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METHODS FOR MINIMIZING THE SAVAGE FUNCTION WITH VARIOUS CONSTRAINTS

In the paper, a decision-making situation under uncertainty is addressed and analyzed, with emphasis on minimizing Savage's regret function. The developed algorithms are based on the generalized gradient projection method with programmed step size adjustment. Assertions about the convergence of the corresponding algorithms are proven.

Keywords: *uncertainty, decisions, Savage function, optimization*.

When making decisions under uncertainty, Savage's criterion is sometimes used, or the criterion of minimizing regrets [1]. Usually, in the literature, this decision situation is described in matrix language. In other words, both the number of decision alternatives and the number of states of nature are finite. Of particular interest are situations where the admissible domain of decision variants is a convex set, and the regrets with respect to each state of nature are expressed by convex functions. In this paper, we propose numerical methods for minimizing Savage's regret function, constructed based on the subgradient projection method with automatic step size adjustment [2, 3]. The convergence of these methods is demonstrated.

Let on the compact and convex set U in E^n – Euclidean space n-dimensional, define the functions, convex and continuous on $U, r(u, Q_1), \dots, r(u, Q_i), \dots, r(u, Q_m)$. The set $Q = \{Q_1, \dots, Q_i, \dots, Q_m\}$ defines the uncertainty situation, in the sense that any element of this set expresses a certain "state of nature", which can be produced uncontrollably by some decision-maker, which can in turn manage with u variants of U.

Thus, for each state $Q_i \in Q$, it will be denoted by $r_i^* = r\left(u_{(i)}^*, Q_i\right) = \min_{u \in U} r(u, Q_i)$ – the minimum value of the function on the domain U, and $u_{(i)}^*$ representing some element of the set $U_{(i)}^*$ – the set of minimum points of the function $r(u; Q_i)$.

Remark. In a certain context, $r(u,\ Q_i)$ could be considered to express some cost or loss caused by the application of decision u, when "nature" is pronounced with its state Q_i .

It will be defined with $\bar{r}(u,\ Q_i) = (r(u,\ Q_i) - r_i^*)$ – the function of regrets, which corresponds to the state Q_i , and with $R_s(u) = \max_{i \in I_m}(\bar{r}(u,\ Q_i))$, hereafter called the Savage function. Here $I_m = \{i: i = 1, 2, \dots, m\}$.

Remark. It is elementary to note that both the regret function $\bar{r}(u, Q_i)$ and the Savage function $R_s(u)$ are convex on U.

It will be additionally assumed that the sub gradients of the functions $r(u,\ Q_i)$, $i\ \in\ I_m$, also exist for all border points U, which, at the same time, are uniformly bounded in U. That is, there exists the constant C_1, for which $\|r'_u(u,Q_i)\| \le\ C_1$ for all $u \in U$ and all $i \in\ I_m$, $r'_u(u;Q_i)$ being any subgradient of the function $r'_u(u;Q_i)$, which corresponds to point *u*.

Let the set U is relatively simple, in the sense that for any point $z \in E^n$, the projection of z onto U can be determined exactly [2]. This projection will be denoted $\Pi_U(z)$.

We will consider m+1 iterative processes, which can run in parallel. The first m processes will be called internal and will determine approximations of the values r_i^*, $i \in\ I_m$; and the (m+1)-th one will be called an external process, oriented to provide approximations of the value

$$R_S^* = \min_{u\in U} R_S\,(u)\,. \tag{1}$$

So, the m+1 parallel (optimization) processes are described as follows:

$$\begin{cases} u_{(i)}^{k+1} = \Pi_U\big(u_{(i)}^k - h_{(i)k}\cdot \eta_{(i)}^k\big), i \in I_m, \\ u^{k+1} = \Pi_U\big(u^k - h_k \cdot \eta^k\big). \end{cases} \tag{2}$$

