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Structure of geometrical nonlinearities in problems of liquid sloshing in tanks of non-cylindrical shape

Limarchenko O. S.[1], Cattani C.[2], Pilipchuk V.[3]

[1] *Taras Shevchenko National University of Kyiv*
4-E Academician Glushkov Avenue, 01033, Kyiv, Ukraine
[2] *University of Salerno*
Via Giovanni Paolo II, Fisciano (SA), Italy
[3] *Wayne State University*
42 W. Warren Avenue, Detroit, MI 48202, USA

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Structure of geometrical nonlinearities in mathematical model of liquid sloshing in tanks of non-cylindrical shape is under consideration. In contrast to the case of cylindrical reservoir, some new types of nonlinearities occur in mathematical statement of the problem. They are connected with four main reasons. First, they are determined by new normal modes, which correspond to non-cylindrical shape of the tank and take into account some nonlinear properties of the problem (for example, they follow tank walls above level of a free surface). Second, determination of the potential energy of the liquid includes tanks geometry in close vicinity of cross-section of undisturbed free surface of the liquid and tank walls. Third type of manifestation of geometrical nonlinearities is connected with compensation of elevation of liquid level due to non-cylindrical type of tank shape for providing law of mass conservation. The fourth type of nonlinearities is connected with simultaneous manifestation of physical and geometrical nonlinearities. Investigation showed that mostly manifestation of nonlinear properties of liquid sloshing, connected with geometrical nature, is predetermined by inclination and curvature of tank walls in close vicinity of contact of undisturbed liquid with tank walls. We illustrated some general properties of geometrical nonlinearities by the example of three cases of tanks, namely, cylindrical, conic, and paraboloidal tank, which is selected such that its walls have the same inclination near free surface of the liquid as conic tank, but in this case curvature is manifested supplementary.

Keywords: *nonlinear dynamics of liquid, liquid sloshing, geometrical nonlinearities, numerical example*

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1. Introduction

Many types of engineering structures are connected with transport of liquid cargoes. Such transport systems as rockets carriers, spacecrafts, aircrafts, cisterns, chemical reactors, oil storage systems are frequently used in practice. In most cases liquid does not completely fill tanks of such systems and oscillations of a liquid free surface appears on transporting. Moreover, for many systems liquid is consumed during operation (fuel, oxidant, *etc.*). This results in the property that masses of oscillating liquid and its frequency parameters vary in time. It is known that for a large relative mass of liquid its wavy motion can essentially influence dynamics of a transport object. This makes supplementary demands to a control system (operator) and can cause undesirable modes of motion and even emergency situations. Particularly dangerous consequences can take place for resonant development of oscillations. For elimination of such development of processes various dampers are usually applied, which insertion results in considerable increase of mass of a tank. At the same time part of these modes can be prevented by reasonable selection of mode of motion of transport object. For this selection it is necessary to have highly reliable models of processes, which is convenient for both theoretical analysis and numerical realization. Theoretical and experimental investigations are evidence of the fact that models based on ideas of linearization of processes don't reflect many significant properties and, therefore, their reliability is strictly bounded. Complication of operation modes of modern transport objects and raising the level of requirements of accuracy of their functioning result in necessity of realization of simulation on the basis of nonlinear description of dynamic phenomena, taking into account complete variety of interacting factors, which influence dynamic behavior of structures with liquid.

Recent investigations showed that main achievements in modeling of liquid sloshing are connected with theoretical methods, which use modal discretization (representation of solutions by means of decompositions with respect to normal modes of oscillations of a system), i.e., which reduces the initial discrete-continuum model to a certain discrete model. In addition for different stages of realization of this approach application of numerical methods is used.

The analysis of literature of resent years testifies that on investigation of dynamics of structures with liquid insufficient attention was paid to theoretical investigations of problems, when a carrying body realizes combined motion with liquid. Moreover, the case, when motion of a carrying body is not specified, but is determined in the process of solving the problem about combined motion of a body with liquid under given forces and torques, is also not adequately explored. Namely, this statement of the problem makes it possible to determine characteristics of influence of liquid filling on motion of a carrying body. It is worthy to note also that transient processes are important but scantily explored cases in this area [1–4].

Analysis of literature sources showed also that until now investigations of liquid sloshing in tanks of non-cylindrical shape are still developed insufficiently. Problems of liquid sloshing in non-cylindrical tanks are much more complicated than in the case of reservoirs with vertical walls. In the case of tanks with inclined walls it is necessary to modify considerably mathematical description of the problem, because for non-cylindrical tanks it contains essential contradiction in physical and mathematical sense of the problem. First of all problem statement does not contain information about behavior of tank walls above undisturbed position of liquid free surface. However, in the case of nonlinear problem liquid should follow tank walls for certain height, defined by amplitudes of excitation of waves. It was shown that this property is a part of requirements of solvability conditions for the boundary value of the problem of liquid sloshing. Further investigations showed that for solving the problem of oscillation of liquid in tanks of non-cylindrical shape it is necessary to use non-cylindrical parametrization of the domain, occupied by liquid. At the same time new types of nonlinearities, predetermined by geometry, manifest strongly for this type of problems. For example, the first normal mode in conic reservoir with right cone angle should be plain. However, in the case of nonlinear problem such normal mode violates conservation law for mass and demands compensation of liquid volume by vertical displacement of middle point of liquid free surface.

Objective of investigation of the present article is determination of influence of inclination and curvature of tank walls on formation of waves in tanks of cylindrical, conic and paraboloidal shapes and analysis of distinctive properties of waves generation, caused by geometric factors.

2. Mathematical statement of the problem

The problem about oscillations of ideal incompressible liquid with a free surface in immovable cavity of arbitrary geometrical shape is stated in the form of the following boundary value problem

$$\Delta\varphi = 0 \quad \text{in} \quad \tau; \tag{1}$$

$$\frac{\partial\varphi}{\partial n} = 0 \quad \text{on} \quad \Sigma; \tag{2}$$

$$\frac{\partial \eta}{\partial t} = -\frac{\dfrac{\partial \varphi}{\partial n}}{\|\boldsymbol{\nabla} \eta\|} \quad \text{on} \quad S; \tag{3}$$

$$\frac{\partial \varphi}{\partial t} + \frac{1}{2} \left(\boldsymbol{\nabla} \varphi\right)^2 + U = 0 \quad \text{on} \quad S. \tag{4}$$

Here motion is described in the Cartesian reference frame $Oxyz$ connected with the tank, φ is the velocity potential of liquid, τ is the domain occupied by liquid, $\frac{\partial}{\partial n}$ is external normal derivative to a surface, Σ is boundary of contact of liquid with tank walls in perturbed motion (for convenience we introduce also Σ_0, which corresponds to boundary of contact of liquid with tank walls in unperturbed motion and $\Delta\Sigma$ variation of the contact boundary caused by liquid perturbation, $\Sigma = \Sigma_0 + \Delta\Sigma$), S is a free surface of liquid in its perturbed motion (S_0 is a free surface of liquid in unperturbed motion), $\eta(x, y, z, t)$ is the equation of a free surface of liquid, $\|\boldsymbol{\nabla}\eta\|$ is norm of the gradient of the function η, U is the function of potential energy of liquid, t is time. For simplification of the following analysis of some general properties of the boundary problem (1)–(4) we shall limit our consideration on this stage by the case of an immovable tank.

In the case when the tank cavity represents a cylindrical domain one succeeded to resolve the equation of the free surface relative to the variable z, which corresponds to vertical direction. Then the equation of a free surface takes the form of

$$\eta(x, y, z, t) = z - \xi(x, y, t),$$

the corresponding linear problem admits analytical solution on the basis of the method of variables separation, and the problem (1)–(4) becomes essentially simpler. In the case of cylindrical domains occupied by liquid it is possible to construct efficient algorithms for solving nonlinear problems of dynamics tanks with liquid with a free surface, including the case of translational and rotational motion of the carrying body [2]. The most substantial results in this direction were obtained on the basis of variational algorithms of statement and solving of nonlinear problems of dynamics of tanks with liquid. Thus, for tanks of cylindrical shapes various problems of dynamics of steady and transient modes of motion of reservoirs with liquid were investigated.

In spite of the fact that the question about spreading results, obtained for cylindrical tanks on the cases of tanks of complex geometry seemed for many researchers as a question of numerical realization, until now cases of tanks of non-cylindrical shape are investigated rather superficially. Attempt of creation of unified highly universal approach for solving problems of nonlinear dynamics of liquid in cavities with inclined walls was made in publications [3], where the problem about motion of liquid in tank was formulated for non-Cartesian parametrization of the liquid domain. However, until now this method have not found practical application and investigations in this direction are not continued.

For further progress in solving applied problems of this class of objects it is necessary to analyze deeper mechanical and mathematical essence of the problem. This analysis will be done from point of view of further application of the variational method of solving the problem, which is based on formulation of the mechanical problem on the basis of the Hamilton-Osctrogradsky variational principle. It is known that this approach was successfully applied for solving nonlinear problems of dynamics of reservoirs of cylindrical shape with liquid with a free surface [2]. The mechanical analysis of this approach shows that a part of the problem conditions (kinematical constraints) should be satisfied before solving the variational problem on the stage of construction of decompositions of desired variables, and dynamic boundary conditions and motion equations for the carrying body are obtained from the variational principle as natural. It is significant to note that in the subsequent procedure of problem solving no supplementary requirements are stated for implementation of procedure of satisfying kinematical requirements of the problem. For cylindrical domains the solvability condition of the Neumann problem for the Laplace equation (for the problem about oscillations of liquid with a free surface (1)–(4)) is quite trivial. This condition is equivalent to requirement of conservation of liquid volume in its perturbed motion for every normal mode of oscillations. Analysis, conducted in the present article, shows that simple transfer of this form of the solvability condition on the case of oscillations of liquid in non-cylindrical cavities is insufficient.

