

## Applying the concept of generating polynomials to the antenna synthesis problem by power criterion

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The antenna synthesis problem according to the prescribed power radiation pattern with the equality norm condition is considered. It is solved by the approach based on the concept of generating polynomials. The variational formulation, supplied by the Lagrange method of multipliers, is applied. The Lagrange–Euler equation for obtained functional is a nonlinear integral equation of the Hammerstein type. The polynomial approach is described for a generalized equation of this type, which holds for different types of antennas. The modified Newton method is used for numerical solving of the respective systems of integro-transcendental equation. The approach is applied to the concrete problems related to the linear antenna, equidistant antenna array, and the circular aperture antenna. The numerical results are obtained and analyzed.

**Keywords:** *antenna synthesis problem, power criterion, Hammerstein equation, generating polynomial*

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### 1. Introduction

The antenna synthesis problems by the power criterion belong to the phase optimization problems [1]. Such problems are characterized by the fact that the arguments of the complex functions (phases) are unknown and they play role of the optimizing parameters. Ones of the first papers in this field were [2,3], in which the linear antenna synthesis problem according to the given (desired) amplitude radiation pattern was formulated as a variational problem for minimizing the mean square difference between the modulus (amplitudes) of obtained and desired patterns. The Lagrange–Euler equation for the minimized functional was a nonlinear integral equation. Since the Lagrange–Euler equation describes not only the extremal points of the considered functional, but also all its possible stationary points, the solution of this equation may be nonunique. Moreover, the number of solutions may change when certain physical parameters are varied (the solutions branched). The process was first described for this problem in [4]. This approach was developed and generalized for different types of antenna, in particular, for the antenna array [5]. The theory was described in the book [6].

The basic formulation of the antenna synthesis problem according to power radiation pattern was probably first proposed in [7] as a modification of the synthesis problem according to the amplitude pattern. Later the power formulation was developed and described in applications to the one- and two-dimensional problems in [8,9]. Recently this formulation was supplemented by the norm equality condition stated for the desired and obtained radiation patterns [10], what has improved its physical and mathematical accordance. The nonlinear integral equation was obtained and solved numerically

by the modified Newton method. Because of the nonuniqueness of solutions to this equation, not all of them were obtained.

At the same time the polynomial approach was proposed for the synthesis problems according to the amplitude radiation pattern [11,12], which was based on the exact presentation of solutions to the nonlinear integral equations for these problems by complex polynomials of low degree. This approach was generalized and described in detail in [1]. Its application to the antenna synthesis problem in basic power formulation is given in [13].

In this paper the polynomial approach is used for the power synthesis problem in the supplemented formulation. The general mathematical theory is presented, and the numerical results are shown for the concrete antennas: linear antenna, linear antenna array, circular aperture antenna. Before, the analogous problem for the linear antenna was formulated and partially solved by the immediately applying the Newton method in [10]. Some new results were announced in [14,15].

## 2. Problem formulation

The main relation which is used in the antenna theory can be written in the operator form

$$Au = f, \quad (1)$$

where  $A$  is the linear operator which connects the current (or field)  $u$  on the antenna and the radiation pattern  $f$  created by it. The definition domain of the function  $u$  depends on the type of antenna. In the cases considered here, it has the form of a limited interval (linear antenna of the length  $2a$ ), discrete points numbered from  $-M$  to  $M$ , where the radiators are located (linear antenna array) and the plane circle of the radius  $R$  (circular aperture antenna). The pattern  $f$  is determined in the certain angle domain of the far zone. The operator  $A$  has in our cases the form of integral Fourier transform, discrete Fourier transform, or Hankel (Fourier-Bessel) transform, respectively.

The antenna synthesis problem according to the power pattern was formulated in [7,8] as the minimization problem for the functional (power criterion)

$$\sigma(u) = \|F^2 - |f|^2\|_2^2 + \alpha \|u\|_1^2. \quad (2)$$

Here  $\|u\|_1, \|f\|_2$  denote the mean square norms in the definition spaces of the functions  $u$  and  $f$ , respectively,  $F^2 \geq 0$  is the desired power (squared amplitude) pattern,  $\alpha > 0$  is a given real number (weight factor). The first addend in (2) provides the proximity of the obtained power pattern to the described one inside the main lobe, whereas the second one minimizes the level of the side lobes.

According to the standard variation theory, this functional was reduced in [8] to the Lagrange-Euler equation

$$\alpha f = 2AA^*[(F^2 - |f|^2) f]. \quad (3)$$

The symbol  $A^*$  denotes the operator conjugated to  $A$ . This equation describes all (not only extremal) stationary points of  $\sigma(u)$ .

Note that equation (3) has the obvious trivial solution  $f \equiv 0$  always. In order to avoid this undesirable property as well as to limit the tendency to proportionally decreasing the pattern  $f$  with increasing  $\alpha$ , in [10,16] the functional  $\sigma(u)$  was supplemented by the additional condition

$$\|f\|_2^2 = \|F\|_2^2. \quad (4)$$

Applying the Lagrange multipliers method to the optimization problem (2), (4) leads to the unconditional minimization of the functional

$$\sigma_\mu(u) = \sigma_0 + \alpha \|u\|_1^2 - \mu \|f\|_2^2, \quad (5)$$

where  $\sigma_0 = \|F^2 - |f|^2\|_2^2$ . The Lagrange multiplier  $\mu$  is unknown. It must be determined from the demand that condition (4) holds.

