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Generalized analytic functions and summary representation method and their applications. In commemoration of the 110th anniversary of H. M. Polozhii's birth

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Анотація. Статтю присвячено науковим ідеям та результатам видатного українського математика Георгія Миколайовича Положія (1914–1968). Розглянуто основні ідеї теорії узагальнених аналітичних функцій, методу сумарних зображень та нові результати застосування в математичному моделюванні нелінійних квазіідеальних процесів у LEF-пластах.

Abstract. The article is dedicated to the memory and scientific legacy of the prominent Ukrainian mathematician Heorhii Mykolaiovych Polozhii (1914–1968). The main ideas of the theory of generalized analytic functions, the summary representation method are described and some new application results in mathematical modelling of nonlinear quasi-ideal processes in LEF layers are considered.

1 Introduction

April 23, 2024 marked the 110th anniversary of the birth of the prominent Ukrainian mathematician Heorhii Mykolaiovych Polozhii (1914–1968). His scientific contribution is related to classical problems in the theory of analytic functions, the development of mathematical models for fluid flow in porous media (filtration theory), and advancements in modern computational mathematics.

Prof. Polozhii achieved outstanding results by founding two new directions: the theory of (p, q)-analytic functions and the summary representation method. Thanks to personalities like Prof. Polozhii, from the mid-20th century onward, Taras Shevchenko National university of Kyiv took a leading position in the field of filtration theory and related issues of mathematical physics. It is also worth noting that the impetus for the development of the Kyiv scientific school in the mathematical theory of filtration was the grandiose project of constructing the Dnipro cascade of hydroelectric power plants. The issue of land reclamation in the southern steppes also played an important role in stimulating applied work and involving a large number of young promising researchers.

The article highlights Prof. Polozhii's main scientific contributions and also outlines some new results arising from his classic ideas. Details about H. M. Polozhii's life and scientific work can be found in [1–3].

Key words: generalized analytic functions, filtration theory, computational mathematics.

2 The theory of (p, q)-analytic functions and filtration theory

Prof. Polozhii developed the theory of (p, q)-analitic functions while working on his PhD thesis. In particular, the following results were obtained: he formulated the concept of conjugate kernels, constructed a generalized Cauchy integral and a generalized Cauchy-type integral, investigated the differential properties of (p, q)-analytic functions; classified the isolated singularities and proved the main residue theorem; investigated the issue of uniqueness and (p, q)-continuation for (p, q)-analytic functions, and considered the roots separation problem for the equation $f(z) = A = \text{const}$; proved the region conservation theorem, the boundary congruence theorem, and the univalent neighborhood conservation theorem.

This approach has become widely used in applied mathematics and mechanics, particularly in filtration theory, axisymmetric theory of elasticity, gas dynamics, momentless theory of shells, etc. Using p-analytic functions, Prof. Polozhii managed to solve a number of classical problems in the theory of axisymmetric potential, which had not been previously solved through quadratures. A significant example is the problem of the axisymmetric potential of a spherical disk rather than a flat one, which is a remarkable achievement given the long-term attention this problem received from numerous mathematicians. Polozhii's main ideas are outlined in the monograph [4].

Recall that an analytic function $f(z) = u(x, y) + iv(x, y)$ of a complex variable $z = x + iy$ in the Cauchy–Riemann sense satisfies the Cauchy–Riemann conditions at each point of a connected domain $A \subset \mathbb{C}$:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \tag{2.1}$$

Polozhii suggested the generalization of the definition (2.1) introducing a function $f(z) = u(x, y) + iv(x, y)$ of a complex variable $z = x + iy$ which satisfies the system of equations

$$p\frac{\partial u}{\partial x} - q\frac{\partial u}{\partial y} - \frac{\partial v}{\partial y} = 0, \quad q\frac{\partial u}{\partial y} + p\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0 \tag{2.2}$$

with given real-valued differentiable functions p, q of variables x, y, and $p > 0$.