Hence, construction of a nonlinear discrete resolving model for the problem about oscillations of ideal liquid with a free surface in a tank of non-cylindrical shape will be done according to the following scheme.

1. Analysis of solvability conditions.
2. Construction of decompositions of desired variables, which hold linear kinematic constraints.
3. Construction of decompositions of desired variables, which hold nonlinear kinematic boundary conditions;
4. Construction of resolving system of motion equations relative to amplitude parameters of liquid motion and parameters of translational motion of the carrying body.

Let us consider a problem about oscillations of liquid with a free surface in a reservoir, which cavity is of revolution shape. We consider the case, when reservoir is movable and can perform finite translational movements. For specification of liquid motion we introduce non-Cartesian parametrization of the domain, occupied by liquid, according to the following scheme

$$\alpha = \frac{r}{f(z)}; \quad \beta = \frac{z}{H}, \tag{5}$$

where $r = f(z)$ is the equation of the generatrix of a body of revolution, H is filling depth of liquid. Here we suppose that the origin of the reference frame is in the center of the undisturbed free surface of liquid, the axis Oz is directed upward, (r, θ, z) is the system of cylindrical coordinates, which according to the relations (5) is substituted for the new non-Cartesian system of coordinates (α, θ, β) ($\alpha \in [0, 1]$; $\theta \in [0, 2\pi]$ and for unperturbed state $\beta \in [-1, 0]$). For the accepted system of parametrization the domain of liquid takes cylindrical shape and the equation of a free surface can be resolved relative to the coordinate β and it takes the form of

$$\beta = \frac{1}{H}\xi(\alpha, \theta, t). \tag{6}$$

The problem about motion of the bounded volume of liquid takes now the form of

$$\Delta\varphi = 0; \tag{7}$$

$$\frac{\partial\varphi}{\partial n} = \frac{1}{\sqrt{1+f'^2}}\left(\frac{\partial\varphi}{\partial r} - f'\frac{\partial\varphi}{\partial z}\right) = 0 \quad \text{for} \quad r = f(z), \tag{8}$$

$$\frac{\partial\xi}{\partial t} + \frac{1}{f^2}\frac{\partial\xi}{\partial\alpha}\frac{\partial\varphi}{\partial\alpha} + \frac{1}{\alpha^2 f^2}\frac{\partial\xi}{\partial\theta}\frac{\partial\varphi}{\partial\theta} - \underline{\frac{\alpha f'}{f}\frac{\partial\xi}{\partial\alpha}\frac{\partial\varphi}{\partial z}} - \frac{\partial\varphi}{\partial z} = 0$$
$$\text{for} \quad \beta = \frac{1}{H}\xi(\alpha, \theta, t). \tag{9}$$

We notice that the underlined term appeared in the kinematical boundary condition (9) due to non-cylindrical shape of the domain, occupied by liquid, and reflects non-Cartesian property of the accepted parametrization.

The equations (7)–(9) don't represent the complete formulation of the problem about motion of liquid with a free surface, but they present the system of kinematical restrictions of the problem. Similar to the approach used for cylindrical tanks we shall obtain dynamical boundary conditions and motion equations for the carrying body from the Hamilton-Ostrogradscky variational principle.

Translational motion of the reservoir relative to a conventionally immovable reference frame is described by the vector of displacements $\boldsymbol{\varepsilon}$.

3. Analysis of the solvability condition of the problem

Taking into account that the problem (1)–(4) as well as the corresponding linear boundary problem [1] has no exact analytical solution for arbitrary cavities of revolution, we must come from the fact that boundary conditions of the problem (1)–(4) will be realized approximately. According to the general theory of solvability of the Neumann boundary problems for the Laplace equation the solvability condition for the problem (1)–(4) can be given in the following form

$$\int\limits_{\Sigma_0} \frac{\partial\varphi}{\partial n}\, d\Sigma + \int\limits_{\Delta\Sigma} \frac{\partial\varphi}{\partial n}\, d\Sigma + \int\limits_{S} \frac{\partial\varphi}{\partial n}\, dS = 0. \tag{10}$$

Let us analyze essence of every terms in the expression (10). The first addend represents requirement of satisfying (in weak sense) the non-flowing condition on an unperturbed boundary of contact of liquid with tank walls Σ_0. Therefore, holding this boundary condition of non-flowing on Σ_0 should be performed with improved accuracy. In our appearance on realization of different procedures of solving of this class of problems insufficient attention was paid to this question. Sometimes normal modes of oscillations with errors of satisfying of boundary non-flowing condition about 20% and more were applied for numerical realization. Correspondingly, such violation of non-flowing conditions, and, therefore, the solvability condition, results in instability of realization of numerical procedures.

The second addend corresponds to the requirement of realization in weak sense of the non-flowing condition on the boundary $\Delta\Sigma$, i.e., on wave crests of liquid above level of the undisturbed free surface of liquid. This physically evident kinematic boundary condition is not consequence of statement of the linear problem about oscillations of liquid in a tank, which is usually applied for construction of decompositions of desired variables. Normally this condition was not taken into consideration on analysis of nonlinear oscillations of liquid in tanks of non-cylindrical shape at all, although we suppose that this condition is the dominant one in the analysis of the physical sense of the considered problem. In accordance with the maximum principle for harmonic functions the solution tends to violate realization of non-flowing condition, and this corresponds to overflow of liquid through the tank wall (namely this causes "loss" of liquid in different methods of point-wise discretization). In spite of the property that this condition is expressed by a linear mathematical relation, according to its nature it is nonlinear, because it corresponds to realization of the kinematic condition on a nonlinear perturbed surface, and it is evident that this condition does not enter the linear statement of the problem.

In order to analyze the third addend in the solvability condition of the problem we perform its transformation. It is known that one of the form of the kinematic boundary condition on a free surface has the form

$$\frac{\partial\varphi}{\partial n} = \frac{1}{\sqrt{1+(\boldsymbol{\nabla}\xi)^2}}\frac{\partial\xi}{\partial t} \quad \text{on} \quad S.$$

After substituting this condition into the third addend of the condition (7) we obtain

$$\int\limits_{S} \frac{\partial\varphi}{\partial n}\, ds = -\int\limits_{S_0} \frac{\partial\xi}{\partial t}\, dS = -\frac{\partial}{\partial t}\int\limits_{S_0} \xi\, dS,$$

i.e., the integral over a perturbed free surface of liquid S can be transformed to the unperturbed free surface S_0. The latter form of the third addend corresponds to requirement of the liquid volume conservation in its perturbed motion. This requirement holding will be considered below, where we shall show that implementation of the requirement of liquid volume conservation for every separately taken normal mode of liquid oscillation, which corresponds to linearized requirement, is not sufficient for holding on a whole of the requirement of volume conservation in its perturbed motion.

Thus, the analysis of solvability conditions of the nonlinear boundary problem shows, that for correct solving of the nonlinear problem it is necessary to

a) satisfy with high precision the non-flowing requirement for liquid on moisten in the unperturbed state boundary of contact liquid –tank;
b) satisfy requirement of non-flowing of liquid on tank walls over the level of an unperturbed free surface, where crests of nonlinear waves reach;
c) satisfy requirements of liquid volume conservation in its perturbed nonlinear motion;
d) provide holding all these requirements on the stage of construction of decompositions of desired variables, which satisfy all kinematic boundary conditions of the problem, before solving the variational problem.

4. Construction of decompositions of desired variables (linear requirements)

As it follows from analysis of solvability conditions of the problem for successful implementation of algorithm of construction of the nonlinear finite-dimensional model of dynamics of liquid with a free surface in cavity of revolution it is necessary to construct a system of coordinate functions, which satisfy with high accuracy non-flowing requirements on the moisten boundary Σ, which consists of the unperturbed moisten boundary Σ_0 and its certain prolongation $\Delta\Sigma$, until which wave crests can rise.

Traditionally the problem about determination of this system of coordinate functions was identified with the classical problem about determination of normal frequencies and modes of oscillations of ideal liquid with a free surface in cavities of different geometrical shape. As it follows from the mentioned above analysis the system of coordinate functions for solving the nonlinear problem does not coincide with normal modes of oscillations, since it must supplementary satisfy non-flowing conditions on $\Delta\Sigma$. At the same time the problem about determination of normal frequencies and modes of oscillations of liquid with a free surface has independent theoretical and applied significance. First of all namely on the basis of this problem calculation of normal frequencies of oscillations of a free surface of liquid and hydrodynamic coefficients of the motion equations is done. Here these coefficients (the so-called virtual masses) are expressed in an open form as quadratures of normal modes of oscillations. Normal frequencies and virtual masses of liquid are the basic parameters for construction of linear dynamic systems for control of bodies with liquid. At present different methods for determination of normal frequencies and modes of oscillations of liquid in reservoirs are developed [4], which give suitable results for practice. Practically all these approaches use the traditional boundary eigenvalue problem

$$\Delta\varphi = 0 \text{ in } \tau_0; \quad \frac{\partial\varphi}{\partial n} = 0 \text{ on } \Sigma_0; \quad \frac{\partial\varphi}{\partial z} = \lambda\varphi \text{ on } S_0, \tag{11}$$

or its variational analog

$$\delta I = 0, \quad \text{where} \quad I = \int\limits_{\tau_0} (\boldsymbol{\nabla}\varphi)^2 d\tau - \lambda \int\limits_{S_0} \varphi^2 dS. \tag{12}$$

As it was shown in [4], solution of the problem (11) or its variation analog (12) contains analytical singularity on the contour L_0. The presence of this singularity mainly clarifies sufficiently rapid manifestation of numerical instability on increase of the number of coordinate functions, which approximate a solution of the problem (11). Preliminary analytical isolation of the singularity on the contour L_0 and further search of the problem solution in the form of sum of singular and regular components essentially complicates the algorithm of problem solving and reduces its practical applicability. The presence of this mathematical singularity is predetermined by existence of angular on the contour L_0 and alteration of the form of boundary condition in a vicinity of L_0. Partial reason of manifestation of this singularity is caused by linearization of the problem, as a result of which certain mechanical contradiction appears, i.e., the problem is solved for the immutable domain τ_0, however, for points of boundary S_0 of the domain τ_0 non-zero normal velocities are possible. This contradiction is absent in the nonlinear statement of the problem, where singular properties of the solution are manifested weaker.