The Lagrange-Euler equation for functional (5) has the form

$$\alpha f = 2AA^*[W(|f|, \mu) \exp(i \arg f)], \tag{6}$$

where  $W(|f|, \mu) = (F^2 - |f|^2 + \mu/2)|f|$ . When the function  $f$  and parameter  $\mu$  are found, the function  $u$  which provides the stationary value of  $\sigma_\mu(u)$  can be calculated by the formula

$$u = \frac{2}{\alpha} A^*[W(|f|, \mu) \exp(i \arg f)]. \tag{7}$$

Since equation (7) can have nonunique solutions, all of ones must be found and then that providing the minimum to functional  $\sigma_\mu$  must be chosen between them.

For all types of antenna considered here the equation (6) is a nonlinear integral equation of the Hammerstein type and can be written in the general form

$$\alpha f(\xi) = 2 \int_a^b K(\xi, \xi') W(|f|, \mu) \exp(i \arg f(\xi')) d\xi', \tag{8}$$

where  $a, b$  are the integration limits, different for different types of antenna,

$$K(\xi, \xi') = \frac{s(\xi)q(\xi') - s(\xi')q(\xi)}{\tau(\xi) - \tau(\xi')}, \tag{9}$$

$$W(|f|, \mu) = \left( F^2(\xi) - |f(\xi)|^2 + \frac{\mu}{2} \right) |f(\xi)|p(\xi). \tag{10}$$

The functions  $s(\xi)$ ,  $q(\xi)$ ,  $\tau(\xi)$  are assumed to be real continuous ones such that the sets of functions  $\{\tau^n(\xi)s(\xi)\}$ ,  $\{\tau^n(\xi)q(\xi)\}$ ,  $n = 0, 1, \dots$  are linearly independent,  $p(\xi)$  is the weight function (if needed).

Additional condition (4) of the pattern norm equality has in these notations the form

$$\int_a^b F^2(\xi)p(\xi)d\xi = \int_a^b |f(\xi)|^2p(\xi)d\xi. \tag{11}$$

For simplicity, we assume below  $\int_a^b F^2(\xi)p(\xi)d\xi = 1$ .

The above equations can be modified also for the case when the condition (4) is not needed (as in [7,8]). Then the parameter  $\mu$  should be set zero in (5), as well as in the definition of the function  $W(|f|)$  (10). Of course, the equality (11) must be also omitted.

### 3. Polynomial representation of solutions

We apply to (8) the approach described in [17] for a little simpler equation. As in [17], we confine ourselves to the case when the solutions to equation (8) have no zeros in the interval  $\xi \in [a, b]$ , and write them in the form

$$f(\xi) = |f(\xi)| \frac{P_N(\tau)}{|P_N(\tau)|}, \tag{12}$$

where

$$P_N(\tau) = \prod_{k=1}^N (1 - \eta_{Nk}\tau) \tag{13}$$

is the polynomial of a finite degree  $N$  with the complex pairwise nonconjugate zeros  $\eta_{Nk}^{-1}$ :

$$\eta_{Nk} - \bar{\eta}_{Nm} \neq 0, \quad k, m = 1, 2, \dots, N; \quad (14)$$

the dashed symbol means the complex conjugation. An arbitrary complex factor with unite modulus is omitted in (12).

After this notation the exponent in equation (8) obtains the form

$$\exp(i \arg f(\xi)) = \frac{P_N(\tau)}{|P_N(\tau)|}. \quad (15)$$

According to Theorem 1 from [17], the function  $f(\xi)$  of the form (12), having no zeros at  $\xi \in [a, b]$ , is a solution to equation (6) if and only if the complex parameters  $\eta_{Nk} = \eta'_{Nk} + i\eta''_{Nk}$ , the function  $|f(\xi)|$ , and the real number  $\mu$  satisfy the following system of integro-transcendental equations:

$$Z(|f|, \eta', \eta'', \mu) \equiv \alpha |f(\xi)| - 2 \int_a^b K(\xi, \xi') W(|f|, \mu) \frac{\operatorname{Re}[\bar{P}_N(\tau) P_N(\tau')]}{|P_N(\tau)| |P_N(\tau')|} d\xi' = 0, \quad (16a)$$

$$\Phi(|f|, \eta', \eta'', \mu) \equiv \int_a^b \frac{\tau^{n-1} s(\xi) W(|f|, \mu)}{|P_N(\tau)|} d\xi = 0, \quad n = 1, 2, \dots, N; \quad (16b)$$

$$\Psi(|f|, \eta', \eta'', \mu) \equiv \int_a^b \frac{\tau^{n-1} q(\xi) W(|f|, \mu)}{|P_N(\tau)|} d\xi = 0, \quad n = 1, 2, \dots, N; \quad (16c)$$

$$\Upsilon(|f|) \equiv \int_a^b |f(\xi)|^2 d\xi - 1 = 0. \quad (16d)$$

At fixed  $N$ , the solutions to equation (8) form up the “equivalent” groups, inside each of them one or several parameters  $\eta_{Nk}$  are replaced with their complex conjugated  $\bar{\eta}_{Nk}$ . All modulus  $|f(\xi)|$  of the radiation patterns are the same in such a group; consequently, the values of the first addend  $\sigma_0(u)$  in functional (5) are the same in the group. The number of solutions in each of such group is  $2^N$ .

The solutions to equation (8) with different polynomial degree  $N$  can exist simultaneously. The number of solutions can change when the functions  $s(\xi)$ ,  $q(\xi)$ ,  $\tau(\xi)$  depend on a certain physical parameter  $c$  and this parameter is varied. Such changing is called the branching of solutions. This effect is explained by the fact that at the branching points  $c = c_j$  the polynomial degree  $N$  is changed. As a rule, the value of  $N$  changes by 1 at the branching points. In exclusive cases, connected with symmetry of the problem, the value of  $N$  may change by 2 or does not change of all [1]. At the branching points some additional equations can be satisfied together with (16).

