MATHEMATICAL MODELING AND COMPUTING, Vol. 1, No. 1, pp. 84–96 (2014)

Models of mass transfer in gas transmission systems

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(Received 1 July 2014)

The models of gas movement in pipelines and gas filtration processes in complex porous media are considered in entire and fractional derivatives. The method for linearization of equations, which are included in the mathematical model of mass transfer, is suggested as well as an iterative scheme for solving initial systems of nonlinear differential equations is constructed. The finite-element model of the problem with the use of the Petrov-Galerkin method and Grunwald-Letnikov scheme concerning derivatives of the fractional order are implemented. The research of the models is carried out as well as comparative analysis of the numerical results is done.

Keywords: *mathematical model, unsteady gas movement, derivative of fractional order, linearization, numerical method*

2000 MSC: 35B, 65P

UDC: 519.6:539.3

1. Introduction

A gas transportation system as a unit of gas mains also includes gas storage facilities. Gas is transported through pipelines at high rates and under great pressures. Gas movement is influenced by the state of the internal walls (roughness), the change of flow direction (pipe bends) and the variation of pipeline cross-sections. At current rates of the gas transportation, vorticity flows detach from the pipe walls, get in to the inner area of the flow, and constantly change the velocity distribution in the cross-section that is turbulent processes take a place.

Gas storage is conducted in depleted deposits. Layers, which are collectors of deposits, are porous formations in sandstone and exist (it can be considered so) in fractal media. In fractal media, as opposed to a continuous one, a random wandering part moves away from the start base slower, as not all directions of movement are available for it. Hindering of filtration and diffusion processes in fractal media, as well as reducing the velocity of the gas in pipelines are so essential, that the physical quantities begin to change slower than the first derivative and accounting for this effect is possible only in the integral-differential equation containing derivative with respect to time of the fractional order.

The data mentioned above make us develop the theory and methods of mathematical and computer modeling of processes and systems in view of the mentioned effects.

In classical mathematical models, integro-differential equations and systems of ordinary and partial derivatives, integrals and derivatives have the order that is expressed in whole numbers [1–4]. Currently, the widespread use of fractional integrals and fractional derivatives is constrained by the lack of their clear physical interpretation, which, for example, the ordinary integral or ordinary derivative has [5–7].

One of the goals of this work is to show the degree of adequacy of models with fractional derivatives. A construction of a model of any physical process is associated with certain assumptions as for physical processes, so for the construction of mathematical tools. In particular, to simplify the mathematical model of the gas flow in pipelines, it is assumed that the change in gas density in time can be neglected. It is obvious that for slow process this is permissible. However, in the rapid change of the process, such assumption can lead to the loss of the adequacy over some space-time intervals. Further, in terms of computational mathematics, some problems arise due to the fact that the calculation involves large and small numbers that results in loss of significant value precision.

The aim of this paper is a construction and study of models of mass transfer in complex media, methods of linearization of equations, which are included in the mathematical model, and the construction of iterative schemes for solving the initial system of nonlinear differential equations, as well as testing the results obtained.

2. Definition of fractional derivatives

Derivatives of entire orders are local characteristics of functions that describe the physical process. If a course of the process is described in some space-time neighbourhood only, then with the help of derivatives of entire orders, the process under study can be quite adequately and accurately described. However, in nature, behaviour of many physical processes depends on their history. Then the use of derivatives of entire orders requires the construction of certain iterative procedures that in a sense would take into account the history of this process. This leads to the complication of the corresponding mathematical model and algorithms of necessary calculations. A solution to this problem is to build models using fractional calculus. Derivatives of fractional order are non-local characteristics of functions: they depend not only on the function behaviour in the neighbourhood of the point in question.

In the literature, it is known many ways to introduce the fractional calculus. In particular, the most commonly used is the fractional derivative operators in terms of Caputo and Riemann-Lowville. Operator of fractional derivative in terms of Caputo is determined as follows [5–7]:

$$ {}^{c}D_{\tau}^{\alpha} = \frac{{}^{c}\partial^{\alpha}}{\partial \tau^{\alpha}}\varphi(\tau) := \frac{1}{\Gamma(m+1-\alpha)} \int\limits_{0}^{\tau} \frac{\frac{\partial^{m+1}}{\partial \xi^{m+1}}\varphi(\xi)}{(\tau-\xi)^{\alpha-m}} d\xi \tag{1} $$

where $m = [\alpha]$, $[\cdot]$ is an integer part of a real number, and in terms of Riemann-Lowville

$$ D_{\tau}^{\alpha} = \frac{{}^{c}\partial^{\alpha}}{\partial \tau^{\alpha}}\varphi(\tau) := \frac{1}{\Gamma(m+1-\alpha)} \frac{\partial^{m+1}}{\partial \xi^{m+1}} \int\limits_{0}^{\tau} \frac{\varphi(\xi)}{(\tau-\xi)^{\alpha-m}} d\xi. \tag{2} $$

Between the Caputo's and Riemann-Lowville's derivatives the following relationship has a place [12]

$$ {}^{C}D_{\tau}^{\alpha}\varphi = D_{\tau}^{\alpha}\varphi - \sum_{k=0}^{m} \frac{\tau^{k-\alpha}}{\Gamma(k-\alpha+1)} \frac{\partial^{k}}{\partial \tau^{k}}\varphi. \tag{3} $$

As it is seen from the last correlations, operators of fractional derivatives depend on the values of functions right from the reference point.