In connection with development of analytical and numerical-analytical methods for solving nonlinear problems of dynamics of bodies with liquid, where application of normal modes of oscillations as coordinate functions is assumed, solutions of the problem similar to (11) must meet requirements of not only integral character (as in the case of the linear theory), but of differential character too. Sense of supplementary requirements consists in the property that the non-flowing boundary condition on $\Delta\Sigma$, i.e., on prolongation of the surface Σ_0 outside the domain τ, must hold. Practically this must be provided in the mentioned methods of solving of nonlinear problems by existence and vanishing derivatives of the following type $\dfrac{\partial^{k+1}\varphi}{\partial n \partial l^k}$, where $\boldsymbol{l}$ is the unit tangent vector to surface Σ_0 on angular contour ($k = 1, 2, \ldots$ depend upon order of maximally taken nonlinearities on modeling of motion of liquid with a free surface).

It is evident that in the general case solutions of the problem (11) does not hold this requirements, since the initial statement of the boundary problem contains only conditions with differential operators of the first order. In this connection it is expedient to certain extent to refuse from the traditional problem of determination of normal frequencies and modes and to construct approximately the system of coordinate functions $\overline{\psi}_i$, which is close to solutions of the problem (11) ψ_i with correspondingly close parameters $\overline{\lambda}_i$ and λ_i, but which in addition holds with high accuracy non-flowing condition on Σ_0 and on the surface $\Delta\Sigma$.

For realization of this goal we suggest two techniques, i.e., successive refinement of solution of the problem (11) and the method of auxiliary domain for reduction of influence of singular points on behavior of the solution.

As it is known, success of application of direct methods of solving of variational problems depends essentially on properties of coordinate functions. However, in most cases it is not possible to construct efficient coordinate functions, which hold exactly in advance a part of conditions of the problem (11). Therefore researchers mainly use the harmonic polynomials $w_k^{(m)}$ for solving problems for different simply connected domains as coordinate functions, which do not hold specificity of geometrical shape of a cavity at all.

For partial account of geometrical and physical characteristics of normal modes (desired solution) we suggest the following algorithm of successive refinement of coordinate functions. After solving the variational problem (12) and the algebraic eigenvalue problem we propose to select the obtained eigenfunctions (normal modes of oscillations of a free surface of liquid) as new coordinate functions and repeat the procedure of solving of the problem (12). In the case of necessity this step can be repeated. However, for next step of successive refinement of coordinate functions it is not necessary to calculate quadratures again, since the above described techniques is equivalent to the following calculation scheme.

Let

$$(A - \lambda B)\, x = 0 \tag{13}$$

be the algebraic eigenvalue problem, which is obtained by numerical realization of the variational problem (12) and $\mathcal{D}$ be the matrix composed of eigenvectors, i.e., solutions of the algebraic problem (13). Then, one step of the above described successive process is equivalent to transition from the problem (13) to the problem

$$[\mathcal{D}^T (A - \lambda B)\mathcal{D}]\, x = 0. \tag{14}$$

Moreover, with considering non-degeneracy of the matrix $\mathcal{D}$ (otherwise solutions will be dependent) eigenvalues of the problems (13) and (14) coincide. Since solutions of the problem (11) possess property of orthogonality, then the matrix $B^* = \mathcal{D}^T B \mathcal{D}$ tends to a diagonally scattered structure (non-zero elements are located only on the main diagonal, part of rows are completely zero). This property of the matrix B^* creates additional preconditions for more precise numerical solving the problem (13).

Actually efficiency of the described technique is based on imperfections of the numerical solution of (13). Moreover, advantages of this approach manifest vividly on tending to boundary of divergence of calculation procedure of the problem (12) on increase of the number of coordinate functions. Thus, in the case of solving the problem about oscillations of liquid with a free surface in spherical cavity with $H = 0.5R$ for $N = 12$ coordinate functions with the given calculation accuracy $\varepsilon = 10^{-6}$ and for single iteration we obtain results, which are practically coincide with solutions of the problem with $\varepsilon = 10^{-13}$ without iterations. However, for $\varepsilon = 10^{-19}$ successive refinement of solving the problem (13) for $N = 12$ is not practically manifested. This technique becomes apparent most efficiently for $23 \leqslant N \leqslant 28$, when for filling levels of liquid close to $H = R$ calculation convergence of solutions of the problem (13) appears. In this case results of calculations testify the described iteration procedure possesses properties of contracting mapping for errors of solving the problem (13), i.e., for certain parameters after reaching calculation instability on the initial step as early as on the first step of the iteration process recovery to domain of stable calculation takes place. Results of numerical experiments in the domain of convergence of calculations ($23 \leqslant N \leqslant 28$ for sphere) application of the iterative scheme makes it possible to improve the solution of the problem (13), obtained according to the classical scheme 1.5 –2 times, and on distance from a free surface it is improved 3 –10 times (in some cases 50 times). Let us note that property of solution continuability along the surface $\Delta\Sigma$, as a rule, become worst after application of this iteration procedure.

For the purpose of construction of coordinate functions for solving the nonlinear problem of dynamics of liquid with a free surface we apply the following technique for reduction of influence of singularity on angular contour on character of behavior of the solution. Numerical experiments showed that solving the problem by the variational method with application of regular coordinate functions presence of singularity in statement of the boundary problem (11) becomes apparent in poor convergence, which further results in instability of numerical procedure.

Hence, we propose to solve the problem about determination of coordinate functions close to normal modes of oscillations, but having the mentioned above supplementary properties in the following way. We solve the problem (12) for the domain $\tau + \Delta\tau$, where $\delta\tau$ is selected on the basis of requirements for validity range of the nonlinear theory for desirable boundaries of satisfying of non-flowing condition of wave crests on $\Delta\Sigma$. Here singularities in the constructed solution become apparent in a vicinity of the point A'. However, in a vicinity of the point A, which is an inner point of the surface $\Sigma + \Delta\Sigma$ domain for the domain $\tau + \Delta\tau$, the solution possesses regular properties. Next important stage consists in method of projection of the obtained values of ψ_i' and λ_i' onto the surface S_0, i.e., with reference to the domain τ. If we use regular solutions ψ_i' for domain τ' as coordinate functions of the problem (12) for the domain τ, then singularities in the problem statement (12) result in weak convergence for the domain $\Delta\Sigma$ including the point A. Therefore, we propose to consider values ψ_i' on the surface S_0 as $\overline{\psi}_i$. Moreover, as numerical experiments showed according to character of behavior $\overline{\psi}_i$ are close to ψ_i' and ψ_i, and eigenvalues, determined for $\overline{\psi}_i$ by the Rayleigh method, differ from λ_i very slightly. We named this technique for determination of coordinate functions, which partially hold requirements of nonlinear statement of the problem (liquid traces tank walls above level of undisturbed free surface) as method of auxiliary domain.

Table 1.

Sphere $H = 0.5R$, $m = 1$			
Variant	*1*	*2*	*3*
δ_c	$1.6 \cdot 10^{-4}$	$7.4 \cdot 10^{-5}$	$6.1 \cdot 10^{-5}$
δ_b	0.172	4.64	$0.956 \cdot 10^{-3}$
λ	1.20772	1.20772	1.20782

Sphere $H = 0.5R$, $m = 2$			
Variant	*1*	*2*	*3*
δ_c	$4.7 \cdot 10^{-5}$	$7.2 \cdot 10^{-5}$	$3.2 \cdot 10^{-5}$
δ_b	0.212	0.942	$1.4 \cdot 10^{-3}$
λ	2.30753	2.30753	2.30781

Let us show application of the methods of successive refinement and auxiliary domains by the numerical examples (Table 1). We consider spherical tank filled by liquid with level $H = 0,5R$. Let us analyze results of determination of the first normal modes of oscillations ($k = 1$) for peripheral numbers $m = 1$ and $m = 2$. The solution of the problem was constructed on the basis of decompositions by $N = 28$ harmonic polynomials. The algebraic eigenvalue problem was solved for $\varepsilon = 10^{-19}$. Here λ are eigenvalues, δ_c is ratio error of realization of non-flowing condition at angular point, δ_b is ratio error of realization of non-flowing condition at the point $\Delta z = 0,2R$ higher the level of a free surface of liquid. Ratio error was calculated according to the following formula

$$\delta = \frac{\partial \varphi}{\partial n}\Big|_{\Sigma} \Big/ \max \frac{\partial \varphi}{\partial n}\Big|_{S_0}.$$

The variant *1* corresponds to classical realization of the variational method, the variant *2* corresponds to the method of successive refinement and the variant *3* corresponds to the method of an auxiliary domain applied in the aggregate with the method of successive refinement.

