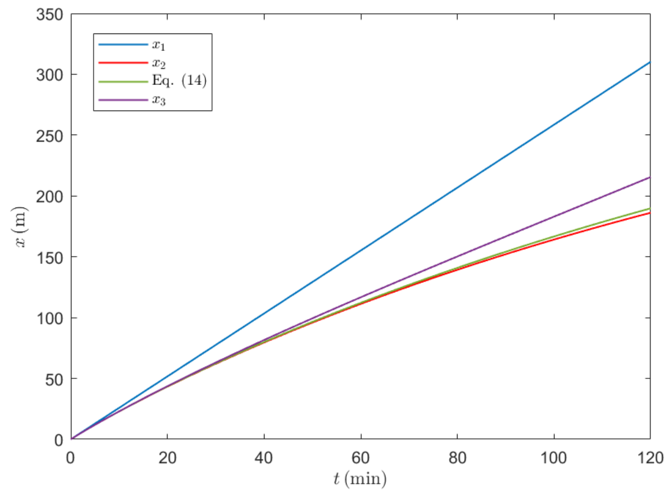

Fig. 5.1. The approximate solution of (2.1) using SAM at different iteration steps for the infiltration function (2.2) applied to the "Knight" soil.

5 Numerical examples

Regarding the two-term equation, see Section 4.1, in practise only the first few terms are needed for an accurate approximation as stated by Philip and Farrell (1964) [23]. Usually, the three-term approximation

$$x(t) \simeq \frac{qt}{c} - \frac{2qS}{3c^2}t^{3/2} + \frac{q\left(\pi S^2 - 4Ac\right)}{8c^3}t^2 \tag{5.1}$$

is sufficient and we examine its applicability in the next numerical example. The main reason is that since the infiltration function (2.2) is for short times then the next terms in (5.1), given the soils properties, may be neglected.

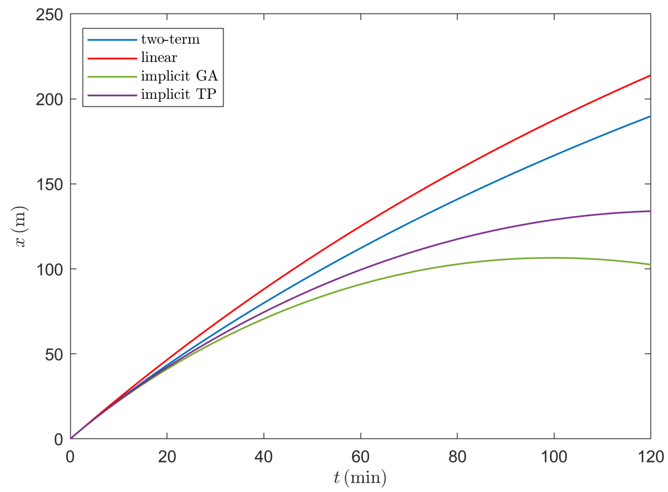

Fig. 5.2. The approximate solutions of (2.1) for the different infiltration functions applied to the "Knight" soil.

We consider the so called "Knight" soil with properties $K_s = 10^{-5}\mathrm{ms}^{-1}$, $S = 7.07 \times 10^{-4}\mathrm{ms}^{-1/2}$

and $A = 0.36K_s$ (Cook et al., 2013 [4]). The infiltration process is modelled through the flow depth $c = 0.1$m and the flow rate $q = 4.31 \times 10^{-3}\mathrm{m}^2\mathrm{s}^{-1}$. In Figure 5.1 we plot the solutions x_1, x_2, x_3 and (5.1). In x_2 and x_3 the last terms are omitted. We see that the differences are not crucial, less that 4% for $t = 60$min, and the behavior of the solution does not change drastically from x_2 to x_3.

In Figure 5.2 we plot all approximate solutions (up to three terms) for soil properties as in the first example. Equation (2.4) is excluded since the coefficients κ and α are not known for the particular soil. We observe that for short times all approximate solutions behave similar and the effect of the different infiltration functions is observed afterwards.

6 Conclusions

In this work we proposed to solve the infiltration advance problem using two iterative methods. We considered the successive approximation method and the variational iteration method and showed that they coincide when zero initial infiltration is assumed. We derived representations of the advance position for five (but not limited to) different infiltration functions after few iteration steps and simple calculations. We compared the results with existing formulas for three of the five infiltration functions. The representations derived for the implicit functions appear for the first time to our knowledge. As future work we plan to examine with experimental data their applicability as time increases.

Declarations

Conflict of Interest.

The authors declare no conflict of interest.

Funding.

This research received no external funding.

Author Contributions.

Conceptualization: G.K. and L.M.; Data curation: G.K.; Formal analysis: L.M.; Investigation: G.K. and L.M.; Methodology: L.M.; Visualization: L.M.; Writing - original draft: G.K. and L.M.; Writing - review and editing: G.K. and L.M.

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Received 05.03.2024
Revised 24.04.2024