3. Modelling gas flow process in pipelines

3.1. Case of integer differentiation

In these times, an enough common model of the gas movement process in pipelines in unsteady non-isothermal mode is an interconnected system of differential equations in partial derivatives [3,4,8]

$$ \frac{\partial(\rho v)}{\partial t} + \frac{\partial}{\partial x}\left(p + \rho v^{2}\right) = -\rho\left(\frac{\lambda v\,|v|}{2D} + g\frac{dh}{dx}\right), $$

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho \upsilon) = 0, \qquad (4)$$
$$\frac{\partial}{\partial t}(\rho E) + \frac{\partial}{\partial x}\rho\upsilon\left(E + \frac{p}{\rho}\right) = \frac{4k(T_{gr} - T)}{D} - \rho\upsilon g\frac{dh}{dx}.$$

In the equation (4) we denote: ρ, υ, p are the density, the velocity and the pressure of gas respectively; λ is the coefficient of hydraulic resistance; k is the coefficient of heat transfer from the pipe to the ground; T_{gr} is the temperature of the ground; T is the gas temperature; g is the acceleration of gravity; D is the diameter of the pipe; h is the difference of pipeline height marks; t is time; x is current coordinate, $x \in [0, l]$; l is the length of the pipeline; E is the total energy of the unit of mass;

$$E = i - \frac{p}{\rho} + \frac{\upsilon^2}{2}; \qquad (5)$$

$$di = \frac{\partial i}{\partial T}dT + \frac{\partial i}{\partial p}dp = C_p dT + \left[\frac{1}{\rho} - T\left(\frac{\partial(1/\rho)}{\partial T}\right)_p\right]dp; \qquad (6)$$
$$C_p = \left(\frac{\partial i}{\partial T}\right)\rho.$$

In formulae (5) and (6), i is the change in internal energy; C_p is the specific heat at constant pressure. For closing of the system of equations, the gas law is used [3]

$$p = \rho\chi RT.$$

To calculate the compressibility factor χ, which describes the difference between natural gas and ideal one, a considerable amount of empirical dependences are constructed, in particular [10]

$$\chi = \frac{1}{1 + fp},$$

where p is measured in atmospheres, $f = (24 - 0.21t°\mathrm{C}) \cdot 10^{-4}$, $t°\mathrm{C}$ is the gas temperature Celsius scale; R is the gas constant.

In the isothermal case, a common mathematical model of gas flow in the pipeline is the following

$$\begin{cases} \dfrac{\partial p}{\partial x} + \dfrac{\lambda\rho\upsilon^2}{2D} + \dfrac{\partial(\rho\upsilon)}{\partial t} = 0; \\ \dfrac{\partial(\rho\upsilon)}{\partial x} + \dfrac{1}{c^2}\dfrac{\partial p}{\partial t} = 0. \end{cases}$$

The first equation of the last formula is obtained under assumption that the change in gas density in time can be neglected. Otherwise, the system has the form

$$\begin{cases} \dfrac{\partial p}{\partial x} + \dfrac{\lambda\rho\upsilon^2}{2D} + \rho\dfrac{\partial \upsilon}{\partial t} = 0; \\ \dfrac{\partial(\rho\upsilon)}{\partial x} + \dfrac{1}{c^2}\dfrac{\partial p}{\partial t} = 0, \end{cases} \qquad (7)$$

where c is the speed of sound in the gas. In practice, values of the pressure p is of the 10^6 order, and the value of speed is about of ten orders. To ensure the stability of numerical methods, in the algorithm for solving the system (7) is beneficial to make replacements that should level the order of numbers. Using a real gas law, and denoting $f = \ln(p)$ and $\gamma = \chi RT$, the system (6) can be written as follows

$$\begin{cases} \gamma \dfrac{\partial f}{\partial x} + \dfrac{\lambda \upsilon^2}{2D} + \dfrac{\partial \upsilon}{\partial t} = 0, \\ \upsilon \dfrac{\partial f}{\partial x} + \dfrac{\partial \upsilon}{\partial x} + \dfrac{\gamma}{c^2} \dfrac{\partial f}{\partial t} = 0. \end{cases} \tag{8}$$

3.2. Modelling the process of gas movement in the pipeline in fractional derivatives

Consider the problem formulated above with the neglect of the Coriolis force at the constant value of the compressibility factor χ using Caputo's fractional derivative of the order α with respect to time. Under these assumptions, the system (2) is written

$$\begin{cases} D_{0+}^{\alpha} \omega(x,t) + \dfrac{\partial p}{\partial x} + a\omega - bp = \Theta(x,t), \\ \dfrac{\partial \omega}{\partial x} + \dfrac{1}{c^2} D_{0+}^{\alpha} p(x,t) = \Psi(x,t). \end{cases}$$

The process of gas movement in a horizontal pipeline of the length l is considered under given boundary conditions on the function of the pressure

$$p(0,t) = p_{ok}(t), \quad p(l,t) = p_{kk}(t),$$

or gas consumptions

$$\omega(o,t) = \omega_{ok}(t), \quad \omega(l,t) = \omega_{kk}(t).$$

As the initial condition there is taken the known stationary pressure distribution

$$p(x,0) = p_{om}(x)$$

or gas consumptions

$$\omega(x,0) = \omega_{om}(x).$$

which are derived from the original system. Otherwise, we may get inconsistent boundary conditions.