The proposed generalization (2.2) allows a researcher to analytically solve problems in which the following elliptic system occurs:

$$a_1\frac{\partial u}{\partial x} + b_1\frac{\partial u}{\partial y} + a_2\frac{\partial v}{\partial x} + b_2\frac{\partial v}{\partial y} = A_1 u + A_2 v,$$
$$c_1\frac{\partial u}{\partial x} + d_1\frac{\partial u}{\partial y} + c_2\frac{\partial v}{\partial x} + d_2\frac{\partial v}{\partial y} = B_1 u + B_2 v,$$

where a_i, b_i, c_i, d_i $(i = 1, 2)$ are known functions of variables x, y.

An important consequence of the theory of (p, q)-analytic functions is the method of majorant regions. This method is based on the topological properties of (p, q)-analytic functions, which makes it possible to combine two real-valued solutions $u(x, y)$ and $v(x, y)$ of the corresponding system of differential equations (3) into one complex-valued function $f(z) = u(x, y) + iv(x, y)$ of a complex variable $z = x + iy$.

Thanks to this, a number of problems in filtration theory and torsion theory for solids of revolution were solved. The essence of the method lies in using the topological properties of (p, q)-analytic functions to prove variational-topological comparison theorems. These theorems allow for the approximate determination of the integral characteristics of boundary value problems (for example, in filtration theory such characteristics include fluid flow, output velocity, and back pressure). By obtaining these characteristics, estimates from above and below can be determined as the region changes. Majorant regions, of course, should be simple and at the same time provide a sufficiently accurate two-sided estimate of the desired characteristic. If high precision is not required, the method is quite effective.

Mathematically, this method is based on Polozhii's theorem on conservation of a region for linear elliptic systems [5]. Below we formulate an important special case of Polozhii's theorem for p-analytic functions defined by an elliptic system

$$\frac{\partial\varphi}{\partial x} = \frac{1}{p}\frac{\partial\psi}{\partial y}, \quad \frac{\partial\varphi}{\partial y} = -\frac{1}{p}\frac{\partial\psi}{\partial x}$$

with a given function $p = p(x, y) > 0$.

Theorem 2.1. *Let $f = \varphi + i\psi \neq const$ is a p-analytic function in a domain D of the complex plane $z = x + iy$, and let the characteristic function p and its first partial derivatives are Hölder continuous. Then the set $f(D)$ is a region (i.e. a non-empty, connected, and open set).*

The theorem has mane it possible to extend the variational theorems of filtration theory and the method of majorant regions to the case of filtering in a homogeneous medium.

Later, this direction was further developed by I. I. Liashko, I. M. Velykoivanenko, and others, addressing numerous problems (see, e.g. [6–12]). Moreover, these results were utilized in the calculation of the Dnipro alluvial dams, the filtration characteristics of the dams at the Kyiv and Kaniv hydroelectric power plants, and many other hydrotechnical structures [13].

3 The summary representation method

Another direction developed by H. M. Polozhii is the summary representation method for solving problems of mathematical physics, as described in his book [14]. This method is a discrete analogue of the integral representation method. Such problems arise in the approximation of boundary value problems formulated by means of partial differential equations. Typically, these problems are reduced through discretization to large systems of linear algebraic equations. H. M. Polozhii proposed a method that allows one to find solutions of two- and three-dimensional boundary value problems for partial differential equations, either in explicit form, or by reducing them to relatively small systems of linear algebraic equations (through the so-called P-transformations, that is, matrices of a special type that are discrete analogues of complete integral transformations).

Considering the limited computational power of electronic computing machines at that time, the summary representation method was a significant advancement. Its computational efficiency inspired numerous works by followers, leading to its further development and widespread application. In particular, I. M. Lyashenko and V. I. Didenko developed the summary representation method for the Helmholtz equation in domains consisting of several rectangles. I. I. Lyashko, I. M. Velikoivanenko and A. A. Hlushchenko applied the method to the problem of non-pressure flat filtration. H. M. Polozhii, A. A. Skorobohatko and B. M. Bublyk found a solution of the first main biharmonic problem in the theory of elasticity for a semicircle and a rectangular plate.