Here:

$$\eta_{(i)}^k = \begin{cases} r'_u\big(u_{(i)}^k, Q_i\big) / \big\|r'_u\big(u_{(i)}^k, Q_i\big)\big\|, \text{if}\ r'_u\big(u_{(i)}^k, Q_i\big) \ne 0, \\ 0, \text{otherwise}, \end{cases} \tag{3}$$

$$\eta^k = \begin{cases} \Big(R_S^k\big(u^k\big)\Big)'_u / \Big\|\Big(R_S^k\big(u^k\big)\Big)'_u\Big\|, \text{if}\ \Big(R_S^k\big(u^k\big)\Big)'_u \ne 0, \\ 0, \text{otherwise}. \end{cases} \tag{4}$$

where $\Big(R_S^k\big(u^k\big)\Big)'_u$ represents an arbitrary subgradient of the function

$$R_S^k(u) =\ \max_{i\in I_m}\big[r(u,Q_i) - r\big(u_{(i)}^k, Q_i\big)\big], \tag{5}$$

calculated in the point u=u^k .

Number strings $h_{(i)k}, h_k$ they will respect the conditions [2]:

$$h_{(i)k}, h_k\ \ge 0\ ;\ h_{(i)k}, h_k\ \to\ 0\ ;\ \sum_{k=0}^{\infty} h_{(i)k}\,, \sum_{k=0}^{\infty} h_k = \infty\,. \tag{6}$$

Let U^* – the set of minimum points of the Savage function on the domain U. In compliance with the conditions listed above, the following statement occurs.

Theorem 1.

$$\lim_{k\to\infty}\min_{u^*\in U^*}\big\|u^k - u^*\big\| = 0\,, \lim_{k\to\infty} R_S^k\big(u^k\big) = R_S^*, \tag{7}$$

where R_S^* and $R_S^k(u)$ are as defined in (1) and (5), respectively.

Demonstration.

For this, the following two lemmas will be useful:

Lemma 1. Let some vector field $U \subseteq E^n$ be defined on the convex set $G: U \to 2^U$, which maps to each element $u \in U$ a non-empty set of vectors $G(u)$ (concisely, $u \to G(u)$) and there exists a compact set, denoted by U^*, which satisfies the following condition: for a certain number $\varepsilon > 0$, there exists the number $\delta > 0$, such that for all $u \in U \setminus V(U^*, \varepsilon/2)$, $u^* \in U$ and $g(u) \in G(u)$ the inequality occurs:

$$\frac{(u - u^*, g(u))}{\|u - u^*\| \cdot \|g(u)\|} \ge \delta. \tag{8}$$

If the string were considered:

$$u^{k+1} = \Pi_U\left(u^k - h_k \cdot \eta^k\right) \quad (9)$$

with $\eta^k = g(u^k)/\|g(u^k)\|$, $h_k \geq 0$, $h_k \to 0, \sum_{k=0}^{\infty} h_k = \infty$, then for any point $u^\circ \in U$ $\left(\|u^\circ\| < \infty\right)$, starting with some K, for all $k \geq K$, $u^k \in V(U^*, \varepsilon)$ – the spherical neighborhood of radius ε of the set U^*; that is $V(U^*, \varepsilon)$ represents the union of all neighborhoods of radius ε, $V(u^*, \varepsilon), \forall\, u^* \in U^*$.

Proof of Lemma 1.

The expression will be analyzed $\|u^{k+1} - u^*\|^2$, where $u^* \in U^*$.

Obviously, it takes place:

$$\|u^{k+1} - u^*\|^2 \leq \|u^k - h_k \cdot \eta^k - u^*\|^2 = \|u^k - u^*\|^2 - 2h_k(u^k - u^*, \eta^k) + (h_k)^2.$$

Two stages will be considered.

Stage 1. We will argue the existence of a substring $\{u^{k_i}\}_{i\geq 1}$, which is totally contained in the set $V(U, \varepsilon/2)$. For this purpose, the method of reduction to the absurd will be applied.

According to inequality (8), we get $\|u^{k+1} - u^*\|^2 \leq \|u^k - u^*\|^2 - 2\delta h_k\|u^k - u^*\| + h_k^2 \leq$ $\leq \|u^k - u^*\| - \delta\varepsilon h_k + h_k^2$, for all $k \geq K_1$, where K_1 is a certain fixed number.