If the boundary-initial conditions are constant, i.e. $p_{ok}(t) = p_{ok} \equiv const$, $p_{kk}(t) = p_{kk} \equiv const$ and $\omega_{ok}(t) = \omega_{ok} \equiv const$, $\omega_{kk}(t) = \omega_{kk} \equiv const$, then instead of the functions p and ω, it is expedient to introduce into consideration the following functions

$$p \leftrightarrow p + \frac{x}{l} h_p - p_{0k}, \quad \omega \leftrightarrow \omega + \frac{x}{l} h_\omega - \omega_{0k},$$

where

$$h_p = p_{0k} - p_{kk}, \quad h_\omega = \omega_{0k} - \omega_{kk}.$$

In this case, the boundary conditions are zero. Under these conditions, it is advisable to carry out the separation of variables based on Fourier series with respect to sine, i.e.

$$\begin{Bmatrix} p(x,t) \\ \omega(x,t) \end{Bmatrix} = \sum_{n=1}^{\infty} \begin{Bmatrix} p_n(t) \\ \omega_n(t) \end{Bmatrix} \sin \frac{n\pi x}{l}. \tag{9}$$

Under such boundary conditions, the coefficients of the series (5), which describes the pressure distribution, are calculated according to the formula

$$\tilde{p}_n(t) = -\left(\frac{c}{l}\right)^2 \frac{l}{\nu_n} \left[d_1 \varsigma_{1n}(t) + d_2 \varsigma_{2n}(t) + d_3 \varsigma_{3n}(t) \right] - \left(\frac{c}{l}\right)^2 \frac{l}{\nu_n} \varsigma_{4n}(t) + c^2 \varsigma_{5n}(t).$$

Here the following denotations are introduced

$$\nu_n = \frac{1}{n\pi i}, \quad s_1 = \frac{1}{2}\left(-a - \sqrt{a^2 - 4\kappa_n}\right), \quad s_2 = \frac{1}{2}\left(-a + \sqrt{a^2 - 4\kappa_n}\right),$$

$$d_1 = \hat{\hat{\nu}}_n\left(ah_\omega - bh_p\right) + \hat{\nu}_n\left(\frac{1}{l}h_p + bp_{ok} - a\omega_{ok}\right) - a\nu_n\,\hat{\nu}_n\,h_\omega,$$

$$d_2 = \hat{\hat{\nu}}_n h_\omega + \hat{\nu}_n\left(\omega_{om} - \omega_{ok}\right) - \nu_n l\left(\frac{1}{l}p\hat{\nu}_n\,h_\omega + \frac{a}{c^2}\left(\hat{\hat{\nu}}_n\,h_p + \hat{\nu}_n\left(p_{om} - p_{ok}\right)\right)\right),$$

$$d_3 = -\nu_n l\frac{1}{c^2}\left(\hat{\hat{\nu}}_n h_p + \hat{\nu}_n\left(p_{om} - p_{ok}\right)\right), \quad \kappa_n = \frac{1 - bl\nu_n}{(\nu_n l/c)^2},$$

$$\zeta_{1n} = \frac{1}{s_1 s_2} - \frac{e^{s_1\tau}}{s_1(s_2 - s_1)} + \frac{e^{s_2\tau}}{s_2(s_2 - s_1)}, \quad \zeta_{2n} = \frac{e^{s_2\,\tau} - e^{s_1\,\tau}}{(s_2 - s_1)}, \quad \zeta_{3n} = \frac{s_2\,e^{s_2\,\tau} - s_1\,e^{s_1\,\tau}}{(s_2 - s_1)},$$

$$\xi_{4n}(t) = \frac{1}{l}\sum_{i=1}^{I} p_{st,i}e^{-\frac{x_i}{\nu_n l}}\left[\left\{\begin{array}{ll} 0, & t < t_{1i} \\ H_{1n}(t - t_{1i}), & t > t_{1i}\end{array}\right. - \left\{\begin{array}{ll} 0, & t < t_{2i} \\ H_{1n}(t - t_{2i}), & t > t_{2i}\end{array}\right.\right],$$

$$\xi_{5n}(t) = \frac{1}{l}\sum_{j=1}^{J}\frac{q_j}{F}e^{-\frac{x_j}{\nu_n l}}\left[\left\{\begin{array}{ll} 0, & t < t_{1j} \\ H_{2n}(t - t_{1j}) + aH_{1n}(t - t_{1j}), & t > t_{1j}\end{array}\right. -\right.$$

$$\left. - \left\{\begin{array}{ll} 0, & t < t_{2j} \\ H_{2n}(t - t_{2j}) + aH_{1n}(t - t_{2j}), & t > t_{2j}\end{array}\right.\right].$$

$$\mathrm{H}_{1n}(t) = \int_0^\infty\left[\frac{1}{s_1 s_2} - \frac{e^{s_1\,\tau}}{s_1(s_2 - s_1)} + \frac{e^{s_2\tau}}{s_2(s_2 - s_1)}\right]\left[1 - \frac{1}{\pi\alpha}\int_0^\infty \frac{1}{\upsilon}e^{-t\upsilon^{1/\alpha}}\sin(\tau\upsilon)\,d\upsilon\right]d\tau,$$

$$\mathrm{H}_{2n}(t) = \int_0^\infty\left[\frac{e^{s_2\,\tau} - e^{s_1\,\tau}}{(s_2 - s_1)}\right]\left[1 - \frac{1}{\pi\alpha}\int_0^\infty \frac{1}{\upsilon}e^{-t\upsilon^{1/\alpha}}\sin(\tau\upsilon)\,d\upsilon\right]d\tau,$$

$$\mathrm{H}_{3n}(t) = \int_0^\infty\left[\frac{s_2\,e^{s_2\,\tau} - s_1\,e^{s_1\,\tau}}{(s_2 - s_1)}\right]\left[1 - \frac{1}{\pi\alpha}\int_0^\infty \frac{1}{\upsilon}e^{-t\upsilon^{1/\alpha}}\sin(\tau\upsilon)\,d\upsilon\right]d\tau.$$