If a new solution with polynomial  $P_{N+1}(\tau)$  is branched off from the solution with polynomial  $P_N(\tau)$  at  $c = c_j$ , then the system (16) must be supplemented by two following equations:

$$\int_a^b \frac{\tau^N(\xi) s(\xi) W(|f|, \mu)}{|P_N(\tau)| (1 - \eta_{N+1, N+1} \tau)} d\xi = 0, \quad (17a)$$

$$\int_a^b \frac{\tau^N(\xi) q(\xi) W(|f|, \mu)}{|P_N(\tau)| (1 - \eta_{N+1, N+1} \tau)} d\xi = 0. \quad (17b)$$

The parameter  $\eta_{N+1,N+1}$  in  $P_{N+1}(\tau)$  is real at the branching point; all other ones coincide with those in  $P_N(\tau)$ :  $\eta_{N+1,k} = \eta_{N,k}$ ,  $k = 1, 2, \dots, N$ . Thus, we have two additional equations (17) and two new unknown real values  $\eta_{N+1,N+1}$  and  $c_j$ . Other cases of the branching points we will comment in the particular examples.

#### 4. Numerical algorithm

The integro-transcendental equation system (16) can be solved by the modified Newton method [1]. The vector of unknowns in this method is formed as the aggregate of all real values: discretized values of the function  $|f(\xi)|$ ,  $\xi \in [a, b]$ ; parameters  $\eta'_{Nk}, \eta''_{Nk}$ ,  $k = 1, \dots, N$ ; and the Lagrange multiplier  $\mu$ .

On each step of this method, we must solve the linear equation system with respect to the increments of the unknowns:

$$D \times \begin{bmatrix} \delta|f| \\ \delta\eta' \\ \delta\eta'' \\ \delta\mu \end{bmatrix} = - \begin{bmatrix} Z(|f|, \eta', \eta'', \mu) \\ \Phi(|f|, \eta', \eta'', \mu) \\ \Psi(|f|, \eta', \eta'', \mu) \\ \Upsilon(|f|) \end{bmatrix}. \tag{18}$$

Here the matrix  $D$  can be written in the following schematic form

$$D = \begin{bmatrix} \partial Z/\partial|f| & \partial Z/\partial\eta' & \partial Z/\partial\eta'' & \partial Z/\partial\mu \\ \partial\Phi/\partial|f| & \partial\Phi/\partial\eta' & \partial\Phi/\partial\eta'' & \partial\Phi/\partial\mu \\ \partial\Psi/\partial|f| & \partial\Psi/\partial\eta' & \partial\Psi/\partial\eta'' & \partial\Psi/\partial\mu \\ \partial\Upsilon/\partial|f| & 0 & 0 & 0 \end{bmatrix}. \tag{19}$$

The first block-row in this matrix contains the first (linear) variations of the integrand function in equation (16a), caused by perturbations of the following values: function  $|f(\xi)|$ , real and imaginary parts of parameters  $\eta_{Nk}$ , and parameter  $\mu$ , respectively. This block has  $L$  rows, where  $L$  is the dimension of discretized argument  $\xi \in [a, b]$ . The second and third block-rows contain first variations of the integrands in equation (16b) and (16c), respectively, caused by the perturbations of the same values. Each of them has  $N$  rows. The last block-row is the variation of the integrand in equation (16d) caused by perturbation of the function  $|f(\xi)|$ ; this equation is independent from other unknowns. Finally, the first block-column has  $L$  columns, each of second and third ones – by  $N$  columns, and the last block-column – one column. Total dimension of the matrix  $D$  is  $(L + 2N + 1) \times (L + 2N + 1)$ . The symbol  $\delta$  means the increment of the respective value on the step.

In the case when the parameter  $\mu$  is fixed and equation (16d) does not participate in the system (16), the last block-row in system (18) together with the last block-row and block-column in its matrix (19) should be omitted.

#### 5. Particular cases of the antenna; linear antenna

For the linear antenna of the length  $2a$  the operator  $A$  in (1) has the form of the integral Fourier transform of the finite (compactly supported) function

$$f(\xi) = [Au](\xi) \equiv \sqrt{\frac{c}{2\pi}} \int_{-1}^1 u(x) \exp(ic\xi x) dx, \tag{20}$$

where  $x$  is the normed linear coordinate along the antenna,  $u(x)$  is the current distribution on the antenna,  $\xi = \sin \theta / \sin \theta_0$  is the generalized angular coordinate normed by the condition that real angle interval  $\theta \in [-\theta_0, \theta_0]$ , where the desired power pattern  $F^2$  is given, corresponds to the generalized

interval  $\xi \in [-1, 1]$ ,  $c = ka \sin \theta_0$  is the given physical parameter proportional to the antenna length,  $k$  is the wavenumber.

The functional to be minimized is

$$\sigma_\mu(u) = \int_{-1}^1 (F^2(\xi) - |f(\xi)|^2)^2 d\xi + \alpha \int_{-1}^1 |u(x)|^2 dx - \mu \int_{-1}^1 |f(\xi)|^2 d\xi. \quad (21)$$

The equality condition for the pattern norm is

$$\int_{-1}^1 F^2(\xi) d\xi = \int_{-1}^1 |f(\xi)|^2 d\xi. \quad (22)$$

The operator  $AA^*$  in (6) is described by the integral transform

$$[AA^*v](\xi) = \frac{c}{2\pi} \int_{-1}^1 v(\xi') \frac{\sin(c(\xi - \xi'))}{\xi - \xi'} d\xi'. \quad (23)$$

Its integration limits and the kernel coincide with those in (8) if  $[a, b] = [-1, 1]$ ,  $s(\xi) = \sin(c\xi)$ ,  $q(\xi) = \cos(c\xi)$ ,  $\tau(\xi) = \xi$ ,  $p(\xi) \equiv 1$ .