Because $h_k \to 0$, it follows that the number exists K_2, so that $h_k < \delta\varepsilon/2$, for all $k \geq K_2$. Then, for all $k \geq K_0 = max\{K_1, K_2\}$, the series of inequalities is true:

$$\|u^{k+1} - u^*\|^2 \leq \|u^k - u^*\|^2 - h_k(\delta\varepsilon - h_k) \leq \|u^k - u^*\|^2 - \frac{\delta\varepsilon}{2}h_k \leq \|u^{K_0} - u^*\|^2 - \frac{\delta\varepsilon}{2}\sum_{l=K_0}^{k} h_k.$$

The right-hand side of this series of inequalities, for certain sufficiently large values of *k*, becomes negative, moreover, it decreases indefinitely. An obvious contradiction is obtained.

Therefore, there exists such a substring $\{u^{k_i}\}_{i\geq 1}$ of the string $\{u^k\}_{k\geq 0}$, which is completely contained in the set $V(U^*, \varepsilon/2)$.

Stage 2. From the fact that $h_k \to 0$, it follows that and $u^{k+1} - u^k \to 0$.

Without restricting the generality, it can be considered that the previously mentioned number K_0 is such that $\|u^{k+1} - u^k\| \leq \frac{\varepsilon}{2}$ for all $k \geq K_0$. Let *l* be any number from the set of natural numbers for which:

$$u^{k_l} \in V\left(U^*, \frac{\varepsilon}{2}\right), \text{and } u^{k_l+1} \notin V\left(U^*, \frac{\varepsilon}{2}\right) \text{ and } \|u^{k_l+1} - u^{k_l}\| \leq \varepsilon/2.$$

Then: $\min\limits_{u^*\in U^*}\|u^{k_l+1} - u^*\| = \min\limits_{u^*\in U^*}\|u^{k_l+1} - u^{k_l} + u^{k_l} - u^*\| \leq \|u^{k_l+1} - u^{k_l}\| + \min\limits_{u^*\in U^*}\|u^{k_l} - u^*\| < \varepsilon.$

But, because, at the same time $h_k < \delta\varepsilon/2$, for $k \geq K_0$, results:

$$\|u^{k_l+2} - u^*\|^2 \leq \|u^{k_l+1} - u^*\|^2 - \frac{\delta\varepsilon}{2}h_{k_l+1} < \|u^{k_l+1} - u^*\|^2, \dots, \|u^{k_l+j} - u^*\|^2 \leq \ \dots\ < \|u^{k_l+1} - u^*\|^2.$$

So, if $u^{k_l+j} \notin V\left(U^*, \frac{\varepsilon}{2}\right)$, it is obtained that $\min\limits_{u^*\in U^*}\|u^{k_l+j} - u^*\| \leq \min\limits_{u^*\in U^*}\|u^{k_l+1} - u^*\| < \varepsilon$, for all $j = \overline{2, J_l}$, where $J_l = K_{l+1} - K_l$.

Lemma 1 is proved.

Lemma 2. Let $i \in I_m = \{1, 2, \dots, m\}$, and $\{a_i\}$ and $\{b_i\}$ – two arbitrary numeric strings. The following inequality holds:

$$\left|\max_{i\in I_m} a_i - \max_{i\in I_m} b_i\right| \leq \max_{i\in I_m}|a_i - b_i|.$$

Demonstration. Let $a_{i_1} = \max\limits_{i\in I_m} a_i$, $b_{i_2} = \max\limits_{i\in I_m} b_i$.

1. If $i_1 = i_2$, the inequality is obvious.
2. Let $i_1 \neq i_2$.

For $a_{i_1} \geq b_{i_2}$, results: $0 \leq a_{i_1} - b_{i_2} \leq a_{i_1} - b_{i_1}$.

If $b_{i_2} \geq a_{i_1}$, it is obtained that $0 \leq b_{i_2} - a_{i_1} \leq b_{i_2} - a_{i_2}$.

So, $\left|a_{i_1} - b_{i_2}\right| \leq \max\{\left|a_{i_1} - b_{i_1}\right|, \left|a_{i_2} - b_{i_2}\right|\} \leq \max_{i \in I_m}|a_i - b_i|$.

Lemma 2 is proved.

Proof of Theorem 1.

From the previous assumptions it follows that the function $R_s(u)$ is convex and continuous on U.