Since the originals of coefficients of series of the pressure function decomposition in the series (9) are found, then on the basis of the additive property of the Laplace-Carson transform, it can be considered that the found solution is the solution of the formulated problem with respect to the pressure.

4. Modelling the gas filtration process in complex porous media

4.1. The use of partial derivatives of entire orders

To describe the filtration of gas and liquid in complex porous medium, there are used equations in partial derivatives [3]

$$\frac{\partial}{\partial x}\left(\frac{kh}{\mu z}\frac{\partial p^j}{\partial x}\right) + \frac{\partial}{\partial y_1}\left(\frac{kh}{\mu z}\frac{\partial p^j}{\partial y_1}\right) + \frac{\partial}{\partial y_2}\left(\frac{kh}{\mu z}\frac{\partial p^j}{\partial y_2}\right) = 2mh\left(\frac{\partial}{\partial t}\left(\frac{p}{z}\right) + 2qp_{at}\right). \tag{10}$$

In the last equation, $j = 2$ for gas and $j = 1$ for incompressible liquid; $k = k\left(x, y_1, y_2, t\right)$, $m = m\left(x, y_1, y_2\right)$ and $h = h\left(x, y_1, y_2\right)$ are coefficients of permeability, porosity, and the thickness of the medium, respectively; μ is the dynamic viscosity of substance; p_{at} is the atmospheric pressure; q is the density of withdrawing.

Gas is withdrawn out of porous media through I wells, which are located in the points (x_i^0, y_i^0) over some period of time $t \in [t_{1i}, t_{2i}]$, $(i = \overline{1, I})$. Then the density of the withdrawing is determined according to the formula

$$q = \frac{1}{V}\sum_{i=1}^{I} q_i\left(x, y, t\right)\delta\left(x - x_i^0\right)\left(y - y_i^0\right)\left[\eta\left(t - t_{1i}\right) - \left(t - t_{2i}\right)\right].$$

Here q_i is the gas withdrawing out of the $i-$th gas well in the instant of time t; $\delta(x)$ is Dirac delta function; $\eta\left(t - t_{ji}\right)$ is Heaviside unit function.

4.2. The use of derivatives of fractional order

From the practice it is well known that the process of mass transfer in porous media at the absence of sources strongly depends on the history of the process. In particular, this takes place during the extraction of fluids. A special place takes the process of storing gas in underground storages. To describe such processes it is expedient to apply the fractional calculus. In particular, the process of gas and liquid filtration is described by means of equation with a fractional derivative with respect to the time variable as follows:

$$\frac{\partial}{\partial x}\left(\frac{kh}{\mu\chi}\frac{\partial p^j}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{kh}{\mu\chi}\frac{\partial p^j}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{kh}{\mu\chi}\frac{\partial p^j}{\partial z}\right) = 2mh\left(\frac{\partial^\alpha}{\partial t^\alpha}\left(\frac{p}{\chi}\right) + 2qp_{at}\right). \qquad (11)$$

Here α is the order of the fractional derivative.

4.3. The main problem

The main problem is to find the solution $p(x, y_1, y_2, t)$ of equation (11) using the known values of the pressure $p(x_i, y_{1\,i}, y_{2\,i}, t_0)$ at the given points of the medium. Herewith it is necessary that the condition of substance mass balance in the storage should be executed

$$M = \int_V \rho dv,$$

where the integration is performed with respect to the volume of the storage V; M is the mass of gas in the storage; ρ is the gas density related to the pressure of the equation of the gas state.

5. Methods for solving formulated problems

Methods for solving problems of mathematical physics can be divided into several major classes: analytical, numerical, asymptotic, iterative.

5.1. Analytical methods are based mainly on the use of integral transforms with respect to some chosen variable. Integral transformations allow us to reduce the order of differential equation, which significantly facilitates the finding its solution.

5.2. Numerical methods are based on the replacement of differential or integral operators for the corresponding discrete analogue. In such cases, the problem is reduced to solving algebraic equations.

5.3. Asymptotic methods are used in a case where it is necessary to have a sufficiently exact solution of the problem in the space-time neighborhood of a point, and finding the complete solution either impossible or associated with considerable mathematical difficulties. Asymptotic methods of problem solving to some extent can be referred to analytical methods.

5.4. In the basis of iterative methods there lie assignments by appropriate means the initial approximation of the desired solution and the construction of algorithm of its specification. It should be noted that the accuracy and the convergence of the iterative procedure are significantly influenced by an initial approximation. Therefore, as an initial solution is desirable to choose the solution of a corresponding simplified problem.