The constant factor  $c/(2\pi)^{-1}$  we will omit below for simplicity.

The Lagrange-Euler equation for the functional (21) has the form

$$\alpha |f(\xi)| - 2 \int_{-1}^1 \frac{\sin(c(\xi - \xi'))}{\xi - \xi'} \left( F^2(\xi) - |f(\xi)|^2 + \frac{\mu}{2} \right) |f(\xi)| \exp(i \arg f(\xi')) d\xi' = 0. \quad (24)$$

The optimal current distribution is calculated by the amplitude radiation pattern  $|f(\xi)|$ , polynomial  $P_N(\xi)$  and the Lagrange multiplier  $\mu$ , obtained from this equation, via the formula

$$u(x) = \frac{2}{\alpha} \int_{-1}^1 \left( F^2(\xi) - |f(\xi)|^2 + \frac{\mu}{2} \right) |f(\xi)| \frac{P_N(\xi)}{|P_N(\xi)|} \exp(-icx\xi) d\xi. \quad (25)$$

The system of integro-transcendental equations for determination of  $|f(\xi)|$ , parameters  $\eta_{Nk}$  in  $P_N(\xi)$  and the Lagrange multiplier  $\mu$  has for the case of linear antenna the form

$$Z(|f|, \eta', \eta'', \mu) \equiv \alpha |f(\xi)| - 2 \int_{-1}^1 \frac{\sin(c(\xi - \xi'))}{\xi - \xi'} W(|f|, \mu) \frac{\operatorname{Re}[\bar{P}_N(\xi) P_N(\xi')]}{|P_N(\xi)| |P_N(\xi')|} d\xi' = 0, \quad (26a)$$

$$\Phi(|f|, \eta', \eta'', \mu) \equiv \int_{-1}^1 \frac{\xi^{k-1} \cos(c\xi) W(|f|, \mu)}{|P_N(\xi)|} d\xi = 0, \quad k = 1, 2, \dots, N; \quad (26b)$$

$$\Psi(|f|, \eta', \eta'', \mu) \equiv \int_{-1}^1 \frac{\xi^{k-1} \sin(c\xi) W(|f|, \mu)}{|P_N(\xi)|} d\xi = 0, \quad k = 1, 2, \dots, N; \quad (26c)$$

$$\Upsilon(|f|) \equiv \int_{-1}^1 |f(\xi)|^2 d\xi - 1 = 0. \quad (26d)$$

This system was solved numerically for  $F^2(\xi) \equiv 1/2$  with different  $\alpha$ . The numerical results are presented in Figs. 1, 2 ( $\alpha = 0.4$ ) and Figs. 3, 4 ( $\alpha = 0.9$ ). Fig. 1 and Fig. 3 demonstrate the behavior of  $\eta_{Nk}$  versus the parameter  $c$  for different  $N$ . Only one representative (with  $\text{Im} \eta_{Nk} \geq 0$ ) of each equivalent group of solutions is shown. The two-digit labels of curves correspond to the indices  $Nk$ ; the curves describing different solutions related to the polynomials  $P_N(\xi)$  with the same  $N$  are indicated as  $N'k, N''k$ , etc. Recall that we consider only the solutions having no zeros at  $\xi' \in [-1, 1]$ . As it was mentioned above, in this case, owing symmetry of the problem, there are possible branching points not only with changing  $N$  by 1 but also by 2 or without changing  $N$ .

At small values of parameter  $c$  there exists only the real solution  $f_0(\xi)$  of equation (24) corresponding  $N = 0$ . The first branching point of this solution is  $c = c_1$ . Two solutions  $f_1(\xi)$  with  $N = 1$  and imaginary parameters  $\pm \eta_{11}$  arise here; at the branching point these parameters are real. At this point, the values of  $c_1$ , real function  $|f_0(\xi)| = f_0(\xi)$  and real parameter  $\mu$  are calculated from the transcendental equation system (16) with additional equation (17), having in our case the form

$$\begin{aligned} \alpha f_0(\xi) - 2 \int_{-1}^1 \frac{\sin(c(\xi - \xi'))}{\xi - \xi'} W(f_0, \mu) d\xi' &= 0, \\ \int_{-1}^1 \cos(c\xi) W(f_0, \mu) d\xi &= 0, \\ \int_{-1}^1 f_0^2(\xi) d\xi - 1 &= 0. \end{aligned}$$

At the point  $c = c_2$  two solutions with  $N = 2$  branch off from the solution  $f_1(\xi)$  (with  $N = 1$ ); we denote them by  $f_2(\xi)$ . Each solution has two imaginary parameters  $\eta_{21}$  and  $\eta_{22}$ . At the branching point  $\eta_{21} = \eta_{11}$  and the real values  $\eta_{11}, \eta_{22}, c_2, \mu$  and function  $|f_1(\xi)|$  are determined from the following system:

$$\begin{aligned} \alpha |f_1(\xi)| - 2 \int_{-1}^1 \frac{\sin(c(\xi - \xi'))}{\xi - \xi'} W(|f_1|, \mu) \frac{\text{Re}[\bar{P}_1(\xi)P_1(\xi')]}{|P_1(\xi)||P_1(\xi')|} d\xi' &= 0, \\ \int_{-1}^1 \frac{\cos(c\xi)W(|f_1|, \mu)}{|1 - \eta_{11}\xi|} d\xi &= 0, \quad k = 1, 2, \dots, N, \\ \int_{-1}^1 \frac{\xi \sin(c\xi)W(|f_1|, \mu)}{|1 - \eta_{11}\xi|(1 - \eta_{22}\xi)} d\xi &= 0, \quad k = 1, 2, \dots, N; \\ \int_{-1}^1 \frac{\xi \cos(c\xi)W(|f_1|, \mu)}{|1 - \eta_{11}\xi|(1 - \eta_{22}\xi)} d\xi &= 0, \quad k = 1, 2, \dots, N; \\ \int_{-1}^1 |f_1(\xi)|^2 d\xi - 1 &= 0. \end{aligned}$$