Let U^* – the set of minimum points of the function $R_s(u)$, and u^* an arbitrary element of U^*. Based on continuity $R_s(u)$ on U, it follows that for anything $\varepsilon > 0$ the number can be indicated $\Delta > 0$, such that:

$$R_s(u) - R_s(u^*) \geq 2\Delta \tag{10}$$

for all $u \in U \setminus V(U^*, \varepsilon/2)$.

Based on the convergence of each internal process [3, 4], it follows that:

$$\min_{u^*_{(i)} \in U^*_i} \left\| u^k_{(i)} - u^*_{(i)} \right\| \to 0 \text{ and } r(u^k_{(i)}, Q_i) \to r^*_i.$$

So, starting with some K^1, for all $k \geq K^1$ and all $i \in I_m$ one would obtain, for Δ indicated above, the inequality: $r\left(u^k_{(i)}, Q_i\right) - r^*_i < \Delta/2$.

For any $u \in U$ will be considered

$$\left|R^k_s(u) - R_s(u)\right| = \left|\max_{i \in I_m}\left[r(u, Q_i) - r\left(u^k_{(i)}; Q_i\right)\right] - \max_{i \in I_m}[r(u, Q_i) - r^*_i]\right|.$$

Noting $a_i = \left[r(u, Q_i) - r(u^k_{(i)}, Q_i)\right]$, $b_i = [r(u, Q_i) - r^*_i]$, for each u fixed in U, and calling on Lemma 2, one obtains:

$$\left|R^k_s(u) - R_s(u)\right| \leq \max_{i \in I_m}(|a_i - b_i|) = \max_{i \in I_m}\left(\left|r\left(u^k_{(i)}, Q_i\right) - r^*_i\right|\right) < \frac{\Delta}{2}. \tag{11}$$

From inequalities (10) and (11), for all $u \in U \setminus V(U^*, \varepsilon/2)$, it follows that

$$R^k_s(u) - R_s(u^*) \geq \frac{3}{2}\Delta, \text{ for all } k \geq K^1.$$

Since inequality (11) also holds for $u = u^*$, it follows that for $k \geq K^1$ and all $u \in U \setminus V(U^*, \varepsilon/2)$ the relationship will take place:

$$R^k_s(u) - R^k_s(u^*) \geq \Delta. \tag{12}$$

Obviously, for $\forall k = 0,1, \ldots$, fonction $R^k_s(u)$ is convex on U and, from its construction it follows that at any point $u \in U$ there is its subgradient [3], $g^k_s(u)$ and $\left\|g^k_s(u)\right\| \leq C_1$. The set U being compact suggests the existence of the constant C_2, for which $\max_{u,v \in U}\|u - v\| \leq C_2$.

So, for $k \geq K^1$, from (12) it follows that $-\Delta \geq R^k_s(u^*) - R^k_s(u) \geq (u^* - u, g^k_s(u))$ for all $u \in U \setminus V(U^*, \varepsilon/2)$, or, it follows that

$$(u - u^*, g^k_s(u)) \geq \Delta. \tag{13}$$

From (13) and the recently established, we get:

$$\frac{\left(u - u^*, g^k_s(u)\right)}{\|u - u^*\| \cdot \left\|g^k_s(u)\right\|} \geq \frac{\Delta}{\|u - u^*\| \cdot \left\|g^k_s(u)\right\|} \geq \frac{\Delta}{C_1 \cdot C_2} = \delta > 0. \tag{14}$$

So for everyone $k \geq K^1(K^1$, of course it depends ε) and all $u \in U \setminus V(U^*, \varepsilon/2)$ inequality occurs

$$\frac{(u - u^*, g_s^k(u))}{\|u - u^*\| \cdot \|g_s^k(u)\|} \geq \delta.$$

Noting $\eta^k = g_s^k(u^k)/\|g_s^k(u^k)\|$ and parsing the string u^k for $k \geq K^1$, the conditions of Lemma 1 are observed. Therefore, the number will be found $K_0 \geq K^1$, so that all $u^k \in V(U^*, \varepsilon)$, as soon as $k \geq K_0$. By virtue of the fact that the number ε > 0 can be taken however small, it follows that $\lim_{k\to\infty} \min_{u^* \in U^*} \|u^k - u^*\| = 0$ and, because the functions $R_s^k(u)$ are continuous on U, it follows that $\lim_{k\to\infty} R_s^k\left(u^k\right) = R_s(u^*) = R_s^*$.