6. Numerical experiments

6.1. Testing of the application of fractional calculus is performed for the following model problem.

Find the solution of the differential equation

$$\frac{\partial^2 f}{\partial x^2} - \frac{1}{a^2} D_t^{\alpha} f(t) = 0$$

under the zero initial condition and $f(0,t) = \sqrt{\pi}/a\sqrt{t}$, $\lim\limits_{x\to\infty} f(x,t) = 0$. Here D_t^{α} is fractional derivative in terms of Riemann-Lowville [1, 2]. In the case $\alpha = 1$ Riemann-Lowville fractional derivative transforms into the normal derivative with respect to time and the equation (1) takes the form

$$\frac{\partial^2 f}{\partial x^2} - \frac{1}{a^2}\frac{\partial f}{\partial t} = 0.$$

Under imposed boundary conditions, the solution of the last equation is a function

$$f(x,t) = \frac{\sqrt{\pi}}{a\sqrt{t}} \exp\left(-\frac{x^2}{4a^2 t}\right).$$

An analytical solution of the problem in terms of Riemann-Lowville fractional derivatives has the form

$$f(t) = \frac{1}{a}\int\limits_0^{\infty} \frac{1}{\sqrt{\rho}} e^{-t\rho} e^{-\frac{x}{a}\rho^{\frac{\alpha}{2}}\cos\left(\frac{\alpha\pi}{2}\right)} \cos\left[\frac{x}{a}\rho^{\frac{\alpha}{2}} \sin\left(\frac{\alpha\pi}{2}\right)\right] d\rho.$$

The function $f(x,t)$ values calculated for different values of the fractional derivative are given in Table 1.

Table 1. Values of the solution of formulated problem for different values of the order of fractional derivative with respect to dimensionless coordinates and time.

t	$t=0.001$				$t=0.01$				$t=0.1$			
x/α	1.00	0.80	0.90	0.99	1.00	0.80	0.90	0.99	1.00	0.80	0.90	0.99
0.001	1230.08	2692.70	754.77	1019.02	1057.37	1995.54	943.19	954.30	369.54	475.12	434.64	377.35
0.002	43.88	75.55	238.33	-161.79	757.64	778.89	850.72	676.83	357.42	503.27	430.20	363.75
0.003	0.17	111.54	-118.4	-159.65	434.70	568.96	459.45	375.76	338.11	455.35	384.77	341.04
0.004	0.00	-63.76	-98.40	-110.12	199.71	298.88	226.18	158.08	312.81	395.37	339.94	314.01
0.005	0.00	-72.54	-46.30	-65.34	73.47	147.66	93.29	46.30	283.04	340.41	299.42	283.54

Analysis of the results presented in Table 1 shows that the order of the fractional derivative has a significant impact on the solution of the original problem. Hence a need for additional research and the use of prior information to determine the order of the fractional derivative in physical processes simulation appears.

6.2. Numerical experiment to determine the pressure distribution along the pipe according to the system (8) is carried out in the pipeline of the length of 100 km, of the Ø1.388 m for the following values of parameters: $\lambda = 0.009$, $\rho_0 = 0.682\,\mathrm{kg/m^3}$, $T = 313\,\mathrm{K}$, $R = 506.7\,\mathrm{J/(kg\cdot K)}$, $z = 0.87$. Boundary conditions are imposed on the gas pressure, which varies in time from 60 to 70 bar on the left edge of the pipeline and from 44.37 to 50 bar on the right side. The step with respect to time

$dt = 10\,\mathrm{s}$, a number of discrete with respect to coordinate $k_x = 8$. Numerical results are presented in tabular form.

Table 2. Values of the pressure for different values of time and coordinates.

	0	12.5	25	37.5	50	62.5	75	87.5	100
0	60.000	57.929	55.893	53.891	51.923	49.987	48.084	46.213	44.374
100	60.496	58.391	56.273	54.286	52.289	50.336	48.375	46.476	44.573
200	60.996	58.774	56.541	54.483	52.417	50.463	48.503	46.634	44.761
300	61.496	59.193	56.879	54.764	52.644	50.667	48.686	46.818	44.948
400	61.996	59.630	57.253	55.090	52.923	50.915	48.907	47.020	45.136
500	62.496	60.077	57.646	55.439	53.229	51.186	49.146	47.232	45.324
600	62.996	60.529	58.051	55.801	53.549	51.470	49.395	47.449	45.511
700	63.496	60.983	58.461	56.169	53.877	51.760	49.650	47.668	45.699
800	63.996	61.439	58.874	56.540	54.209	52.053	49.907	47.889	45.886
900	64.496	61.895	59.289	56.913	54.542	52.348	50.165	48.111	46.074
1000	64.996	62.351	59.704	57.287	54.877	52.644	50.424	48.333	46.261
2000	69.996	66.916	63.858	61.027	58.226	55.602	53.012	50.555	48.137
3000	69.996	67.439	64.879	62.331	59.788	57.292	54.807	52.401	50.013
4000	69.996	67.516	65.035	62.529	60.027	57.512	55.008	52.500	50.013
5000	69.996	67.516	65.035	62.529	60.028	57.512	55.008	52.501	50.013

Table 3. Values of the velocity for different values of time and coordinates.