At the point  $c = c_3$  two solutions with  $N = 2$  branch off from real one  $f_0(\xi)$ ; we denote them by  $f_2(\xi)$ . This is the case of branching with changing the polynomial degree by two. Each solution has two complex parameters  $\eta_{21}$  and  $\eta_{22}$ . Each of these pairs are calculated separately (curves 2'1, 2'2 in Fig. 11). At the branching point  $\eta_{21} = \eta_{22}$  and they are imaginary. The real values  $\eta_{21}, c_3$ , function

$|f_0(\xi)| = f_0(\xi)$  and  $\mu$  are determined from the following system:

$$\begin{aligned} \alpha|f_0(\xi)| - 2 \int_{-1}^1 \frac{\sin(c(\xi - \xi'))}{\xi - \xi'} W(|f_0|, \mu) d\xi' &= 0, \\ \int_{-1}^1 \frac{\cos(c\xi) W(|f_0|, \mu)}{(1 - \eta_{21}^2 \xi^2)} d\xi &= 0, \\ \int_{-1}^1 \frac{\xi \sin(c\xi) W(|f_0|, \mu)}{(1 - \eta_{21}^2 \xi^2)} d\xi &= 0, \\ \int_{-1}^1 |f_0(\xi)|^2 d\xi - 1 &= 0. \end{aligned}$$

At the point  $c = c_4$  the solution  $f_1(\xi)$  branches without changing the polynomial degree: two new solutions  $f_{1'}(\xi)$  with  $N = 1$  branch off from it. Each of them have one complex parameter  $\eta_{1'1}$  (curve 1'1 in Fig. 1). The equation system for the branching point  $c_4$  and imaginary (at this point)  $\eta_{11} = \eta_{1'1}$  is obtained from the condition

$$D(c) \equiv \det(\mathcal{F}') = 0, \tag{27}$$

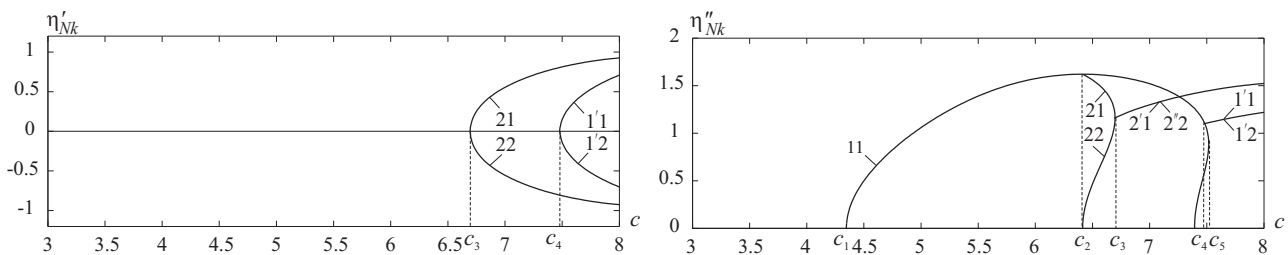
where

$$\mathcal{F}' = \left\{ \begin{array}{cc} \left\{ \frac{\partial \Phi_{Nj}}{\partial \eta'_{Nk}} \right\}_{j,k=1}^N & \left\{ \frac{\partial \Phi_{Nj}}{\partial \eta'_{Nk}} \right\}_{j,k=1}^N \\ \left\{ \frac{\partial \Psi_{Nj}}{\partial \eta'_{Nk}} \right\}_{j,k=1}^N & \left\{ \frac{\partial \Psi_{Nj}}{\partial \eta'_{Nk}} \right\}_{j,k=1}^N \end{array} \right\}$$

is the Jakobi matrix of these functions. Condition (27) gives the additional transcendental equation to system (16):

$$\eta_{11} \int_{-1}^1 \frac{\xi \sin(c\xi) W(|f|, \mu) d\xi}{(1 - \eta_{11}^2 \xi^2)^{3/2}} - \int_{-1}^1 \frac{\xi^2 \cos(c\xi) W(|f|, \mu) d\xi}{(1 - \eta_{11}^2 \xi^2)^{3/2}} = 0.$$

Fig. 2 and Fig. 4 show the main part  $\sigma_0$  of functional (21) reached on the real solution  $f_0(\xi)$  (dashed line) and on the solutions with different values of polynomial degree for given function  $F^2(\xi) \equiv 1/2$ ,  $\alpha = 0.4$  and  $\alpha = 0.9$ , respectively. In the synthesis problem according to given radiation pattern [12], the minimal value of the functional considered there was reached on the solution with the largest possible value of  $N$ . For the functional (21), which consider here, such properties does not take place. This fact is explained existing additional term in (21).



**Fig. 1.** Parameters  $\eta_{Nk}$  for the linear antenna.  $F^2(\xi) \equiv 1/2$ ,  $\alpha = 0.4$ .



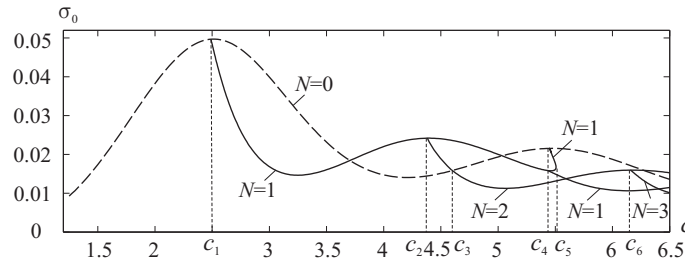


Fig. 2. The main part  $\sigma_0$  of functional (21) for the linear antenna.  $F^2(\xi) \equiv 1/2$ ,  $\alpha = 0.4$ .