Theorem 1 is proved.

Next, the problem addressed will be analyzed in a more general context than the one formulated in aspect (1). Namely, whether, in addition, it is necessary to comply with all the restrictions of the form:

$$F_j(u) \leq 0, j \in J_t = \{1,2, \dots, t\}.$$

All functions $F_j(u) \leq 0, j \in J_t$, are assumed convex and continuous on the convex compact U.

Considering $F(u) = \max_{j \in J_t} F_j(u)$, the researched model would appear as follows:

$$\begin{cases} R_s(u) \to min \\ F(u) \leq 0, \\ u \in U. \end{cases} \tag{15}$$

Two cases will be analyzed.

Case A. Assuming the existence of the solution to the problem (15), and thus to each of the *m* internal problems, the "lenient" variant will be investigated, in which the fulfillment of the restriction will be required

$$F(u) \leq \bar{\varepsilon},\ \ \bar{\varepsilon} > 0. \tag{16}$$

The quantity $\bar{\varepsilon}$ will be called threshold or tolerance level.

Therefore, in this case, instead of problem (15), the problem in the form (1), (16) will be solved, the solutions of which could be considered approximate solutions of the model (15).

Remark. Internal issues are resolved with the same "threshold of tolerance" although each of them might have its own threshold of tolerance.

The proposed numerical algorithm includes all actions (2)–(6) with the following specification for (3) and (4):

$$\eta_{(i)}^k = \begin{cases} r_u'(u_{(i)}^k, Q_i)/\|r_u'(u_{(i)}^k, Q_i)\|,\ if\ F\left(u_{(i)}^k\right) \leq \bar{\varepsilon} \text{ and } r_u'(u_{(i)}^k, Q_i) \neq 0, \\ g(u_{(i)}^k)/\|g(u_{(i)}^k)\|,\ if\ F(u_{(i)}^k) > \bar{\varepsilon} \text{ and } g\left(u_{(i)}^k\right) \neq 0, \\ 0, \text{ in the rest of the cases. Here } i \in I_m. \end{cases} \tag{17}$$

Each of the internal problems is solved with the necessary precision [4], to guarantee the fulfillment of inequality (12) for all $k \geq K^1$ and all $u \in U \setminus V(U^*, \varepsilon/2)$ for which $F(u) \leq 0$.

$$\eta^k = \begin{cases} g_s^k(u^k)/\|g_s^k\left(u^k\right)\|, if\ F(u^k) \leq \bar{\varepsilon} \text{ and } g_s^k(u^k) \neq 0, \\ g(u^k)/\|g(u^k)\|, if\ F\left(u^k\right) > \bar{\varepsilon} \text{ and } g\left(u^k\right) \neq 0, \\ 0, \text{ in the rest of the cases.} \end{cases} \tag{18}$$

It will be admitted that the subgradients $g_s^k(u)$ and $g(u)$ of the functions $R_s^k(u)$ and $F(u)$, correspondingly, are uniformly bounded on the set U of the same constant C_1.

In the conditions listed, it has their affirmation:

Theorem 2. For any element $u^0 \in U$ and any number $\varepsilon > 0$, there is such a number $\bar{\varepsilon} > 0$, so that, except for a finite number, all terms of the series $\{u^k\}$, generated by the algorithm (2), (3), (6) are contained in the set $V(U^*, \varepsilon)$, where U^* is the totality of the solutions to the problem (15).

Demonstration.

Be the number $\varepsilon > 0$ however small (but considered fixed) and the crowd $U \setminus V(U^*, \varepsilon/2) \neq \emptyset$. Function $F(u)$ being continuous on U is also uniformly continuous on this set. For this reason, there is obviously a positive number $\bar{\varepsilon}$, however small, for which in turn there is the number $\Delta > 0$, so that for all $k \geq K^1$ (see how (12) was obtained in the theorem 1):

$$R_s^k(u) - R_s^k(u^*) \geq \Delta, \text{ if } F(u) \leq 0 \text{ and, always,}$$

$$R_s^k(u) - R_s^k(u^*) \geq \frac{\Delta}{2}, \text{ if } F(u) \leq \bar{\varepsilon}.$$

Analogously, how the inequality (14) was obtained, is deduced:

$$\frac{(u - u^*, g_s^k(u))}{\|u - u^*\| \cdot \|g_s^k(u)\|} \geq \frac{\Delta}{2C_1 \cdot C_2} = \delta_1 > 0,$$

fair inequality for all $u \in U \setminus V(U^*, \varepsilon/2)$, $k \geq K^1$ and for which the relationship takes place $F(u) \leq \bar{\varepsilon}$.