The summary representation method was firts proposed in [4] for solving main boundary value problems in the theory of axisymmetric potential. Those problems are associated with partial differential equation

$$\frac{\partial^2 U}{\partial x^2} + \frac{1}{x}\frac{\partial U}{\partial x} + \frac{\partial^2 U}{\partial y^2} - cU = f(x, y), \quad (x, y) \in D, \tag{3.1}$$

where $c = \text{const} > 0$, $f(x, y)$ is a known function, D is a rectangle.

On a uniform grid

$$x_k = x_0 + ik, \quad y_j = y_0 + jh_1, \quad k = 1, 2, \ldots, n, \quad j = 1, 2, \ldots, m,$$
$$h > 0, \quad h_1 > 0,$$

the traditional three-point approximation for the second derivatives and the right-hand difference approximation for the first-order derivative $\partial U/\partial x$ are applied. It is shown that when using the

P-transformation with respect to the variable x to find the eigenvalues of the corresponding matrix of type Π and its fundamental matrix, the Sturm–Liouville problem arises for the finite-difference equation (see [14, p. 51]):

$$(a+k)v_{k+1}+v_k+(a+k-1)v_{k-1}-\lambda(a+k)v_k=0, \\ k=1,2,\dots,n, \quad a=\text{const}, \tag{3.2}$$

with the boundary value conditions

$$v_0\cos\alpha+v_1\sin\alpha=0, \quad v_{n+1}\cos\beta+v_n\sin\beta=0, \quad \cos\alpha\cos\beta\neq 0, \tag{3.3}$$

if the rectangle D is not adjacent to the y-axis, and with the boundary value conditions

$$v_0\neq\infty, \quad v_{n+1}\cos\beta+v_n\sin\beta=0, \quad \cos\beta\neq 0, \tag{3.4}$$

if the rectangle D is adjacent to the y-axis ($a=0$).

The solution of the problem (3.2), (3.3) or the problem (3.2), (3.4) can be obtained by introducing special functions of a discrete argument λ associated with the so-called Pollaczek polynomials [15, 16].

In [17], the construction of the summary representation formulas for the equation (3.1) is considered under the condition that the derivative $\partial U/\partial x$ is approximated by the central difference derivative. It is shown that in this case the eigenvalues of the corresponding matrix of type Π are closely related to the Legendre polynomials.

The equation (3.1) with partial derivatives in the rectangle D corresponds to the finite-difference equation

$$\frac{1}{h^2}\begin{array}{|c|c|c|}\hline & 1-\frac{1}{2(a+k)} & \\ \hline \gamma^2 & -2(1+\gamma^2)-ch^2 & \gamma^2 \\ \hline & 1+\frac{1}{2(a+k)} & \\ \hline\end{array}\, u=f(x_k,y_j),$$

$$k=1,2,\dots,n; \quad j=1,2,\dots,m; \quad \gamma=h/h_1.$$

Introducing the operator

$$R\big(u_k(y)\big)=\gamma^2u_k(y+h_1)-\big[2(1+\gamma^2)+ch^2\big]u_k(y)+\gamma^2u_k(y-h_1), \\ u_k(y)=u_k(x_k,y),$$

we obtain the finite-difference equation

$$R\big(u_k(y_j)\big)+\Big[1-\frac{1}{2(a+k)}\Big]u_{k-1}(y_j)+\Big[1+\frac{1}{2(a+k)}\Big]u_{k+1}(y_j)= \\ =h^2f_k(y_j), \quad k=1,2,\dots,n. \tag{3.5}$$