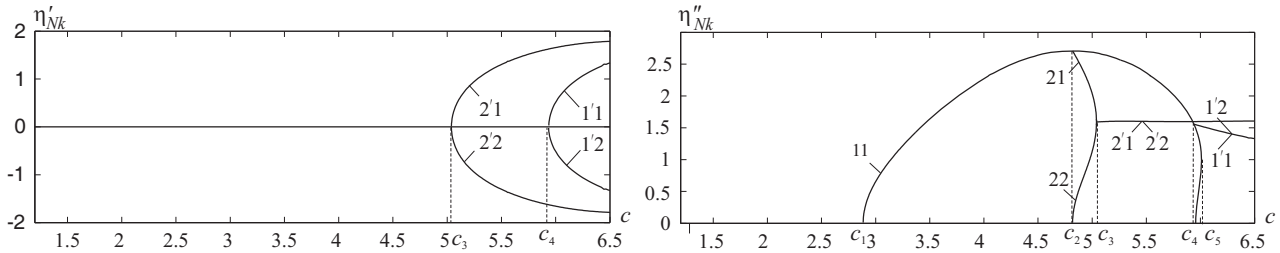


Fig. 3. Parameters  $\eta_{Nk}$  for the linear antenna.  $F^2(\xi) \equiv 1/2$ ,  $\alpha = 0.9$ .

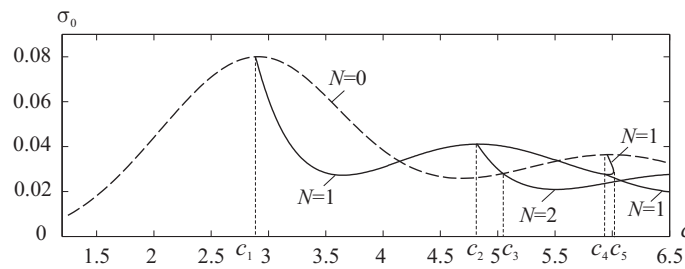


Fig. 4. The main part  $\sigma_0$  of functional (21) for the linear antenna.  $F^2(\xi) \equiv 1/2$ ,  $\alpha = 0.9$ .

### 6. Linear antenna array

The next example is related to the linear antenna array which contains  $T = 2M + 1$  identical radiators equidistantly located with the distance  $d$  between them (for simplicity, we chose the number of radiators as odd one). The radiation pattern of the array has two factors: the pattern of individual radiator and the array factor. As a rule, the pattern of one radiator is known, and it can be taken into account in the given function function  $F^2(\xi)$ . The array factor is described by the discrete Fourier transform

$$f(\xi) = [Au](\xi) \equiv \sqrt{\frac{c}{2\pi}} \sum_{n=-M}^M u_n \exp(icn\xi), \tag{28}$$

where  $u = \{u_n\}$ ,  $n = -M, \dots, M$ , is the  $2M + 1$ -dimensional vector of complex components,  $c = kd \sin \theta_0$ , where, as before,  $[-\theta_0, \theta_0]$  is the interval of angles, in which the function  $F^2(\xi)$  is given. We denote this factor by the same symbol  $f(\xi)$  and call as the pattern. Here, as before, the generalized angular coordinate is defined by the formula  $\xi = \sin \theta / \sin \theta_0$ .

The function  $f(\xi)$  defined by (28) is periodical, its period is  $2\pi/c$ . We formulate the functional in the synthesis problem inside of this period; the integration interval is  $[-1, 1]$ . For this reason the value of  $c$  is limited by the condition  $c \leq \pi$ .

The functional to be minimized in this case is

$$\sigma_{\mu}(u) = \int_{-1}^1 (F^2(\xi) - |f(\xi)|^2)^2 d\xi + \alpha \sum_{n=-M}^M |u_n|^2 - \mu \int_{-1}^1 |f(\xi)|^2 d\xi, \quad (29)$$

where the given real function  $F^2(\xi)$ ,  $\xi \in [-1, 1]$ , is given. The norm equality has the same form (22),  $\mu$  is an unknown Lagrange multiplier.

The adjoint operator  $A^*$  is

$$[A^*v]_n = \sqrt{\frac{c}{2\pi}} \int_{-1}^1 v(\xi) \exp(-icn\xi) d\xi, \quad n = -M, \dots, M, \quad (30)$$

and the operator  $AA^*$  acts as follows

$$[AA^*v](\xi) = \frac{c}{2\pi} \int_{-1}^1 v(\xi') \frac{\sin(cT(\xi - \xi')/2)}{\sin(c(\xi - \xi')/2)} d\xi'. \quad (31)$$

After some transformation (see [1], p.122), the kernel in (31) is reduced to

$$\frac{\sin(cT(\xi - \xi')/2)}{\sin(c(\xi - \xi')/2)} = \frac{\sin(Tc\xi/2) \cos(Tc\xi'/2) - \cos(Tc\xi/2) \sin(Tc\xi'/2)}{(\tan(c\xi/2) - \tan(c\xi'/2)) \cos(c\xi/2) \cos(c\xi'/2)},$$

and the Lagrange-Euler equation for functional (29) obtains the form (8) with the following notations:  $[a, b] = [-1, 1]$ ,  $s(\xi) = \sin(Tc\xi/2)/\cos(c\xi/2)$ ,  $q(\xi) = \cos(Tc\xi/2)/\cos(c\xi/2)$ ,  $\tau(\xi) = \tan(c\xi/2)$ ,  $p(\xi) \equiv 1$ .