Now so be it $F(u) > \bar{\varepsilon}$, so and $F(u) - F(u^*) > \bar{\varepsilon}$.

Or, $-\bar{\varepsilon} > F(u^*) - F(u) \geq (u^* - u, g(u))$. Or, $(u - u^*, g(u)) > \bar{\varepsilon}$.

If it is admitted that $\|g(u)\| \leq C_1$ for all $u \in U$, then it follows that

$$\frac{(u - u^*, g(u))}{\|u - u^*\| \cdot \|g(u)\|} \geq \frac{\bar{\varepsilon}}{C_1 \cdot C_2} = \delta_2.$$

Introducing the notation $\delta = \min\{\delta_1, \delta_2\}$, is obtained $\frac{(u-u^*,\eta^k)}{\|u-u^*\|} \geq \delta$, for all $u \in U \setminus V(U^*, \varepsilon/2)$ and all $k \geq K^1$. The vector η^k is calculated according to (18).

Theorem 2 is proved, since both Lemma 1 and Theorem 1 confirm the correctness of its statement.

Remark. The algorithm stops if, at some iteration k, the vector $\eta^k = 0$. In this situation two cases would be possible:

1. Or in point u^k inequality occurs $F(u^k) \leq \bar{\varepsilon}$ and $g_s^k(u^k) = 0$; thus u^k is one of the optimal solutions of problem (1), (16).

2. Or in point u^k inequality occurs $F(u^k) > \bar{\varepsilon}$ and $g(u^k) = 0$, which means that the point u^k is the minimum point for the function $F(u)$ on the domain U and therefore the problem (1), (16) has no admissible solutions.

Case B. Problem (15) will be solved, using the same ideas and procedures as in case A, only that at each iteration *k* the value of the number $\bar{\varepsilon}$, will be changed, in other words $\bar{\varepsilon}$ will be assigned the value $\bar{\varepsilon}_k$ which, together with the increase of *K*, will tend to zero, respecting, at the same time, some additional requirements. More specifically, the following theorem holds.

Theorem 3. Let $h_{(i)k}, h_k, \bar{\varepsilon}_k$ meet the requirements:

$$h_{(i)k}, h_k \geq 0;\ h_{(i)k}, h_k \to 0;\ \bar{\varepsilon}_k > 0; \bar{\varepsilon}_k \to 0;\ h_{(i)k}/\bar{\varepsilon}_k\,; h_k/\bar{\varepsilon}_k \to 0,$$

$$\sum_{k=0}^{\infty} h_{(i)k}\, \bar{\varepsilon}_k = \infty, \sum_{k=0}^{\infty} h_k \bar{\varepsilon}_k = \infty\,. \tag{19}$$

If problem (15) has a solution, then, under the conditions of compliance with (17)–(19), statements (7) hold.

Demonstration.

Initially, a clarification will be made regarding Lemma 1, for which the same statement holds, only under weaker conditions, namely:

for the points u^k, constructed according to (9), and belonging to the set $U \setminus V\left(U^*, \frac{\varepsilon}{2}\right)$ inequality occurs

$$\frac{(u^k - u^*, g(u^k))}{\|u^k - u^*\| \cdot \|g(u^k)\|} \geq \delta_k > 0;$$

for any $u^* \in U^*$ and any two numeric strings $\{h_k\}, \{\delta_k\}$ who owns the properties:

$$h_k \geq 0, h_k \to 0, \delta_k \to 0, h_k/\delta_k \to 0, \sum_{k=0}^{\infty} h_k \delta_k = \infty.$$