	0	12.5	25	37.5	50	62.5	75	87.5	100
0	11.300	11.798	12.317	12.860	13.426	14.017	14.635	15.279	15.952
100	12.808	12.515	12.784	13.063	13.504	13.970	14.408	14.867	15.027
200	13.443	13.166	13.264	13.360	13.648	13.954	14.265	14.588	14.796
300	13.901	13.648	13.657	13.654	13.824	14.002	14.228	14.460	14.666
400	14.234	14.001	13.966	13.912	14.008	14.103	14.276	14.450	14.658
500	14.486	14.270	14.214	14.135	14.190	14.237	14.380	14.520	14.734
600	14.687	14.486	14.423	14.332	14.368	14.390	14.519	14.641	14.863
700	14.857	14.669	14.606	14.514	14.542	14.553	14.678	14.794	15.025
800	15.006	14.832	14.773	14.684	14.713	14.721	14.849	14.965	15.206
900	15.143	14.981	14.930	14.847	14.880	14.892	15.025	15.147	15.396
1000	15.272	15.122	15.079	15.004	15.045	15.063	15.204	15.333	15.592
2000	16.402	16.362	16.417	16.448	16.584	16.704	16.940	17.170	17.526
3000	13.048	13.621	14.130	14.659	15.120	15.595	15.998	16.409	16.743
4000	12.581	13.137	13.713	14.316	14.944	15.601	16.285	17.002	17.749
5000	12.577	13.133	13.709	14.314	14.943	15.602	16.288	17.006	17.754
6000	12.577	13.133	13.709	14.314	14.943	15.602	16.288	17.006	17.754

The conducted numerical experiments show that for stable calculation of the gas-dynamic parameters (gas store, pressure distribution etc.) the accuracy of setting the boundary conditions must be harmonized with the precision required in solving the particular practical problems.

The deviation of values of the obtained results when using different input models of the process takes place over certain finite space-time intervals. The resulting difference can have a significant impact on the calculation of some gas-dynamic parameters, including the change in store of natural gas in the pipeline.

On a choice of the numerical method to determine the pressure distribution and mass velocity in the pipeline, of special importance are the computational process parameters such as the resistance to fatal errors, errors of sampling, time of calculation, the variation of the boundary conditions.

6.3. Numerical model of gas filtration in a complex porous medium (Fig. 1) is based on the finite element method combined with the iterative procedure, which acts over some time subinterval [2,8,9,10]. The 2-D region is divided into elementary triangular elements.

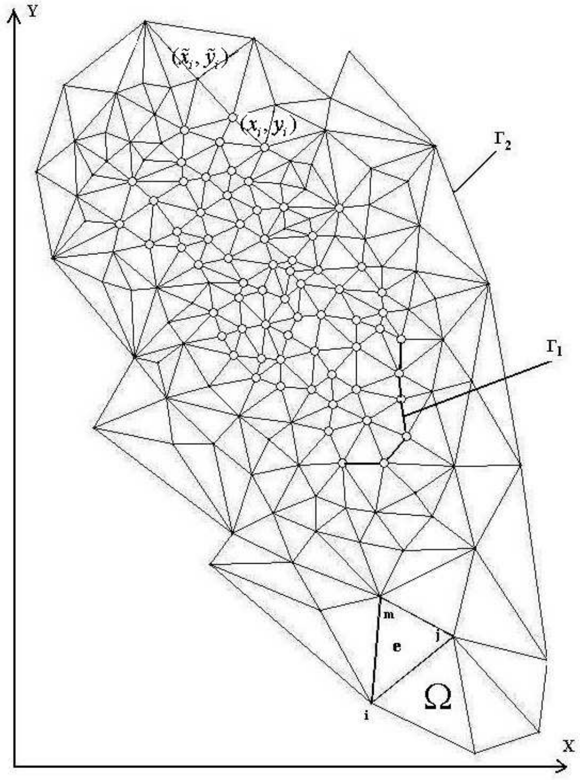

Fig. 1.

Partition of the field is made so that the coordinates of the known values of the pressure (x_i, y_i) coincide with the coordinates of the vertices of triangles, and $(\tilde{x}_i, \tilde{y}_i)$ are nodes of the apices of the triangles, the values of which must be found.

In general, the differential equation of second order partial derivatives with respect to the unknown function p is written in the form

$$-\sum_{i,j=1}^{2} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial p}{\partial x_j}\right) + gp = f, \ x \in \Omega_2 \subset R^2. \tag{12}$$

Here p is the desired function; a_{ij}, g, f are continuous functions in their domains.

Based on the variational approach, from Equation (12) we obtain a system of linear algebraic equations with respect to unknown values of the function

$$\sum_j u_j(A\varphi_i, \varphi_j) = (f, \varphi_j), \quad i, j = 1..n.$$

We can go further in two ways:

1) since the original equation is nonlinear, in the result of the variational approach we obtain a nonlinear algebraic system of equations, the solution of which should be found by means of the approximate or numerical methods;

2) the original equation is linearized and a linear system of algebraic equations is obtained.

In practice, it is more expedient to use the second approach, coupled with the procedure of iteration of the coefficients and the desired solution.