For the considered case the system of integro-transcendental equations for  $|f(\xi)|$ ,  $\eta_{Nk}$ , and  $\mu$  has the form

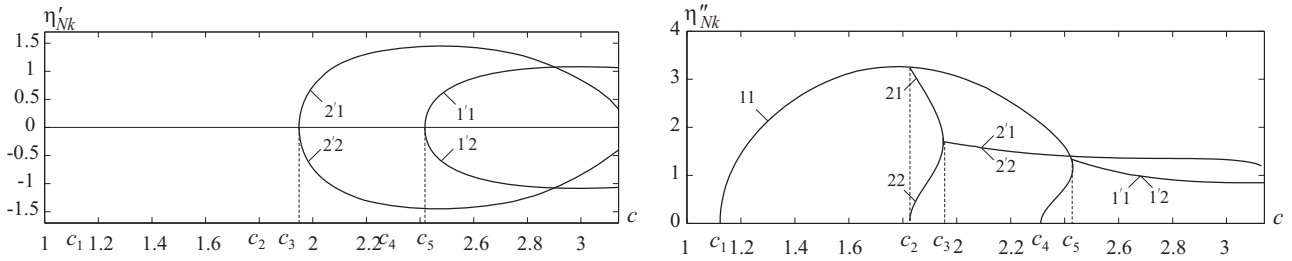
$$\alpha |f(\xi)| - 2 \int_{-1}^1 \frac{\sin(cT(\xi - \xi')/2)}{\sin(c(\xi - \xi')/2)} W(|f|, \mu) \frac{\operatorname{Re}[\bar{P}_N(\tau) P_N(\tau')]}{|P_N(\tau)| |P_N(\tau')|} d\xi' = 0, \quad (32a)$$

$$\int_{-1}^1 \frac{\tau^{k-1} \sin(Tc\xi/2) W(|f|, \mu)}{|P_N(\tau)| \cos(c\xi/2)} d\xi = 0, \quad k = 1, 2, \dots, N; \quad (32b)$$

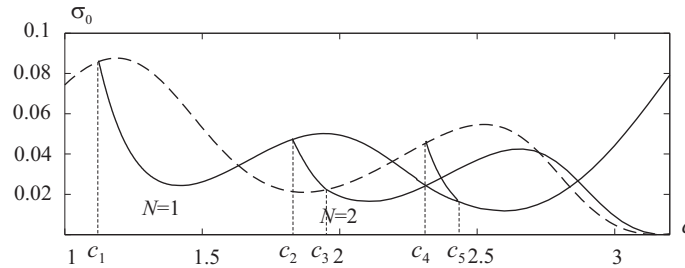
$$\int_{-1}^1 \frac{\tau^{k-1} \cos(Tc\xi/2) W(|f|, \mu)}{|P_N(\tau)| \cos(c\xi/2)} d\xi = 0, \quad k = 1, 2, \dots, N; \quad (32c)$$

$$\int_{-1}^1 |f(\xi)|^2 d\xi - 1 = 0. \quad (32d)$$

The numerical results for the case of linear antenna array are presented in Figs. 5, 6. Fig. 5 demonstrates the behavior of  $\eta_{Nk}$  versus the parameter  $c$  for given function  $F^2(\xi) \equiv 1/2$  and  $\alpha = 0.9$ . Fig. 6 shows the main part of functional (29). The qualitative behaviors of these results are similar to those for the case of linear antenna, except of the condition  $c \leq \pi$ .



**Fig. 5.** Parameters  $\eta_{Nk}$  for the linear antenna array.  $F^2(\xi) \equiv 1/2$ ,  $\alpha = 0.9$ .



**Fig. 6.** The main part  $\sigma_0$  of functional (29) for the linear antenna array.  $F^2(\xi) \equiv 1/2$ ,  $\alpha = 0.9$ .

### 7. Planar circular antenna

The third example relates to the planar circular antenna. In general case the radiation pattern of the planar antenna is calculated from the field on its aperture with accuracy to a constant factor by the formula

$$f(\mathbf{d}) = [Au](\mathbf{d}) \equiv \iint_D u(\mathbf{R}) \exp(ic(\mathbf{r} \cdot \mathbf{d})) \, dr, \tag{33}$$

where  $\mathbf{r}$  and  $\mathbf{d}$  are the radius-vectors of the points on the aperture  $D$  and on the unit sphere (angles in the far zone). In particular case, when the aperture is the circle of the radius  $R$  and the field  $u(\mathbf{r})$  does not depend on the angular coordinate  $\varphi$ , then the pattern depends only on the coordinate  $\theta$  and formula (33) can be rewritten in the one-dimensional form

$$f(\xi) = [Au](\xi) \equiv \int_0^R J_0(cr\xi)u(r)rdr. \tag{34}$$

Here  $c = kR \sin \theta_0$ ,  $\xi = \sin \theta / \sin \theta_0$ ,  $J_0(cr\xi)$  is the Bessel function of zero order. The angular interval, in which the function  $F^2$  is given, is in this case  $\theta \in [0, \theta_0]$ , what corresponds to  $\xi \in [0, 1]$ .