Introducing the matrix of type Π

$$T_8=\rho^{-1}\times$$

$$\times\begin{pmatrix} 0 & 2a+3 & 0 & \dots & \dots & \dots & 0 \\ 2a+3 & 0 & 2a+5 & 0 & \dots & \dots & 0 \\ 0 & 2a+5 & 0 & 2a+7 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & 0 & 2a+2n-3 & 0 & 2a+2n-1 \\ 0 & \dots & \dots & \dots & 0 & 2a+2n-1 & 0 \end{pmatrix}$$

with the diagonal matrix $\rho = \operatorname{diag}\big[2(a+1), 2(a+2), \dots, 2(a+n)\big]$, we can rewrite the system (3.5) in the vector form

$$R\big(\vec{u}(y_j)\big) + T_8\vec{u}(y_j) = h^2\vec{f}(y_j) - \vec{\omega}_8(y_j), \quad j = 1, 2, \dots, m, \tag{3.6}$$

where $\vec{u}(y_j)$ and $\vec{f}(y_j)$ are n-dimensional vectors with the components $u_k(y_j)$ and $f_k(y_j)$ respectively,

$$\vec{\omega}_8(y_j) = \left\{\Big[1 - \frac{1}{2(a+1)}\Big]u_0(y_j),\, 0, \dots, 0,\, \Big[1 + \frac{1}{2(a+n)}\Big]u_{n+1}(y_j)\right\}$$

is an n-dimensional vector.

The equation for eigenvalues λ and eigenvectors $\vec{v} = \{v_1, v_2, \dots, v_n\}$ of the matrix T_8 can be written in the form $T_8\vec{v} - \lambda\vec{v} = \vec{0}$, namely:

$$(2a + 2k + 1)v_{k+1} + (2a + 2k - 1)v_{k-1} - 2(a + k)\lambda v_k = 0,$$
$$k = 1, 2, \dots, n.$$

Setting $a = \nu - 1/2$, where $\nu \geqslant 0$ is an integer number, we get the Sturm–Liouville problem

$$kv_{k+1} + (k-1)v_{k-1} - \lambda/2(2k-1)v_k = 0, \quad k = 1, 2, \dots, n, \tag{3.7}$$

$$v_0 \neq \infty, \quad v_{n+1} = 0, \tag{3.8}$$

for $\nu = 0$, and

$$(k+\nu)v_{k+1} + (k+\nu-1)v_{k-1} - \lambda/2\big[2(k+\nu) - 1\big]v_k = 0, \;\; k = 1, 2, \dots, n, \tag{3.9}$$

$$v_0 = 0, \quad v_{n+1} = 0, \tag{3.10}$$

for $\nu > 0$.

Let $\Lambda = \operatorname{diag}[\lambda_1, \lambda_2, \dots, \lambda_n]$ be a diagonal matrix of the eigenvalues of the matrix T_8 and let P_8 be its fundamental matrix. Introducing the P-transformation of the vector $\vec{u}(y_j)$ by the formula

$$\vec{\widehat{u}}(y_j) = P_8^{\mathrm{T}}\rho\vec{u}(y_j),$$

where P_8^{T} denotes the transpose of the matrix P_8, and applying the main theorem about the matrix of type Π [14, p. 32], we can rewrite the matrix equation (3.6) as follows:

$$R\vec{\widehat{u}}(y_j) + \Lambda\vec{\widehat{u}}(y_j) = h^2\vec{\widehat{f}}(y_j) - \vec{\widehat{\omega}}_8(y_j),$$

or in a more detailed form

$$\widehat{u}_k(y_j + h_1) - \big[2(1+\gamma^{-2}) + ch_1^2 - \lambda_k\gamma^{-2}\big]\widehat{u}_k(y_j) + \widehat{u}_k(y_j - h_1) =$$
$$= h_1^2\widehat{f}_k(y_j) - \gamma^{-2}\widehat{\omega}_{8,k}(y_j), \quad k = 1, 2, \dots, n. \tag{3.11}$$

The general solution of each equation in (3.11) can be written in the form

$$\widehat{u}_k(y_j) = A_k\varphi_k(y_j) + B_k\psi_k(y_j) + \sum_{p=1}^{j-1} G(j-p)\big[h_1^2\widehat{f}_k(y_p) - \gamma^{-2}\widehat{\omega}_{8,k}(y_p)\big],$$
$$j = 0, 1, \dots, m+1, \tag{3.12}$$

where the sum is assumed to be zero for $j = 0, 1$; A_k and B_k are arbitrary constants, and $\varphi_k(y_j)$, $\psi_k(y_j)$, $G(j)$ depend on the value of the quantity

$$\eta_k = 1 + \gamma^{-2}\big(1 - \lambda_k/2\big) + ch_1^2/2$$

and are defined as follows:

	$\varphi_k(y_j)$	$\psi_k(y_j)$	$G(j)$	
$\lvert\eta_k\rvert > 1$	μ_k^j	ν_k^j	$\dfrac{\mu_k^j - \nu_k^j}{\mu_k - \nu_k}$	$\mu_k = \eta_k + \sqrt{\eta_k^2 - 1}, \nu_k = \mu_k^{-1}$
$\lvert\eta_k\rvert = 1$	μ_k^j	$j\mu_k^j$	$j\mu_k^{j-1}$	$\nu_k = \eta_k - \sqrt{\eta_k^2 - 1}$
$\lvert\eta_k\rvert < 1$	$\cos(j\theta_k)$	$\sin(j\theta_k)$	$\dfrac{\sin(j\theta_k)}{\sin\theta_k}$	$\theta_k = \arccos\eta_k$

Taking into account the representation (3.12), for the general solution of the equation (3.11) we have

$$\vec{\tilde{u}}(y_j) = \Phi(j)\vec{A} + \Psi(j)\vec{B} + \\ + \sum_{p=1}^{j-1} G(j-p)\big[h_1^2\vec{\tilde{f}}(y_p) - \gamma^{-2}\vec{\tilde{\omega}}_8(y_p)\big], \quad j = 0, 1, \ldots, m+1, \tag{3.13}$$

where the sum is assumed to be zero for $j = 0, 1$; $\vec{A} = \{A_k\}_{k=1}^n$ and $\vec{B} = \{B_k\}_{k=1}^n$ are n-dimensional vectors of arbitrary constants; $\Phi(j), \Psi(j), G(j)$ are $n \times n$-dimensional diagonal matrices with the elements $\varphi_k(y_i)$, $\psi_k(y_j)$, $G_k(j), k = 1, 2, \ldots, n$, respectively.

Multiplying both sides of the equation (3.13) by P_8, we finally get the summary representation formula for the equation (3.1) in a rectangle D:

$$\vec{u}(y_j) = P_8\Phi(j)\vec{A} + P_8\Psi(j)\vec{B} + \sum_{p=1}^{j-1} P_8 G(j-p)P_8^{\mathrm{T}}\rho\times \\ \times \big[h_1^2\vec{f}(y_p) - \gamma^{-2}\vec{\omega}_8(y_p)\big], \quad j = 0, 1, \ldots, m+1, \tag{3.14}$$

where the sum is assumed to be zero for $j = 0, 1$.

We consider now the following two cases:

(a) $a = -1/2$;

(b) $a = \nu - 1/2$, where ν is a positive integer number.

In case (a), determining the elements of the matrix Λ and the columns of the matrix P_8 comes down to determining the eigenvalues and the eigenvectors (orthogonal with the weight ρ) of the Sturm–Liouville problem (3.7), (3.8).

Considering the recurrence relation for the Legendre polynomials

$$(m+1)P_{m+1}(x) + mP_{m-1}(x) - x(2m+1)P_m(x) = 0,$$

one can observe that for $k = m+1$, $v_{m+1}P_m(\lambda/2)$, $m = 0, 1, \ldots, n-1$, equation (3.7) can be written as follows:

$$(m+1)P_{m+1}(\lambda/2) + mP_{m-1}(\lambda/2) - \lambda/2(2m+1)P_m(\lambda/2) = 0, \\ m = 0, 1, \ldots, n-1.$$

The boundary value conditions (3.8) take the form $v_0 \neq \infty$, $P_n(\lambda/2) = 0$. From this we conclude that the eigenvalues of the matrix T_8, that is the elements of the diagonal matrix Λ, are the doubled zeros of the Legendre polynomial $P_n(x)$.