As it follows from Table 1 usage of the method of successive refinement does not influence frequency, and usage of the method of an auxiliary domain for $\Delta z = \Delta H = 0.2R$ raises the frequency lesser than 10^{-4}. Moreover, in the case $\Delta H = R$; $\lambda = 1.210119$, i.e., distinction by frequency does not exceed 0.2%. This shows potential of approximate determination of frequencies in tanks of revolution according to the following scheme: we solve the problem for maximally required filling level with improved determination of normal modes; then, by known eigenfunctions for the maximal level we determine the frequency for arbitrary intermediate level on the basis of the the Rayleigh method, i.e., by calculation of two quadratures from similar coordinate function for the maximal level taken on cross-section, which corresponds to the required free surface. Here coordinate functions must satisfy non-flowing condition on Σ_0 with high accuracy.

On the basis of analysis of errors of realization of non-flowing conditions, presented in Table 1, it is possible to note that in the case of the classical method and the method of successive refinement behavior of solution above a free surface is unsatisfactory from the point of view of potentials of application of this solution as coordinate functions for solving nonlinear problem. This is perfectly explicable, since statement of the boundary problem (11) obtained on the basis of the linear theory and its hypotheses does not impose restrictions on character of behavior of solution above a free surface of liquid. The solution obtained by the method of auxiliary domain with the acceptable accuracy "follows" the contour above free surface of liquid and values of deviation δ on Σ_0 also decrease considerably.

We analyzed behavior of solution on $\Delta\Sigma$, obtained for the classical method and for the method of auxiliary domain. In the case of the spherical tank with liquid filling $H = 0.5R$ we take into consideration $N = 22$ functions $w_k^{(m)}$ [4] and specify auxiliary domain with $\Delta H = 0.25R$. The law of alteration of ratio error δ of non-flowing condition on Σ_0 for the mode $m = 1$, $k = 1$ is such, that δ reaches the maximum in a vicinity of L_0, i.e., errors on the contour of the generatrix are much smaller than at the corner point. Lower the undisturbed free surface of liquid errors, obtained on the basis of the classical method and the method of auxiliary domain, are approximately of the same order. However, behavior of the solutions on $\Delta\Sigma$ differ fundamentally. On the upper boundary $\Delta\Sigma$ errors δ for the classical method are much greater than by the method of an auxiliary domain (in 800 times for the mode $m = 1$; $k = 1$ and in 1800 times for the mode $m = 2$; $k = 1$), in spite of closeness of both solutions by frequency parameter and values of δ on Σ_0. It is worthy to notice that as a result of usage of procedure of projection of function ψ_i' onto S_0 the property of orthogonality of functions $\bar{\psi}_i$, which correspond to same m and different k, is violated.

Closeness of the obtained coordinate functions and the corresponding frequencies to normal modes and normal frequencies of the problem (11) is fundamentally significant for usage of these methods for solving nonlinear problems due to widely used division of normal modes of liquid oscillations into classes [2]. This conventional division is realized on the basis of mechanical analysis of potential contribution of coordinate functions into formation of wave processes with taking into account values of the corresponding frequencies. Later on this division is used for introduction of hypothesis about ways of simulation of nonlinear terms for amplitudes, which correspond to different normal modes. Efficiency of this techniques was confirmed by the example of cylindrical tank in [2].

In contrast to the classical approach, when the problem statement does not contain information about character of variation of the surface $\Delta\Sigma$ above the level of the undisturbed free surface, the method of an auxiliary domain reflects dependence of the solution of the problem about determination of frequencies and coordinate functions on geometrical characteristics of $\Delta\Sigma$.

The suggested methods of successive refinement of coordinate functions and the method of auxiliary domain according to their essence are engineering approaches for construction of coordinate functions, which do not possess mathematical stringency. Simplicity of numerical realization of the described approaches for arbitrary cavities of revolution, their mechanical clearness and obtaining of good final results of satisfying non-flowing condition much accurate than by classical methods of solving the problem (11) make it possible to recommend the proposed methods for construction of coordinate functions of nonlinear problem and for determination of reductive dependence of the frequency on filling depth.

We suggest the following algorithm for implementation of the described techniques of improved determination of coordinate functions close to normal modes of oscillation of liquid in cavity of revolution.

1. The initial eigenvalue problem was solved by the method of an auxiliary domain.
2. Refinement of the problem solution was realized according to the successive approach.
3. In the case, when into the system of coordinate function we include several functions with the same circular number, then we produced their orthogonalization (functions with different circular numbers are automatically orthogonal).
4. For each function we determine the frequency parameter by the Rayleigh method (these parameters are mainly used for verification and estimation of frequency characteristics).

For solving the nonlinear problem of dynamics of combined motion of a reservoir and liquid, which partially fills its cavity of revolution, we applied the following discretization parameters $n_1 = 10$; $n_2 = 6$; $n_3 = 3$. Moreover, the coordinate functions are arranged in the following way

$$\begin{gathered}\psi_1 = \psi^*_{1,1}\sin\theta; \quad \psi_2 = \psi^*_{1,1}\cos\theta; \quad \psi_3 = \psi^*_{0,1};\\ \psi_4 = \psi^*_{2,1}\sin 2\theta; \quad \psi_5 = \psi^*_{2,1}\cos 2\theta; \quad \psi_6 = \psi^*_{0,2};\\ \psi_7 = \psi^*_{3,1}\sin 3\theta; \quad \psi_8 = \psi^*_{3,1}\sin 3\theta;\\ \psi_9 = \psi^*_{1,2}\sin\theta; \quad \psi_{10} = \psi^*_{1,2}\cos\theta, \end{gathered} \tag{15}$$

where $\psi^*_{m,k}$ is solution of the problem about refined determination of coordinate functions closed to normal modes of oscillations for the circular number m, to which the k-th eigenvalue corresponds (if eigenvalues are put in ascending order).

Results of numerical determination of coordinate functions (15) for cavities of spherical, parabolic, conic and cylindrical (for verification) shapes showed that this procedure gives errors about 10^{-5} on tank walls below undisturbed free surface of liquid and errors about 10^{-5} on tank walls above undisturbed free surface of liquid. So, all coordinate functions with sufficiently high accuracy “follow” the profile of $\Delta\Sigma$. Moreover, it is necessary to note that on solving the nonlinear problem amplitudes of oscitation of normal modes are about 0.1–0.3, which reduce 3–10 times errors of non-flowing condition on the surface $\Sigma + \Delta\Sigma$.

The suggested technique of determination of coordinate functions is based on solving of the linear problem, but it is supplemented by several requirements, which are outside scope of linear statement of the problem and reflect a part of kinematic requirements and solvability conditions of the nonlinear problem. Finally this makes it possible to construct the system of coordinate functions, which in improved way holds the non-flowing condition on the perturbed moisten boundary of the domain occupied by liquid. Further this system of functions was successfully used for solving of nonlinear problem.

5. Construction of decompositions of variables (nonlinear requirements)

Let us construct decompositions of desired variables, which hold nonlinear kinematic boundary conditions. Let us select decompositions of the desired variables of excitation of a free surface of liquid ξ and the velocity potential Φ in the following form

$$\xi = \bar{\xi}(t) + \sum_i a_i \bar{\psi}_i(\alpha) T_i(\theta); \quad \Phi = \dot{\boldsymbol{\varepsilon}} \cdot \boldsymbol{r} + \varphi_0;$$

$$\varphi_0 = \sum b_i \psi_i(\alpha, \beta) T_i(\theta), \tag{16}$$

where

$$\bar{\psi}_i(\alpha) = \frac{\partial \psi_i}{\partial z}\Big|_{\beta=0}. \tag{17}$$

Due to specificity of a cavity shape we separate the circular coordinate in the decompositions (16). Here $T_i(\theta)$ are trigonometric functions, which selection is predetermined by distribution of functions (15). The functions $\psi_i(\alpha, \beta)$ are constructed by the described above technique, therefore, they are harmonic and satisfy with high accuracy the non-flowing conditions on the moisten border Σ including a part of this domain, where waves crests can reach above the level of the unperturbed free surface. The functions $\bar{\psi}_i(\alpha)$ possess the property of completeness on S_0. We note also that in contrast to the case of cylindrical domains decompositions (16) contain the term $\bar{\xi}(t)$, which is determined from the requirement of conservation of the liquid volume in its perturbed motion.

We consider amplitude parameters a_i as independent variables. Therefore, we shall determine interdependence $b_i = b_i(a_j, \dot{a}_k)$ from the ki8nematic boundary condition on a free surface (9), and the dependence $\overline{\xi} = \overline{\xi}(a_i, t)$ from the requirement of conservation of the liquid volume (in the case of absence of liquid outflow this dependence transforms into the form $\overline{\xi} = \overline{\xi}(a_i)$). To this end we represent $\overline{\xi}$ and b_i as

$$\overline{\xi} = \overline{\xi}^{(1)} + \overline{\xi}^{(2)} + \overline{\xi}^{(3)}; \quad b_i = b_i^{(1)} + b_i^{(2)} + b_i^{(3)} + b_i^{(4)}, \tag{18}$$

where upper indexes correspond to the order of smallness of values, if we accept the order of a_i as a small value.