The functional to be minimized is

$$\sigma_\mu(u) = \int_0^1 (F^2(\xi) - |f(\xi)|^2)^2 \xi d\xi + \alpha \int_0^1 |u(r)|^2 r dr - \mu \int_0^1 |f(\xi)|^2 \xi d\xi. \tag{35}$$

The norm equality has the form

$$\int_0^1 F^2(\xi) \xi d\xi = \int_0^1 |f(\xi)|^2 \xi d\xi.$$

The adjoint operator  $A^*$  is defined as

$$[A^*v](r) = c \int_0^1 v(\xi) J_0(cr\xi) \xi d\xi,$$

and the operator  $AA^*$  is described by the formula

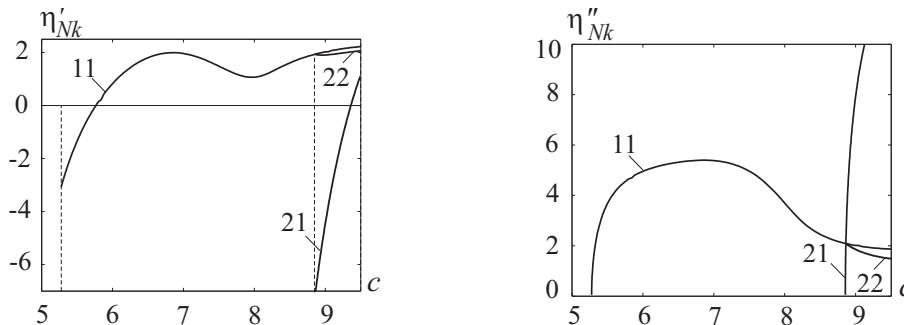
$$[AA^*v](\xi) = c \int_0^1 \frac{\xi J_0(c\xi') J_1(c\xi) - \xi' J_0(c\xi) J_1(c\xi')}{\xi^2 - (\xi')^2} v(\xi') \xi d\xi'.$$

The Lagrange–Euler equation for this case has the form (8) with the following notations:  $[a, b] = [0, 1]$ ,  $s(\xi) = J_0(c\xi)$ ,  $q(\xi) = \xi J_1(c\xi)$ ,  $\tau(\xi) = \xi^2$ ,  $p(\xi) \equiv \xi$ . Similarly as before, the constant factor  $c$  is omitted in this equation.

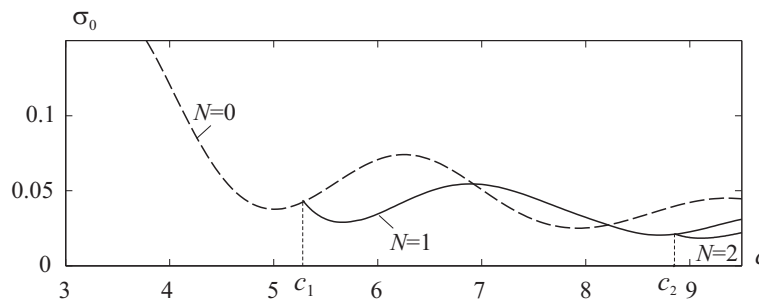
The system of integro-transcendental equations for  $|f(\xi)|$ ,  $\eta_{Nk}$ , and  $\mu$  has is

$$\begin{aligned} \alpha |f(\xi)| - 2 \int_0^1 K(\xi, \xi') W(|f|, \mu) \frac{\operatorname{Re}[\overline{P_N(\tau)} P_N(\tau')]}{|P_N(\tau)| |P_N(\tau')|} d\xi' &= 0, \\ \int_0^1 \frac{(\xi^2)^{n-1} J_0(c\xi) W(|f|, \mu)}{|P_N(\tau)|} d\xi &= 0, \quad n = 1, 2, \dots, N; \\ \int_0^1 \frac{(\xi^2)^{n-1} \xi J_1(c\xi) W(|f|, \mu)}{|P_N(\tau)|} d\xi &= 0, \quad n = 1, 2, \dots, N; \\ \int_0^1 |f(\xi)|^2 \xi d\xi - 1 &= 0. \end{aligned}$$

In this case the solutions can branch only with changing the polynomial degree  $N$  by 1. The numerical results for the case of planar circular antenna are presented in Figs. 7, 8.



**Fig. 7.** Parameters  $\eta_{Nk}$  for the circular aperture antenna.  $F^2(\xi) \equiv 1/2$ ,  $\alpha = 0.9$ .



**Fig. 8.** The main part  $\sigma_0$  of functional (35) for the circular aperture antenna.  $F^2(\xi) \equiv 1/2$ ,  $\alpha = 0.9$ .

## 8. Conclusions

In this work the antenna synthesis problem according to the prescribed power radiation pattern with the equality norm condition was considered. The problem was reduced to the nonlinear integral equation of Hammerstein type. It was solved by the usage of polynomial approach, developed earlier for the synthesis problem according to the given amplitude radiation pattern. This approach leads to the system of integro-transcendental equations, which are solved by the modified Newton method. The concrete problems relating the linear antenna, equidistant antenna array, and the circular aperture antenna are numerically solved and obtained results were analyzed.

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## Застосування концепції породжуючих поліномів до задачі синтезу антен за діаграмою за потужністю

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Розглядається задача синтезу антен за заданою діаграмою напрямленості за потужністю з додатковою умовою рівності норм. Вона розв'язується з використанням концепції породжуючих поліномів. Сформульована варіаційна постановка з наступним використанням методу множників Лагранжа. Рівняння Лагранжа-Ейлера для одержаного функціонала є нелінійним інтегральним рівнянням типу Гаммерштейна. Описано поліноміальний підхід у застосуванні до загального рівняння такого типу, яке охоплює різні типи антен. Для розв'язування інтегро-трансцендентних рівнянь, що одержуються при цьому, використовується модифікований метод Ньютона. Підхід застосовано до конкретних задач, які стосуються лінійної антени, еквідистантної антенної ґратки та кругової апертурної антени. Приведено та проаналізовано одержані числові результати.

**Ключові слова:** *синтез антен, критерій за потужністю, рівняння Гаммерштейна, породжуючий поліном*

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