Each eigenvalue λ_m corresponds to the eigenvector $\vec{v}_m$ of the matrix T_8:

$$\vec{v}_m = \frac{1}{N_m}\big\{P_0,\, P_1(\lambda_m/2),\, P_2(\lambda_m/2), \ldots,\, P_{n-1}(\lambda_m/2)\big\}$$

with $N_m = \left\{ \sum\limits_{k=0}^{n-1} P_k^2(\lambda_m/2)(2k+1) \right\}^{1/2}$. The fundamental matrix P_8 can be written as follows:

$$P_8 = \begin{pmatrix} 1 & 1 & \dots & 1 \\ P_1(\lambda_1/2) & P_1(\lambda_2/2) & \dots & P_1(\lambda_n/2) \\ \dots & \dots & \dots & \dots \\ P_{n-1}(\lambda_1/2) & P_{n-1}(\lambda_2/2) & \dots & P_{n-1}(\lambda_n/2) \end{pmatrix} N^{-1},$$

where $N = \mathrm{diag}[N_1,\, N_2, \dots,\, N_n]$ is a diagonal matrix. Noting that for $a = -1/2$, the first component of the vector $\vec{\omega}_8(y_p)$ is zero (regardless of the value of u_0) and taking into consideration that for $c \geq 0$ the conditions $|\eta_k| > 1$, $k = 1, 2, \dots, n$, are satisfied (indeed, the zeros of the Legendre polynomials lie in the interval $(-1; 1)$), we conclude that the summary representation formula (3.14) gives explicitly the solution of the problem (3.1) in the finite-difference setting for a domain adjacent to the y-axis if the values of the function u are known on the vertical line $x = x_{n+1}$.

In case (b), determining the elements of the matrix Λ and the columns of the matrix P_8 comes down to solving the Sturm–Liouville problem (3.9), (3.10). The Pollaczek polynomials (which are identical to the associated Legendre polynomials) $P_n(x, \tau, A, B, C)$ of degree n and of the variable x can be defined by the recurrence relation (see [16])

$$(n + C)P_n - 2\big[(n - 1 + \tau + A + C)x + B\big]P_{n-1} +$$
$$+ (n + 2\tau + C - 2)P_{n-2} = 0, \quad n = 1, 2, \dots; \quad P_0 = 1, \quad P_{-1} = 0.$$

Putting $v_{k+1} = P_k$ into the relation (3.9), we obtain

$$(k + \nu)P_k + (k + \nu - 1)P_{k-2} - \lambda/2\big[2(k + \nu) - 1\big]P_{k-1} = 0,$$
$$k = 1, 2, \dots, n.$$

The relation above is the same as (3.14) for $C = \nu$, $\tau = 1/2$, $A = B = 0$, $x = \lambda/2$. Therefore, the equation (3.9) will be satisfied if

$$v_k = P_{k-1}\big(\lambda/2,\, 1/2,\, 0,\, 0,\, \nu\big), \quad k = 0, 1, \dots, n, n+1,$$
$$P_{-1} = 0, \quad P_0 = 1.$$

Denoting

$$P_{n,\nu} = P_n\big(x,\, 1/2,\, 0,\, 0,\, \nu\big), \quad P_{0,\nu}(x) = 1, \quad P_{-1,\nu}(x) = 0,$$

we obtain $v_k = P_{k-1,\nu}(\lambda/2)$, $k = 0, 1, \dots, n, n+1$, and the boundary value conditions (3.10) take the form

$$v_0 = P_{-1,\nu}(\lambda/2) = 0, \quad P_{n,\nu}(\lambda/2) = 0.$$

Thus, in case (b), the eigenvalues of the matrix T_8 (that is, the elements λ_m of the matrix Λ) are the doubled zeros of the polynomials $P_{n,\nu}(x)$. Each eigenvalue λ_m corresponds to the eigenvector $\vec{v}_m$ of the matrix T_8:

$$\vec{v}_m = \frac{1}{N_{m,\nu}}\big\{1,\, P_{1,\nu}(\lambda_m/2),\, P_{2,\nu}(\lambda_m/2), \dots,\, P_{n-1,\nu}(\lambda_m/2)\big\}$$

with $N_{m,\nu} = \left\{ \sum\limits_{k=0}^{n-1} P_{k,\nu}^2(\lambda_m/2)\big[2(k + \nu) + 1\big] \right\}^{1/2}$. The fundamental matrix P_8 can be written as follows:

$$P_8 = \begin{pmatrix} 1 & 1 & \dots & 1 \\ P_{1,\nu}(\lambda_1/2) & P_{1,\nu}(\lambda_2/2) & \dots & P_{1,\nu}(\lambda_n/2) \\ \dots & \dots & \dots & \dots \\ P_{n-1,\nu}(\lambda_1/2) & P_{n-1,\nu}(\lambda_2/2) & \dots & P_{n-1,\nu}(\lambda_n/2) \end{pmatrix} N_\nu^{-1},$$

where $N_\nu = \mathrm{diag}[N_{1,\nu},\ N_{2,\nu}, \dots,\ N_{n,\nu}]$ is a diagonal matrix. The general properties of the Pollaczek polynomials [15, 16] imply that all the zeroes of the polynomial $P_{n,\nu}(x)$ lie in the interval $(-1;\,1)$. Therefore, for $c \geq 0$ the conditions $|\eta_k| > 1$, $k = 1, 2, \dots, n$, are satisfied. We conclude that the summary representation formula (3.14) provides explicitly the solution of the problem (3.1) in the finite-difference setting for a domain which is not adjacent to the y-axis.

4 Textbooks on mathematical physics and computational mathematics

In addition to his active scientific work, Prof. Polozhii was engaged in intensive pedagogical activities, creating innovative textbooks on mathematical physics and computational mathematics. Notably, he authored the first Ukrainian textbook on mathematical physics, 'Equations of Mathematical Physics' [18]. The university textbook 'Mathematical Workshop' [19], written by a team of authors under Polozhii's leadership, was translated into German and Polish. His university textbook [20] has been widely recognized and remains popular among students and lectures in Ukraine and abroad.

5 Modeling of nonlinear quasi-ideal processes in LEF layers

Based on the generalization, adaptation, and synthesis of numerical methods of complex analysis, the summary representation method, and decomposition, works [21–23] develop the methodology for mathematical modeling of nonlinear quasi-ideal processes in LEF-reservoirs (i.e., zonally heterogeneous porous oil and gas, aquifer, and shale reservoirs, whose zone geometry is determined taking into account the feedback of process characteristics on the medium's conductivity).

Namely, the methodology is meant for solving nonlinear boundary value problems for systems of elliptic differential equations, where the medium's conductivity coefficient is influenced by the potential field (head, pressure) and the flow function. The proposed approach is applicable to simply, two- and multi-connected curvilinear domains bounded by equipotential lines and streamlines (LEF-regions), using the summary representation method for differential equations with discontinuous coefficients (in the case of layered media) or numerical-analytical representations of solutions (which generalize the summary representation method to cases of heterogeneous and anisotropic media).

For the first time, the summary representation formulas are applied as a component of previously developed (based on complex analysis) computational procedures. The combination of methods from complex analysis (involving conformal mappings) and the summary representation formulas for approximating coordinates of internal nodes of a dynamic grid significantly improves existing methodologies for solving such class of problems. It enhances the efficiency (convergence rate) of the corresponding iterative process, addressing the challenge of achieving the necessary accuracy in initial approximations of sought-after functions. Moreover, it allows for comprehensive consideration (summarily) at each iterative step of the influence not only of surrounding nodes but also all boundary and internal nodes of the dynamic grid, thereby greatly accelerating the attainment of convergence for the sought harmonic functions.