On computation of variation of the liquid volume in its perturbed motion we make use of the above introduced non-Cartesian parameterization α, θ, β

$$\Delta V = \int\limits_{\tau-\tau_0} d\tau = \int\limits_0^{2\pi}\int\limits_0^1\int\limits_{-1}^{\xi/H} [f(H\beta)]^2 d\beta \alpha H \, d\alpha \, d\theta - \int\limits_0^{2\pi}\int\limits_0^1\int\limits_{-1}^{0} [f(H\beta)]^2 d\beta \alpha H \, d\alpha \, d\theta =$$
$$= \int\limits_0^{2\pi}\int\limits_0^1 \left[\int\limits_{-1}^{\xi/H} f^2(H\beta) \; d\beta \right] \alpha H d\alpha \, d\theta.$$

For calculation of the last integral we make use of the formula for integration of expressions over variable volume, which is based on application of the Taylor expansion of the integral with variable limits of integration. After realization of this type of integration in the analytical form and substitution of decompositions for ξ from requirements of volume conservation we obtain the following expressions for $\overline{\xi}^{(i)}$:

$$\overline{\xi}^{(1)} = 0; \quad \overline{\xi}^{(2)} = -\frac{f'(0)}{\pi f(0)} \sum_{i,j} a_i a_j \beta_{ij}^v;$$

$$\overline{\xi}^{(3)} = -\frac{f'^2(0) + f(0)f''(0)}{3\pi f^2(0)} \sum_{i,j,k} a_i a_j a_k \gamma_{ijk}^v. \tag{19}$$

As it is seen from the relations (19), the requirement of volume conservation in amplitude parameters represents certain holonomic constraint, which makes it possible to eliminate parameters $\overline{\xi}$. Moreover, linear term in representation of $\overline{\xi}$ is absent. Character of variation of coefficients β_{ij}^v (a positively defined matrix for non-orthogonalized system and a diagonal one with positive elements for orthogonal system) shows that $\overline{\xi}^{(2)}$ is a function of constant sign and its sign is opposite to sign of $f'(0)$, i.e., $\overline{\xi}_2$ is determined by amplitudes a_i^2 and inclination angle of tank walls in a vicinity of a free surface. The value $\overline{\xi}^{(3)}$ depends on square of inclination of tank walls and curvature of the surface Σ_0 in a vicinity of L_0.

Realization of the procedure of elimination of the kinematic boundary condition on a free surface is similar to the procedure for a cylindrical tank [2]. Distinction of this procedure consists in the property that in derivations it is necessary to keep terms $\overline{\xi}(a_i)$ and take into account additional terms of (9), which appear due to non-cylindrical shape of the tank, and produce decompositions into series with respect to ξ of not only unknown functions, but their products with the Jacobian of transformation to non-Cartesian parametrization, which is also a function of β.

By omitting intermediate derivations we write down dependencies $b_i = b_i(a_j, \dot{a}_k)$, which are similar to the case of cylindrical cavity represent non-holonomic constraints. However, by application of methods of nonlinear mechanics and the Galerkin method they admit elimination of dependent parameters b_i

$$b_p^{(1)} = \dot{a}_p; \quad b_p^{(2)} = \sum_{i,j} \dot{a}_i a_j \gamma_{ijk}^c;$$

$$b_p^{(3)} = \sum_{i,j,k} \dot{a}_i a_j a_k \delta_{ijkp}^c; \quad b_p^{(4)} = \sum_{i,j,k,l} \dot{a}_i a_j a_k a_l h_{ijklp}^c. \tag{20}$$

The relations (19) and (20) include different coefficients, which are determined by quadratures from functions ψ_i and T_i over the unknown free surface of liquid S_0. Their explicit form will be given below.

After determination of dependencies (19) and (20) parameters a_i can be supposed as the complete independent system of variables, which characterizes motion of the limited volume of liquid. Now we can pass to immediate realization of the Hamilton – Ostrogradsky variational principle for unconstrained mechanical system, which motion is set by the generalized coordinates a_i and ε_k.

6. Construction of a resolving system of motion equations

We shall derive the motion equations of the system on the basis of the Hamilton-Ostrogradsky variational principle applied to dynamics of bounded liquid volume and a rigid body with cavity of revolution in their combined motion. For transition from continuum structure of the initial model of the system rigid body – liquid to its discrete model we make use of the Kantorovich method. Here spatial and surface integration in separate terms of the Lagrange function is realized in variables α, θ, β. We note that calculation of volumetric integrals in variables α, θ, β can be reduced to successive integration with analytical derivation of the integral over liquid depth.

Finally this volumetric integrals are reduced to surface integrals over the undisturbed free surface S_0 from expressions in the form of expansions relative to powers of ξ. In contrast to the case of the cylindrical cavity occupied by liquid not only terms, which contain the velocity potential φ_0 are differentiated by β, but also terms, which contain products of the velocity potential and the Jacobian of transition to non-Cartesian parametrisation of the liquid domain τ_0.

On the whole the general procedure of transition from the continuum structure of the initial mechanical system body – liquid to its discrete model (it is based on application of the Kantorovich method to the variational formulation of the problem in the form of the Hamilton-Ostrogradsky variational principle) differs insignificantly from the case of the cylindrical domain occupied by liquid [2]. Therefore, we do not describe this procedure in details. Main distinction consists not in the techniques of such a transition, but in the following two properties. Namely, first, decomposition of the velocity potential relative to coordinate function is realized with respect to the system, which holds the non-flowing condition approximately, second, a group of geometrical nonlinearities defined by the relation (19) appeared, which predetermine additional dependence of all normal modes of oscillations.

As the result of application of the proposed technique we obtain the following Lagrange equation of the second kind, i.e., the motion equations of the system body – liquid in amplitude parameters a_i and parameters of translational motion of the carrying body ε_i

$$\sum_i \ddot{a}_i\left(V^1_{ir}+\sum_j a_j V^2_{irj}+\sum_{j,k} a_j a_k V^3_{irjk}\right)+\ddot{\boldsymbol{\varepsilon}}\cdot\left(\boldsymbol{U}^1_r+\sum_i a_i\boldsymbol{U}^2_{ri}+\sum_{j,k} a_i a_j\boldsymbol{U}^3_{rij}+\sum_{i,j,k} a_i a_j a_k\boldsymbol{U}^4_{rijk}\right)=$$
$$=\sum_{i,j}\dot{a}_i\dot{a}_j V^{2*}_{ijr}+\sum_{i,j,k}\dot{a}_i\dot{a}_j a_k V^{3*}_{ijrk}+\dot{\boldsymbol{\varepsilon}}\cdot\left(\sum_i \dot{a}_i\boldsymbol{U}^{2*}_{ir}+\sum_{i,j}\dot{a}_i a_j\boldsymbol{U}^{3*}_{ijr}+\sum_{i,j,k}\dot{a}_i a_j a_k\boldsymbol{U}^{4*}_{ijkr}\right)-$$
$$-g\left(\sum_i a_i W^2_{ir}+\frac{3}{2}\sum_{i,j} a_i a_j W^3_{ijr}+2\sum_{i,j,k} a_i a_j a_k W^4_{ijkr}\right),\qquad r=1,2,\ldots,N, \quad (21)$$

$$\frac{\rho}{M_{\mathrm{r}}+M_{\mathrm{l}}}\sum_i \ddot{a}_i\left(\boldsymbol{U}^1_i+\sum_j a_j\boldsymbol{U}^2_{ij}+\sum_{j,k} a_j a_k\boldsymbol{U}^3_{ijk}\right)+\ddot{\boldsymbol{\varepsilon}}=$$
$$=\frac{\boldsymbol{F}}{M_{\mathrm{r}}+M_{\mathrm{l}}}+\boldsymbol{g}-\frac{\rho}{M_{\mathrm{r}}+M_{\mathrm{l}}}\sum_{i,j}\dot{a}_i\dot{a}_j\left(\boldsymbol{U}^2_{ij}+2\sum_k a_k\boldsymbol{U}^3_{ijk}\right). \quad (22)$$

The mentioned system of equation represents the nonlinear model of dynamics of tank and liquid, which partially fills its cavity, in the case of translational motion of the carrying body. It is necessary to pay attention on terms with gravity forces in the latter line of the equations (22). In contrast to the case of cylindrical tank here nonlinear terms, caused by geometrical factors manifest. Similar to geometrical nonlinearities in (16), (19) these nonlinearities depend also on inclination and curvature of tank walls in close vicinity of $\beta=0$, i.e. intersection of tank walls with unperturbed free surface of liquid, which is clear from values of the coefficients W^3_{ijk} and W^4_{ijkl}, adduced below. For construction of the system of equations (21), (22) it is necessary to compute the following quadratures from the coordinate functions ψ_i and $\overline{\psi}_i$:

$$N_p=\int_0^{2\pi}\int_0^1 \overline{\psi}_i^2 T_i^2\alpha d\alpha\, d\theta;\quad \beta^v_{ij}=N_i\delta_{ij};\quad \gamma^v_{ijk}=\int_0^{2\pi}\int_0^1 \overline{\psi}_i\overline{\psi}_j\overline{\psi}_k T_iT_jT_k\alpha\, d\alpha\, d\theta;$$

$$\delta^v_{ijkl}=\int_0^{2\pi}\int_0^1 \overline{\psi}_i\overline{\psi}_j\overline{\psi}_k\psi_l T_iT_jT_kT_l\alpha\, d\alpha\, d\theta;$$