Linearization of the left-hand side of Equation (5) is carried out as follows:

$$\frac{\partial}{\partial x}\left(\frac{kh}{\mu\chi}\ \frac{\partial p^2}{\partial\zeta}\right)\approx 2\frac{kh}{\mu\chi}\tilde{p}\frac{\partial}{\partial\zeta}\left(\frac{\partial p}{\partial\zeta}\right)+2\frac{\partial}{\partial\zeta}\left(\frac{kh}{\mu\chi}\cdot\tilde{p}\right)\cdot\frac{\partial}{\partial\zeta}\left(\tilde{p}\right), \tag{13}$$

here $\zeta\in\{x,y\}$ is the spatial coordinate. In the right-hand side of the equation, we take the parameter χ outside the derivative as a constant

$$\frac{\partial^\alpha}{\partial t^\alpha}\left(\frac{p}{\chi}\right)\approx\frac{1}{\chi}\frac{\partial^\alpha p}{\partial t^\alpha},$$

and expand the fractional derivative $\frac{\partial^\alpha p}{\partial t^\alpha}$ according to the Grunwald-Letnikov scheme [11]:

$${}^{GL}D^\alpha_\tau p:=\lim_{\Delta t\to 0}\left(\Delta t\right)^{-\alpha}\sum_{j=0}^{[\tau/\Delta t]}(-1)^j\left(\begin{array}{c}\alpha\\ j\end{array}\right)p\left(\tau-j\Delta t\right). \tag{14}$$

Grunwald-Letnikov operator (11) is approximated over the interval $[0,\ \tau]$ with the subinterval step Δt as follows

$${}^{GL}D^\alpha_\tau p\left(\tau\right)\approx\sum_{j=0}^{[\tau/\Delta t]}c_j^{(\alpha)}p\left(\tau-j\Delta t\right) \tag{15}$$

where $c_j^{(\alpha)}$ is Grunwald-Letnikov coefficients, defined as

$$A_j^{(\alpha)}=(\Delta t)^{-\alpha}(-1)^j\left(\begin{array}{c}\alpha\\ j\end{array}\right). \tag{16}$$

Using the recurrence relation [6]

$$c_j^{(\alpha)}=(\Delta t)^{-\alpha},\quad c_j^{(\alpha)}=\left(1-\frac{1+\alpha}{j}\right)c_{j-1}^{(\alpha)} \tag{17}$$

we can calculate the coefficients $c_j^{(\alpha)}$. For j = 1 we have $c_1^{(\alpha)}=-\alpha(\Delta t)^{-\alpha}$.

Using the linearization (13) and the discretization scheme for the fractional derivative (14)–(17) we transform equation (11) to the form:

$$\tilde{p}\frac{kh}{\mu\chi}\frac{\partial}{\partial x}\left(\frac{\partial p}{\partial x}\right)+\tilde{p}\frac{kh}{\mu\chi}\frac{\partial}{\partial y}\left(\frac{\partial p}{\partial y}\right)=mh\frac{1}{\chi}\frac{\partial^\alpha}{\partial t^\alpha}\left(p\right)+2mhqp_{st}+F\left(\tilde{p},k,h,\mu,\chi\right), \tag{18}$$

where

$$\frac{\partial^\alpha}{\partial t^\alpha}p(t)=\sum_{j=0}^{f}c_j^{(\alpha)}p(t_{f-j})-\sum_{k=0}^{m}\frac{(t_f)^{k-\alpha}}{(k-\alpha+1)}p(t_k),$$

$$F\left(\tilde{p},k,h,\mu,\chi\right)=-\frac{\partial}{\partial x}\left(\frac{kh}{\mu\chi}\cdot\tilde{p}\right)\cdot\frac{\partial\tilde{p}}{\partial x}-\frac{\partial}{\partial y}\left(\frac{kh}{\mu\chi}\cdot\tilde{p}\right)\cdot\frac{\partial\tilde{p}}{\partial y}.$$

To bring the linear equation (18) to the form (12), we use the sampling scheme with respect to time under assumption that $\tilde{p}$ is the iterative approximated value of the solution p at the previous iteration step. Then the parameters in Equation (12) will be as follows

$$d=\frac{c_0}{\tilde{p}}\frac{mh}{\chi},$$

$$f = \left(-2mhp_{st}q + F(\tilde{p}, k, h, \mu, \chi) + \frac{mh}{\chi}\Big(\sum_{j=1}^{i} c_j p(t_j - j\Delta t) - \frac{t^{-\alpha}}{\Gamma(1-\alpha)} p(t_0) \Big) \right) \Big/ \tilde{p}.$$

FEM scheme is iteratively applied to the linear equation (18) for each instant of time $t = i\Delta t$, $i = 1, \ldots, I$, clarifying the compressibility factors, the coefficients of permeability $k = \mu l \Delta Q / S \Delta p$, and the approximate solution $\tilde{p} = p$.

The scheme for solving the differential equation is tested in the numerical experiment on the basis of the data on porous medium of the area $S = 16\,\text{mln.m}^2$, for the following input parameters: $\mu = 0.000011\,\text{Pa·s}$, $h = 18.2\,\text{m}$, $R = 506.7\,\text{J/(kg} \cdot \text{K)}$, $T = 293\,\text{K}$, $z = 0.87$, $m = 0.31$, $k = 1.8e - 12\,\text{m}^2$.

The input information is provided by the pressure values of control, metering and operating wells in a neutral period and by withdrawn gas volumes during gas withdrawing out of the gas storage.

Figures 2–5 show the calculated values of the gas pressure in the zone of the well (Fig. 2), in the adjacent zone of the bed (Fig. 3), in the middle of the bed (Fig. 4) and the middle-layer pressures (Fig. 5) for different values of the order of the fractional derivative α.