Now be the point u^0 given (arbitrary from *U*) and the number $\varepsilon > 0$ however small, but fixed. Because $\bar{\varepsilon_k} > 0$ and $\bar{\varepsilon_k} \to 0$, it follows that for some natural number K^1 and for all, $u^* \in U^*$ and all $u^k \in U \setminus V(U^*, \varepsilon/2)$ for which $k \geq K^1$, two cases are possible:

1. $F(u^k) \leq \bar{\varepsilon_k}$. In this situation there exists the number $\Delta > 0$ such that $R_S^k(u^k) - R_S^k(u^*) \geq \Delta$, (the reasoning being analogous to the one by which the inequality (12) was obtained), which implies the observance of the inequality

$$\frac{(u^k - u^*, g_S^k(u^k))}{\|u^k - u^*\| \cdot \|g_S^k(u^k)\|} \geq \delta = \frac{\Delta}{C_1 \cdot C_2}. \tag{20}$$

2. $F(u^k) > \bar{\varepsilon_k}$. Because $F(u^k) > \bar{\varepsilon_k}$, as in theorem 2, but taking into account that $\bar{\varepsilon} = \bar{\varepsilon_k}$, it is obtained that

$$\frac{(u^k - u^*, g(u^k))}{\|u^k - u^*\| \cdot \|g(u^k)\|} \geq \frac{\bar{\varepsilon_k}}{C_1 \cdot C_2} = \delta_k. \tag{21}$$

For the reason that $\bar{\varepsilon_k} \to 0$, when $k \to \infty$, it follows that a number can be indicated $K \geq K^1$, so that for all $k \geq K$, $\delta_k \leq \delta$. Which means that, for $F(u^k) \leq \bar{\varepsilon_k}$, but also for $F(u^k) > \bar{\varepsilon_k}$, inequality occurs $\frac{(u^k - u^*, \eta^k)}{\|u^k - u^*\|} \geq \delta_k$, as soon as $k \geq K$.

From the conditions of theorem 3, it is obtained:

$\delta_k > 0, \delta_k \to 0$; $h_k/\delta_k \to 0, \sum_{k=K}^{\infty} h_k \delta_k = \infty$ conditions, which together with those obtained above, (20), (21), ensure the convergence of the applied algorithm:

$$\lim_{k\to\infty} \min_{u^* \in U^*} \|u^k - u^*\| = 0, \lim R_S^k(u^k) = R_S^* = \min R_S^*(u).$$

Here min $R_s(u)$ – represents the minimum value of the function $R_s(u)$ in the model (15).

Theorem 3 is proved.

Conclusions. The described algorithms can be used in the decision-making process under conditions of uncertainty, when nature manifests itself through a finite number of possible states, and the controllable factors belong to compact and convex sets, at the same time the decision-making criterion is oriented towards minimizing maximum regret. The concordance between the steps $h_{(i)k}$, h_k and the tolerance threshold $\bar{\varepsilon}_k$ is essential from the point of view of ensuring the convergence of these algorithms both in theory and in application.

Authorship contribution statement.

Anatol Godonoaga – scientific guidance, mathematical formulation of the problem, discussion of the concept and methodology for solving the problem.

Stefan Blanutsa – discussion of the concept and methodology for solving the problem, analysis of the results obtained, selection and analysis of literary sources.

Boris Chumakov – development of a mathematical formulation of the problem, creation and design of algorithms, analysis of the results of computer experiments.

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Methods for Minimizing the Savage Function with Various Constraints

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Introduction. When making decisions under uncertainty, Savage's criterion is sometimes used, or the criterion of minimizing regrets [1]. Usually, in the literature, this decision situation is described in matrix language. In other words, both the number of decision alternatives and the number of states of nature are finite. Of particular interest are situations where the admissible domain of decision variants is a convex set, and the regrets with respect to each state of nature are expressed by convex functions. In this paper, we propose numerical methods for minimizing Savage's regret function, constructed based on the subgradient projection method with automatic step size adjustment [2, 3]. The convergence of these methods is demonstrated.