In particular, the model problems describing stationary filtration processes are studied in the following cases: in curvilinear *simply connected* so-called LEF-domains $G_z \subset \mathbb{C}$ $(z = x+iy)$ bounded by the equipotential lines $L_* = \{z : f_1(x, y) = 0\}$, $L^* = \{z : f_3(x, y) = 0\}$ and by the streamlines $L_0 = \{z : f_4(x, y) = 0\}$, $L^0 = \{z : f_2(x, y) = 0\}$ with the potential field $\varphi = \varphi(x, y)$ satisfying the conditions

$$\varphi|_{L_*} = \varphi_*, \quad \varphi|_{L^*} = \varphi^*, \quad \left.\frac{\partial\varphi}{\partial\vec{n}}\right|_{L_0} = \left.\frac{\partial\varphi}{\partial\vec{n}}\right|_{L^0} = 0,$$

where $-\infty < \varphi_* < \varphi^* < +\infty$ are constants and $\vec{n}$ is the external normal to the respective curve; in *two-connected* LEF-domains bounded by two smooth closed curves $L_* = \{z : f_*(x, y) = 0\}$ (the inner curve), $L^* = \{z : f^*(x, y) = 0\}$ (the outer curve), where, to form simply connected LEF-domains G_z^Γ, a conditional cut Γ is made along a certain sought streamline (then L_0 and L^0 are the boundary streamlines of the domain G_z^{Γ}, which are respectively the upper and lower edges of the cut Γ); in *three-connected* LEF-domains bounded by two internal well contours – equipotential lines $L_* = \{z : f_*(x,y) = 0\}$, $L^* = \{z : f^*(x,y) = 0\}$ and by impermeable external contour $L = \{z : f(x,y) = 0\}$ where two conditional cuts, Γ_* and Γ^*, are made along such streamlines (which are the separation lines of the flow) that are uniquely determined by the 'stopping' points of the flow: $H_* \in L$, $H^* \in L$, $G_z^{\Gamma} = G_z \backslash (\Gamma_* \cup \Gamma^*)$; in LEF-domains with *a free boundary* (depression curve), where an additional condition is specified: $\varphi|_{BC} = g(y)$, $H \geqslant y \geqslant y_* = f^*(x_*)$, $g(y)$ is some known monotonically decreasing function.

For multiply connected LEF-domains, the complexity lies in the incomplete determination of the complex quasipotential field shape, which depends on various factors: the configuration of the physical domain, including the relative placement of wells, methods of conditional cuts to reduce a multiply connected domain to a simply connected one, the relationship between boundary potential values, and so on. In [23] a new approach is proposed for classifying situational states of flow formation, which allows for standardization across all cases of three-connected curvilinear LEF-domains bounded by three equipotential lines, and four-connected curvilinear LEF-domains bounded by three equipotential lines and an impermeable contour. It includes the formulation of inverse problems for quasiconformal mappings, their discrete analogues, and numerical algorithms.

6 Conclusion

Heorhii Mykolayovych Polozhii (1914–1968) is rightfully regarded as one of the foremost and influential figures within the Kyiv scientific school of mathematical modeling and computational mathematics. He is the author of the majorant regions method and the summary representation method. The former allows for the systematic construction of bilateral estimates of filtration characteristics by deforming the filtration domain of complex shapes. The latter is a discrete analogue of integral representation methods. In this method, the solution at each node of the grid domain is formulated in a closed form as a summary representation formula. Depending on the nature of the boundary conditions, these formulas either appear explicitly or contain a small number of unknown parameters (compared to the total number of grid nodes) that are determined by auxiliary systems of linear algebraic equations.

The directions initiated by H. M. Polozhii have become guiding principles for the research endeavors of his numerous followers both in Ukraine and abroad. He was also the founder and first head of the Department of Computational Mathematics at Taras Shevchenko National University of Kyiv, and established a scientific school whose representatives continue to engage in active research to this day.

The article focuses on H. M. Polozhii's ideas in the theory of generalized analytic functions and the summary representations method. Additionally, it describes a new methodology for mathematical modeling of nonlinear quasi-ideal processes in LEF reservoirs based on numerical methods of complex analysis and summary representations.

Declarations

Conflict of Interest.

The authors declare that there is no conflict of interest.

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