$$\gamma^{b0}_{ijp}=\int_0^{2\pi}\int_0^1\left[\left(\alpha A^0_{1i}\frac{\partial\overline{\psi}_j}{\partial\alpha}+\ \alpha^2A^0_{3i}\frac{\partial\overline{\psi}_j}{\partial\alpha}-\alpha A^1_{4i}\overline{\psi}_j\right)T_iT_j+\frac{1}{\alpha}A^0_{2i}\overline{\psi}_jT_i'T_j'\right]\overline{\psi}_pT_p\, d\alpha d\theta;$$

$$\gamma_{ijp}^{b1}=\int\limits_0^{2\pi}\int\limits_0^1\left[\left(\alpha A_{1i}^1\frac{\partial\overline{\psi}_j}{\partial\alpha}+\alpha^2A_{3i}^1\frac{\partial\overline{\psi}_j}{\partial\alpha}-\alpha A_{4i}^2\overline{\psi}_j\right)T_iT_j+\frac{1}{\alpha}A_{2i}^1\overline{\psi}_jT_i'T_j'\right]\overline{\psi}_pT_pd\alpha\,d\theta;$$

$$\delta_{ijkp}^{b1}=\int\limits_0^{2\pi}\int\limits_0^1\left[\left(A_{1i}^1\alpha\frac{\partial\overline{\psi}_j}{\partial\alpha}+\ +\alpha^2A_{3i}^1\frac{\partial\overline{\psi}_j}{\partial\alpha}-\alpha A_{4i}^2\overline{\psi}_j\right)T_iT_j+\frac{1}{\alpha}A_{2i}^1\overline{\psi}_jT_i'T_j'\right]\overline{\psi}_jk\overline{\psi}_pT_kT_pd\alpha\,d\theta;$$

$$h_{ijklp}^{b2}=\int\limits_0^{2\pi}\int\limits_0^1\left[\left(\alpha A_{1i}^2\frac{\partial\overline{\psi}_j}{\partial\alpha}+\alpha^2A_{3i}^2\frac{\partial\overline{\psi}_j}{\partial\alpha}-\alpha A_{4i}^3\overline{\psi}_j\right)T_iT_j+\frac{1}{\alpha}A_{2i}^2\overline{\psi}_jT_i'T_j'\right]\overline{\psi}_k\overline{\psi}_l\overline{\psi}_pT_kT_lT_p\,d\alpha d\theta;$$

$$\beta_{ip}^{b1}=\int\limits_0^{2\pi}\int\limits_0^1A_{4i}^1\overline{\psi}_pT_iT_p\alpha\,d\alpha d\beta;\quad \beta_{ij}^{f2}=\int\limits_0^{2\pi}\int\limits_0^1\overline{\psi}_i\overline{\psi}_jT_iT_j\alpha\,d\alpha d\theta f^2(0);$$

$$\beta_i^{f3}=\int\limits_0^{2\pi}\int\limits_0^1\left[\alpha\left(\frac{\partial\psi_j}{\partial\alpha}\right)^2T_i^2+\frac{1}{\alpha}\psi_i^2T_i'^2+\alpha\,f^2\overline{\psi}_i^2T_i^2\right]\Big|_{\beta=0}d\alpha d\theta;$$

$$\gamma_{ijk}^{f3}=\int\limits_0^{2\pi}\int\limits_0^1\left[\alpha\frac{\partial\psi_i}{\partial\alpha}\frac{\partial\psi_j}{\partial\alpha}T_iT_j+\frac{1}{\alpha}\psi_i\psi_jT_i'T_j'+\alpha\,f^2\overline{\psi}_i^2\overline{\psi}_jT_iT_j\right]\Big|_{\beta=0}\overline{\psi}_kT_kd\alpha d\theta;$$

$$\delta_{ijkl}^{f4}=\int\limits_0^{2\pi}\int\limits_0^1\left[\frac{\alpha}{2H}\left(\frac{\partial^2\psi_i}{\partial\alpha\partial\beta}\frac{\partial\psi_j}{\partial\beta}+\frac{\partial\psi_i}{\partial\alpha}\frac{\partial^2\psi_j}{\partial\alpha\partial\beta}\right)T_iT_j+\frac{1}{2\alpha H}\left(\frac{\partial\psi_i}{\partial\beta}\psi_j+\psi_i\frac{\partial\psi_j}{\partial\beta}\right)T_i'T_j'\right.$$

$$\left.+\frac{\alpha}{2H}\left(2Hff'\frac{\partial\psi_i}{\partial z}\frac{\partial\psi_j}{\partial z}+\ f^2\frac{\partial^2\psi_i}{\partial z\partial\beta}+f^2\frac{\partial\psi_i}{\partial z}\frac{\partial^2\psi_j}{\partial z\partial\beta}\right)T_iT_j\right]\Big|_{\beta=0}\overline{\psi}_k\overline{\psi}_lT_kT_ld\alpha d\theta;$$

$$\alpha_i^{x1}=\int\limits_0^{2\pi}\int\limits_0^1\alpha^2f\big|_{\beta=0}\overline{\psi}_iT_i\cos\theta d\alpha d\theta;\quad \alpha_i^{x2}=\int\limits_0^{2\pi}\int\limits_0^1\left(\alpha f\frac{\partial\psi_i}{\partial\alpha}T_i\cos\theta-f\psi_iT_i'\sin\theta\right)\Big|_{\beta=0}d\alpha d\theta;$$

$$\beta_{ij}^{x2}=\int\limits_0^{2\pi}\int\limits_0^1\left(\alpha f\frac{\partial\psi_i}{\partial\alpha}T_i\cos\theta-f\psi_iT_i'\sin\theta\right)\overline{\psi}_jT_j\,d\alpha d\theta;$$

$$\gamma_{ijk}^{x3}=\frac{1}{2H}\int\limits_0^{2\pi}\int\limits_0^1\left[\left(\alpha Hf'\frac{\partial\psi_i}{\partial\alpha}+\alpha f\frac{\partial^2\psi_i}{\partial\alpha\partial\beta}\right)T_i\cos\theta-\left(Hf'\psi_i+f\frac{\partial\psi_i}{\partial\beta}\right)T_i'\sin\theta\right]\Big|_{\beta=0}\overline{\psi}_j\overline{\psi}_kT_jT_kd\alpha d\theta;$$

$$\delta_{ijkl}^{x4}=\frac{1}{6H^2}\int\limits_0^{2\pi}\int\limits_0^1\left[\left(H^2f''\alpha\frac{\partial\psi_i}{\partial\alpha}+2Hf'\alpha\frac{\partial^2\psi_i}{\partial\alpha\partial\beta}\ +\alpha f\frac{\partial^3\psi_i}{\partial\alpha\partial\beta^2}\right)T_i\cos\theta-\right.$$

$$\left.-\left(H^2f''\psi_i+2Hf\frac{\partial\psi_i}{\partial\beta}+\ f\frac{\partial^2\psi_i}{\partial\beta^2}\right)T_i'\sin\theta\right]\Big|_{\beta=0}\overline{\psi}_j\overline{\psi}_k\overline{\psi}_pT_jT_kT_pd\alpha d\theta;$$

$$\beta_{ij}^{z2}=f^2(0)N_i\delta_{ij};\quad \gamma_{ijk}^{z3}=\frac{1}{2H}\int\limits_0^{2\pi}\int\limits_0^1\left[2ff'H\overline{\psi}_i+f^2\frac{\partial^2\psi_i}{\partial z\partial\beta}\right]\Big|_{\beta=0}\overline{\psi}_j\overline{\psi}_kT_iT_jT_k\alpha\,d\alpha d\theta;$$

$$\delta^{z4}_{ijkl} = \frac{1}{6H^2}\int\limits_0^{2\pi}\int\limits_0^{1}\left[4ff'Hf^2\frac{\partial^2\psi_i}{\partial z\partial\beta} + 2H^2\big(f'^2 + f''f\big)f^2\frac{\partial\psi_i}{\partial z} + f^2\frac{\partial^3\psi_i}{\partial z\partial\beta^2}\right]\Bigg|_{\beta=0}\overline{\psi}_j\overline{\psi}_k\overline{\psi}_k T_i T_j T_k T_l \alpha\, d\alpha d\theta. \tag{23}$$