Goal. In the article, the Savage function is defined as a function that expresses the maximum regret value, assumed to be a convex function with respect to the decision factors. This function measures the effectiveness of each decision relative to the set of states of nature. It is important to note that computing the values of these functions is complex because of the need to know the optimal solution for each state of nature. This difficulty is successfully overcome in the process of solving the problem of minimizing functions on convex sets, thanks to parallel solutions of m "internal" algorithms based on the number of states of nature, and one external algorithm, aimed at minimizing the Savage function. Each of the m+1 algorithms represent modifications of the subgradient projection method with a programmable way of adjusting the step size. Depending on the complexity of the constraints and the required precision, three theorems have been proven, confirming the convergence of the investigated methods.

Results obtained. Constructive numerical algorithms have been developed for determining optimal decision alternatives under uncertainty, when the number of states of nature is finite, the admissible domain of control factors is convex and compact, and the Savage regret function serves as a decision criterion. The convergence of the corresponding algorithms to the set of optimal solutions has been proven, without knowing the exact values of the Savage function. Instead, estimates obtained from parallel runs of algorithms were used, aimed at determining optimal solutions for each state of nature.

Conclusions. Uncertainty poses significant difficulties in designing and making decisions. Any decision made under uncertainty represents a certain risk or a certain regret. In cases where the number of states of nature is finite, the decision domain is convex, the target function with respect to each state of nature is convex, and the Savage regret function is adopted as the decision criterion, the decision-making problem can be successfully solved using numerical algorithms based on the generalized gradient method. The implementation of the algorithm is relatively simple, and the fields of application can be very diverse.

Keywords: uncertainty, decisions, Savage function, optimization.

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Методи мінімізації функції Севіджа з різними обмеженнями

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Вступ. При прийнятті рішень за умов невизначеності іноді використовують критерій Севіджа або критерій мінімізації втрат [1]. Зазвичай у літературі ця ситуація прийняття рішення описується матричною мовою. У цьому випадку як кількість альтернатив рішень, так і кількість станів природи скінчені. Особливий інтерес представляють ситуації, коли і допустима область варіантів рішень є опуклою множиною і втрати щодо кожного стану природи виражаються опуклими функціями. У цій роботі ми пропонуємо чисельні методи мінімізації функції втрат Севіджа, які побудовані на основі методу проєкції субградієнту з автоматичним налаштуванням розміру кроку [2, 3]. Продемонстровано збіжність цих методів.

Ціль. У статті функція Севіджа визначається як функція, що виражає максимальне значення втрат та є опуклою функцією щодо факторів прийняття рішень. Ця функція – міра ефективності кожного рішення щодо множини станів природи. Важливо зазначити, що обчислення значень відповідних функцій досить складне, через необхідність знати оптимальне рішення для кожного стану природи. Ця складність успішно долається у процесі розв'язання задачі мінімізації функцій на опуклих множинах через паралельне застосування m «внутрішніх» алгоритмів за кількістю станів природи та одного зовнішнього алгоритму для мінімізації функції Севіджа. Кожен з m+1 алгоритмів – це модифікація методу проекції субградієнта з програмованим способом регулювання величини кроку. Було доведено три теореми, що підтверджують збіжність досліджуваних методів, залежно від складності обмежень та від необхідної точності результату.

Отримані результати. Були розроблені чисельні алгоритми знаходження оптимальних розв'язків задач за умов невизначеності, коли кількість станів природи скінчена, допустима область дії факторів управління опукла і компактна і як критерій використано функцію втрат Севіджа. Доведена збіжність відповідних алгоритмів без знання точних значень функції Севіджа до множини оптимальних розв'язків. Використані оцінки, отримані в результаті паралельного запуску алгоритмів, орієнтованих на визначення оптимальних розв'язку для кожного стану природи.

Висновки. Невизначеність створює значні труднощі в проектуванні і прийнятті рішень. Будь-яке рішення, прийняте за умов невизначеності, несе певний ризик чи певні втрати. У разі, коли число станів природи скінчене, область розв'язків опукла, цільова функція для кожного стану природи опукла і як критерій прийняття рішення використана функція втрат Севіджа, задача прийняття рішення може бути успішно розв'язана з використанням чисельних алгоритмів, побудованих на основі методу узагальненого градієнта. Реалізація алгоритмів відносно проста, а сфера застосування може бути дуже різноманітна.

Ключові слова: невизначеність, рішення, функція Севіджа, оптимізація.