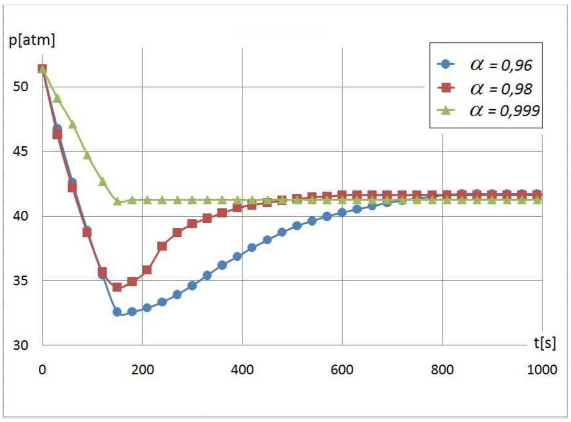

Fig. 2. Values of the gas pressure in the zone of the well for different values of the order of the fractional derivative α.

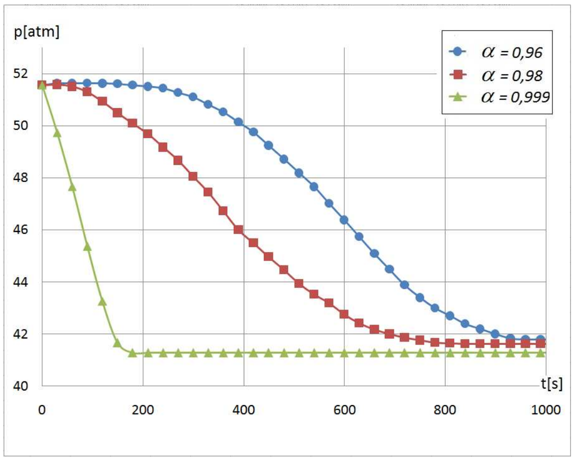

Fig. 3. Values of the gas pressure in the adjacent zone of the bed for different values of the order of the fractional derivative α.

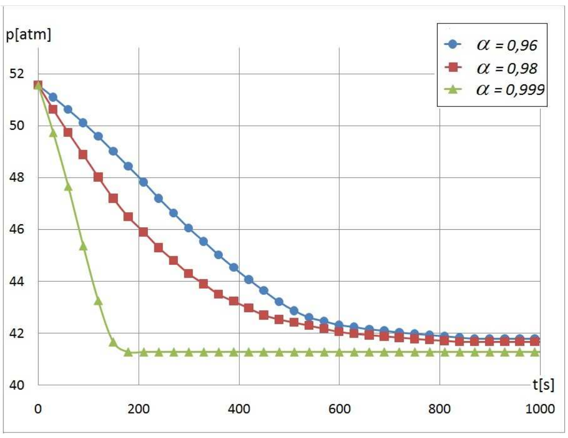

Fig. 4. Values of the gas pressure in the middle of the bed for different values of the order of the fractional derivative α.

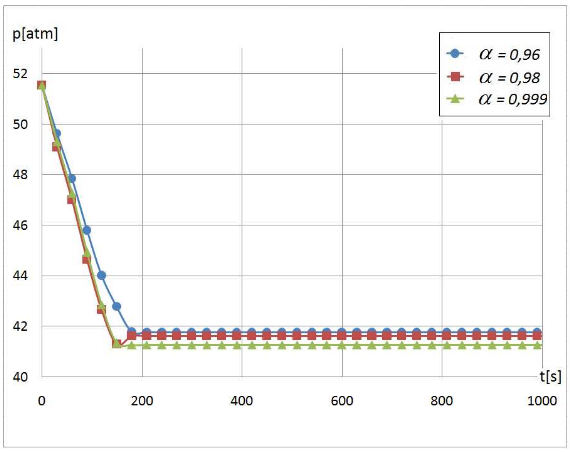

Fig. 5. Values of the middle-layer preassures for different values of the order of the fractional derivative α.

The numerical model of the gas filtration process in complex porous media uses the finite element method combined with the iterative procedure, which acts over each time subinterval and the Grunwald-Letnikov scheme.

7. Conclusions

Analysis of the results of numerical experiment shows that the value of the order of the time derivative α does not affect the dynamics of the middle-layer pressures.

In the well zone and the adjacent zones of the bed, behaviours of the pressure depend on the parameter of the derivative α. The less the parameter α is, the greater the difference between the values of the pressure in the wells and the adjacent zones of the bed is. The experimental results confirm the behavior of the gas pressure in the porous medium in the presence of atypical filtration. If the parameter $\alpha = 1$, then the values of calculated middle-layer pressures coincide with experimental data.

The analysis of Figures 2–4 confirms the physical picture of the pressure distribution in porous medium, namely the depression of the pressure decreases with increasing distance from the outlet.

Conducting numerical experiments on the basis of a real facility and real metering data has shown the effectiveness of suggested in this article approaches of numerical modelling of mass transfer in porous media of complex structure.

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Моделі масопереносу в газотранспортних системах

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Розглянуто моделі руху газу в трубопроводах та фільтрації газу в складних пористих середовищах у цілих та дробових похідних. Запропоновано методику лінеаризації рівнянь, які входять в математичну модель масопереносу та побудовано ітераційну схему розв'язування вихідних систем нелінійних диференціальних рівнянь. Реалізовано скінченно-елементну модель задачі із використанням методу Петрова-Гальоркіна та схему Грюнвальда-Летнікова стосовно похідних дробового порядку. Проведено дослідження моделей та порівняльний аналіз отриманих числових результатів.

Ключові слова: *математична модель, нестаціонарний рух газу, похідні дробових порядків, лінеаризація, числові методи*

2000 MSC: 35B, 65P

УДК: 519.6:539.3