We note that in the mentioned quadratures components of integration by α and θ are always separated, the y-components of vectors are calculated similar to x-components, but it is necessary to substitute $-\cos\theta$ instead of $\sin\theta$ and $\sin\theta$ instead of $\cos\theta$ (in relations (23) expressions for y-components are omitted). On calculation of integrals we use the following denotations δ_{ij} is the Kronecker symbol, $f^{(i)} = f^{(i)}(0)$, where i is the order of differentiation. Numerical algorithm for determination of quadratures was based on the Gauss quadrature formula with 96 points of partition. We introduce also the following functions of the argument α in the relations (23)

$$A^0_{1i} = \frac{1}{f^2}\overline{\psi}_i;\quad A^1_{1i} = \frac{1}{Hf^2}\left(\frac{\partial^2\psi_i}{\partial\alpha\partial\beta} - \frac{2Hf'}{f}\frac{\partial\psi_i}{\partial\alpha}\right)\Bigg|_{\beta=0};$$

$$A^2_{1i} = \frac{1}{2H^2f^2}\left[\frac{\partial^3\psi_i}{\partial\alpha\partial\beta^2} - \frac{4Hf'}{f}\frac{\partial^2\psi_i}{\partial\alpha\partial\beta} + H^2\left(6\frac{f'^2}{f^2} - 2\frac{f''}{f}\right)\frac{\partial\psi_i}{\partial\alpha}\right]\Bigg|_{\beta=0};$$

$$A^0_{2i} = \frac{1}{f^2}\psi_i|_{\beta=0};\quad A^1_{2i} = \frac{1}{Hf^2}\left(\frac{\partial\psi_i}{\partial\beta} - \frac{2Hf'}{f}\psi_i\right)\Bigg|_{\beta=0};$$

$$A^2_{2i} = \frac{1}{2H^2f^2}\left[\frac{\partial^2\psi_i}{\partial\beta^2} - 4H\frac{f'}{f}\frac{\partial\psi_i}{\partial\beta} + H^2\left(6\frac{f'^2}{f^2} - 2\frac{f''}{f}\right)\psi_i\right]\Bigg|_{\beta=0};$$

$$A^0_{3i} = -\frac{f'}{f}\overline{\psi}_i;\ A^1_{3i} = \frac{1}{Hf}\left[\left(-Hf'' + H\frac{f'^2}{f}\right)\frac{\partial\psi_i}{\partial z} - \frac{f'}{f}\frac{\partial^2\psi_i}{\partial z\partial\beta}\right]\Bigg|_{\beta=0};$$

$$A^2_{3i} = \frac{1}{2H^2}\left[\left(-H^2 + 3H^2\frac{f''f'}{f^2} - 2H^2\frac{f'^3}{f^3}\right)\frac{\partial\psi_i}{\partial z} + H\left(-\frac{f'}{f} + \frac{f'^2}{f^2}\right)\frac{\partial^2\psi_i}{\partial z\partial\beta} - \frac{f'}{f}\frac{\partial^3\psi_i}{\partial z\partial\beta^2}\right]\Bigg|_{\beta=0};$$

$$A^1_{4i} = \frac{1}{H}\frac{\partial^2\psi_i}{\partial z\partial\beta}\bigg|_{\beta=0};\quad A^2_{4i} = \frac{1}{2H^2}\frac{\partial^3\psi_i}{\partial z\partial\beta^2}\bigg|_{\beta=0};\quad A^3_{4i} = \frac{1}{6H^3}\frac{\partial^4\psi_i}{\partial z\partial\beta^3}\bigg|_{\beta=0}. \tag{24}$$

Similar to values of the coefficients (23) these expressions contain expressions f' and f''. taken for $\beta = 0$. So, these coefficients include geometrical nonlinearities mostly not in a pure form, but as mixture with physical nonlinearities. On the basis of the mentioned quadratures (23), which are calculated with application of denotations (24), the coefficients of the motion equations (21) and (22) are determined according to the following algorithm:

$$\gamma^c_{ijp} = \frac{1}{N_p}\gamma^{b0}_{ijp};\quad \delta^c_{ijkp} = \frac{1}{N_p}\left(\sum_i \gamma^c_{ijm}\gamma^{b0}_{mkp} + \delta^{b1}_{ijkp} + +\frac{g_2}{\pi g_1}\beta^{b1}_{ip}N_j\delta_{jk}\right);$$

$$h^c_{ijklp} = \frac{1}{N_p}\left[\sum_m\left(\delta^c_{ijkm}\gamma^{b0}_{mlp} + \delta^{b1}_{mlkp}\gamma^c_{ijm} + \frac{g_2}{\pi g_1}\gamma^c_{ijm}\beta^{b1}_{mp}N_k\delta_{kl}\right) + \beta^{b1}_{ip}\gamma^v_{jkl}\frac{g_3}{\pi g_1}\gamma^{b1}_{ijp}N_p\delta_{kl}\right];$$

$$V^1_{ij} = \beta^{f2}_{ij};\quad V^2_{ijk} = \gamma^{f3}_{ijk} + 2\sum_m \gamma^c_{ikm}\beta^{f2}_{mj};$$

$$V^3_{ijkl} = \delta^{f4}_{ijkl} - \frac{g^2}{\pi g_1}N_k\delta_{kl}\beta^{f3}_i\delta_{ij} + \sum_m\left(2\gamma^{f3}_{mjk}\gamma^c_{ilm} + 2\beta^{f2}_{mj}\delta^c_{iklm} + \gamma^c_{jkm}\gamma^c_{ilm}\beta^{f2}_{mm}\right);$$

$$\boldsymbol{U}_i^1 = \boldsymbol{\alpha}_i^1; \quad \boldsymbol{U}_{ij}^2 = \boldsymbol{\beta}_{ij}^2 + \sum_m \gamma_{ijm}^c \boldsymbol{\alpha}_m^1; \quad \boldsymbol{U}_{ijk}^3 = \boldsymbol{\gamma}_{ijk}^3 + \sum_m \left(\delta_{ijkm}^c \boldsymbol{\alpha}_m^1 + \gamma_{ijm}^c \boldsymbol{\beta}_{mk}^2 \right) - \frac{g_2}{\pi g_1} \boldsymbol{\alpha}_i^2 N_j \delta_{jk};$$

$$\boldsymbol{U}_{ijkl}^4 = \sum_m \left(h_{ijklm}^c \boldsymbol{\alpha}_m^1 - \frac{g_2}{\pi g_1} \gamma_{ijm}^c \boldsymbol{\alpha}_m^2 N_k \delta_{kl} + \delta_{ijkm}^c \boldsymbol{\beta}_{ml}^2 + \gamma_{ilm}^c \boldsymbol{\gamma}_{mjk}^3 \right) - \frac{g_3}{\pi g_1} \gamma_{jkl}^v \boldsymbol{\alpha}_i^2 + \boldsymbol{\delta}_{ijkl}^4;$$

$$W_{ij}^2 = e_1 N_i \delta_{ij}; \quad W_{ijk}^3 = e_2 N_i \gamma_{ijk}^v; \quad W_{ijkl}^4 = e_3 \delta_{ijkl}^v + N_i N_k \delta_{ij} \delta_{kl} \left(\frac{e_1 g_2^2}{\pi g_1^2} - \frac{3g^2 e_2}{\pi g_1} \right);$$

$$V_{ijr}^{2*} = \frac{1}{2} V_{ijr}^2 - V_{irj}^2; \quad V_{ijkr}^{3*} = V_{ijkr}^3 - 2V_{irjk}^3;$$

$$\boldsymbol{U}_{ir}^{2*} = \boldsymbol{U}_{ir}^2 - \boldsymbol{U}_{ri}^2; \quad \boldsymbol{U}_{ijr}^{3*} = 2\big(\boldsymbol{U}_{ijr}^3 - \boldsymbol{U}_{rij}^3\big); \quad \boldsymbol{U}_{ijkr}^{4*} = 3\big(\boldsymbol{U}_{ijkr}^4 - \boldsymbol{U}_{rijk}^4\big);$$

$$e_1 = f^2; \quad e_2 = \frac{4}{3} f' f; \quad e_3 = \frac{1}{2}(f'' f + f'^2); \quad g_1 = \pi e_1; \quad g_2 = \pi f f'; g_3 = \frac{\pi}{3}\left(f'^2 + f f''\right). \tag{25}$$

Basic stages of numerical realization of the suggested approach mainly coincide with the case of cylindrical tank. However, distinction consists in the property that approximate determination of coordinate functions close to normal modes of oscillations becomes significant component of the procedure. For determination of parameters of an auxiliary domain $\Delta\tau$ we conventionally select $\Delta z = 0.2$, i.e., excess of the filling level of liquid z related to the radius of the undisturbed free surface. This parameter defines the domain $\Delta\tau$ and the surface $\Delta\Sigma$ on application of the method of an auxiliary domain (for comparison we consider also variants $\Delta z = 0$; $\Delta z = 0.15$; $\Delta z = 0.2$).

We consider the problem about determination of coefficients of the motion equations for nonlinear oscillation of liquid in circular cylindrical tank as a testing one. Good concordance with results of publications [2] was obtained. Coefficients of the motion equations coincide accurate to four significant digits.

7. Numerical examples

We consider absolutely rigid cylindrical tank, inverse conic tank with right cone angle and paraboloidal tank, which is selected such that near undisturbed free surface inclination of tank walls coincides with the case of conic reservoir. Tanks are filled by ideal incompressible homogeneous liquid. Initially system is at rest state. According to the Lagrange theorem its further motion will be vortex free. We consider that reservoir performs only horizontal motion. Motion is caused by force $F_x = k \cdot (M_\mathrm{r} + M_\mathrm{l})$, applied to tank in horizontal direction only during time $t_0 = 1$ (so force has shape of rectangular impulse), where M_l and M_r are masses of liquid and tank correspondingly, k is force factor, which is selected such that liquid oscillations hit nonlinear range of amplitudes of oscillations.

We use the above considered model for determination of elevation of a liquid free surface on tank wall for the same problem of force disturbance of conic, paraboloidal and cylindrical tanks. For numerical examples we make use of the following parameters of the system and its loading: $H = R$, where H is height of liquid, $R = f(0)$ is radius of undisturbed free surface, in all cases we consider $R = 1$ m, $M_\mathrm{r} = 0.2M_\mathrm{l}$, $t_0 = 1$ s, $k = 1$ for conic and cylindrical tanks and $k = 0.6$ for paraboloidal tank. So, we analyze system behavior on active stage of its motion and on interval of system motion by inertia.

The suggested approach makes it possible to determine elevation of a free surface of liquid at every point of tank cross-section, variation in time of every amplitude of oscillation of a free surface a_i, amplitude parameters of translational motion of tank ε_i and liquid force response on tank walls.

Figure 1 shows variation in time of elevation of liquid on tank wall on time interval. Here and after thick solid line corresponds to conic tank, dashed line corresponds to cylindrical tank and thin solid line corresponds to the paraboloidal reservoir. For every graph we consider time interval, which corresponds approximately to 5 periods of normal oscillations, and exceeds considerably time of loading action. Time is given in seconds.

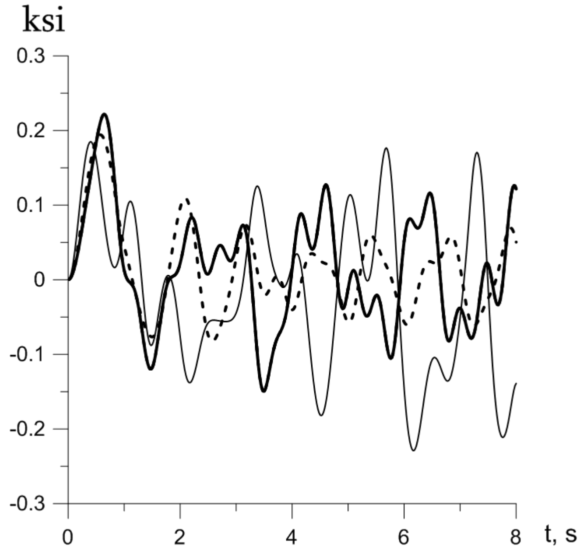

Fig. 1. Variation in time of elevation of liquid on tank wall.

As it is seen from Fig. 1, elevation of a free surface of liquid for conic tank exceeds elevation of a free surface for cylindrical tank especially for liquid motion after removing external loading. On interval of transient mode of system motion this difference is weaker, but it is present also. If we analyze law of variation of these two curves, we can draw conclusion that for conic tank the presence of higher normal modes is considerable. If we consider the paraboloidal tank, we see that initially on active stage of system motion liquid is entrained into notion to lesser extent, however, later on the stage of system motion by inertia amplitudes of oscillations become higher than in the case of cylindrical and conic tanks. If we compare the cases of conic and paraboloidal reservoirs, we can conclude that the presence of high modes of oscillations is also manifested, but it is weaker for paraboloidal reservoir than for conic reservoir. So, this example shows that curvature promotes reduction of excitation of amplitudes of high modes of oscillations.

To analyze nature of this effect let us show variation in time of the first antisymmetric (Fig. 2) and the first axisymmetric (Fig. 3) normal modes. As it is seen from figures difference of amplitudes of the first antisymmetric normal modes is weak, but they differ in frequencies of variation and only in the case of paraboloidal tank influence of high frequency normal modes takes place. It is necessary note also that due to combined character of motion redistribution of energy between translation motion of the system and energy of liquid sloshing occurs. So, for cylindrical and conic tanks we obtain decrease of amplitudes of liquid oscillations. However in the case of paraboloidal reservoir this effect is not manifested clearly.

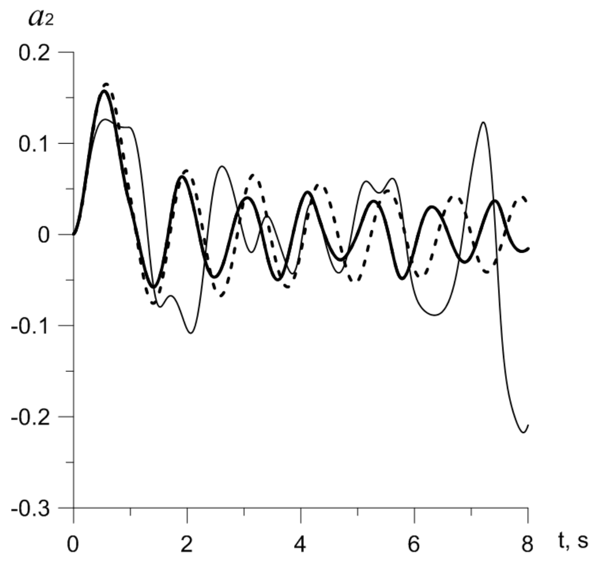

Fig. 2. Variation in time of amplitude of the first antisymmetric normal mode of oscillations of free surface of liquid.

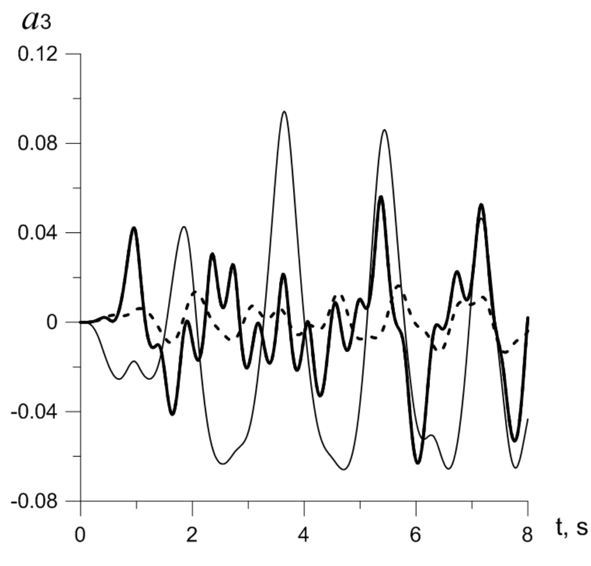

Fig. 3. Variation in time of amplitude of the first axisymmetric normal mode of oscillations of free surface of liquid.

If we analyze variation in time of the first axisymmetric normal modes for these three shapes of tanks, we can conclude that this difference is strong both in variation of amplitudes and frequencies. It is necessary to note that in contrast to amplitudes of the first antisymmetric normal modes the first axisymmetric normal modes are excited only due to nonlinear mechanisms. There are at least three reasons for increase of this difference. First, for conic and paraboloidal tanks (as for variants of non-cylindrical reservoir) potential energy of liquid is represented not only by quadratic terms relative to liquid elevation, but it contains terms of the third and fourth orders, which take into account inclination and curvature of tank walls in a vicinity of a free surface of liquid. Second, expansions of elevations of a free surface contain nonlinear term, which compensate variation of liquid volume due to absence of symmetry above and below undisturbed free surface of liquid. These terms depend again on inclination and curvature of tank walls in a vicinity of a free surface of liquid. Third, increase of wave height is accompanied with two simultaneous effects, namely, wave increases both in vertical and horizontal direction due to inclination of tank wall and increase of wave causes lowering of middle point of a free surface for providing conservation law of liquid volume. The example showed that in the case of presence of walls curvature nonlinearities developed rapidly than in the case of conic reservoir and axisymmetric normal mode attaines greater values of amplitudes. First of all these effects promote excitation of axisymmetric normal modes and they are considerably caused by geometric nonlinearities of the system due to non-cylindrical shape of tank. Therefore, our study makes it possible to draw conclusion that nonlinear mechanisms manifest to greater extent in tank of non-cylindrical shape than in cylindrical tank.

8. Conclusions

We consider problem of modeling of liquid sloshing in conic, paraboloidal and cylindrical tanks, caused by non-stationary loading. Analysis of surface waves generation and contribution of separate normal modes into formation of liquid free surface motion make it possible to draw the following conclusions. Geometrical nonlinearities, caused by non-cylindrical shape of tank, mostly manifests by the presence of inclination of tank walls and their curvature in a vicinity of intersection of tank walls with liquid free surface. This increases total manifestation of nonlinear effects in liquid sloshing. Moreover, mostly these effects are connected with axisymmetric normal modes of oscillations of liquid in tanks.

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Структура геометричних нелінійностей в задачі про коливання рідини в баках нециліндричної форми

Лимарченко О. С.[1], Каттані К.[2], Пилипчук В.[3]

[1] *Київський національний університет імені Тараса Шевченка проспект академіка Глушкова, 4-Е, 01033, Київ, Україна*
[2] *Університет Салерно вул. Іоанна Павла II, Фішіано, Італія*
[3] *Університет Уейн Стейт 42 проспект В. Уоррена, Детройт, Мічиган 48202, США*

Розглянуто структуру геометричних нелінійностей в математичній моделі коливань рідини в баках. На відміну від випадку циліндричного резервуара деякі нові нелінійності проявляються в математичній постановці задачі. Вони пов'язані з чотирма причинами. Перше, вони визначаються новими формами коливань, які відповідають нециліндричній формі бака і приймають до уваги деякі нелінійні властивості задачі (наприклад, вони слідують по стінках бака вище рівня вільноїх поверхні рідини). Друге, іизначення потенціальної енергії рідини включає геометрію бака в в околі перетину незбуреної вільної поверхні рідини і стінок бака. Третій тип прояву геометричних нелінійностей пов'язаний з компенсацією підйомі рівня рідини через нециліндричність форми бака для виконання закону збереження маси. Четвертий тип нелінійностей пов'язаний з одночасним проявом фізичних і геометричних нелінійностей. Дослідження показує, що переважно прояв нелінійних властивостей коливань рідини, викликаний геометричною природою, визначається нахилом і кривизною стінок бака в околі контакту незбуреної рідини з стінками бака. Показано деякі загальні властивості геометричних нелінійностей на прикладі трьох баків: циліндричний, конічний і параболоїдальний бак, який підібраний так, що бак має онаковий нахил біля вільної поверхні рідини, проте в цьому випадку додатково проявляється кривизна.

Ключові слова: *нелінійна динаміка рідини, коливання рідини, геометричні нелінійності, чисельний приклад*

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