MATHEMATICAL MODELING AND COMPUTING, Vol. 3, No. 2, pp. 163–172 (2016)

Generalized electrodiffusion equation with fractality of space-time

Kostrobij P.[1], Markovych B.[1], Viznovych O.[1], Tokarchuk M.[1,2]

[1] *Lviv Polytechnic National University*
12 S. Bandera str., 79013, Lviv, Ukraine
[2] *Institute for Condensed Matter Physics*
of the National Academy of Sciences of Ukraine
1 Svientsitskii str., 79011, Lviv, Ukraine

(Received 30 December 2016)

The new non-Markovian electrodiffusion equations of ions in spatially heterogeneous environment with fractal structure and generalized Cattaneo-type diffusion equation with taking into account fractality of space-time are obtained. Different models of the frequency dependence of memory functions, which lead to known diffusion equations with fractality of space-time and their generalizations are considered.

Keywords: *generalized diffusion equation, nonequilibrium statistical operator, Renyi statistics, multifractal time, spatial fractality, fractality of space-time.*

2000 MSC: 82C05, 82C70

UDC: 538.93

1. Introduction

The fractional derivatives and integrals [1–5] are widely used to study anomalous diffusion in porous media [5–19], in disordered systems [20–31], in plasma physics [32–37], in turbulent [38–40], kinetic and reaction-diffusion processes [40–48], etc. [5, 49]

Currently, together with phenomenological approaches for constructing of the Fokker-Planck equation, the diffusion equation and its generalization — the Cattaneo equation with fractional derivatives, there are two methods of constructing such equations, namely, (1) probabilistic method, which is based on the Chapman-Kolmogorov equation in the stochastic theory of random processes [5, 40, 50], and (2) statistical approach, which is based on the projection operator method (memory functions) [22–28, 44], as well as on the Liouville equation with fractional derivatives [51–64]. In particular, by using this method, the BBGKY hierarchy equations with fractional derivatives [52, 53, 59], transport equation, diffusion equation, and the Heisenberg equation with fractional derivatives [55–57] are obtained. This approach is formulated for non-Hamiltonian systems. If the Helmholtz conditions for coordinate and momentum derivatives of fields of velocities and forces, which act on particles, are fulfilled, the Hamiltonian systems with the time-reversible Liouville equation with fractional derivatives are obtained from non-Hamiltonian systems. In Ref. [65], time-irreversible equations of motion of Hamilton and Liouville for dynamic of classical particles in space with multifractal time are offered. By using the definition of fractional derivative and the Riemann-Liouville integral, the time-irreversible Liouville equation with fractional derivatives (where the time is given on multifractal sets with fractional dimensions) is obtained. In Refs. [66, 67], kinetic equations for systems with fractal structure (in particular, for description of diffusion processes in space of coordinates and momenta) are obtained within the Klimontovich approach. A similar approach for constructing of time fractional generalization for the Liouville equation and the Zwanzig equation (within projection formalism) is proposed in Ref. [68]. An approach with the method of projection operators (memory functions), which is developed in Refs. [22–29, 44], is based on modeling of an frequency dependence of memory functions with using fractional derivatives and integrals [1–5]. For the first time, in Refs. [22–24], Nigmatullin received diffusion equation with fractional derivatives in time for the mean spin density [22], the mean polarization [23], and the charge carrier concentration [24]. In Ref. [25], justification of equations with fractional derivatives is given, and the time irreversible Liouville equation with the fractional time derivative is provided. Within this approach, important results, including microscopic model of a non-Debye dielectric relaxation, which generalizes the Cola-Cola law [28] and the Cola-Davidson law [26], are obtained. In Ref. [29], by using the fractal nature of transport processes of charge carriers, it is studied low-frequency behavior of conductivity with taking into account polarization effects of electrode, which is in good agreement with experimental data.

In our recent work [69], by using the Zubarev nonequilibrium statistical operator method [70–73] and the maximum entropy principle for the Renyi entropy, we consider a way of obtaining generalized (non-Markovian) diffusion equation with fractional derivatives. The use of the Liouville equation with fractional derivatives proposed by Tarasov in Refs. [51–54] is an important and fundamental step for obtaining this equation.

By using the Zubarev nonequilibrium statistical operator method and the maximum entropy principle for the Renyi entropy, we found a solution of the Liouville equation with fractional derivatives at a selected set of observed variables, we chose nonequilibrium average values of particle density as a parameter of reduced description, and then we received a generalized (non-Markovian) diffusion equation with fractional derivatives. In the next section by using [69], new non-Markovian electrodiffusion equations for ions in spatially heterogeneous environment with fractal structure are obtained. Different models of frequency dependence of memory functions are considered, and the electrodiffusion equations with fractality of space-time are obtained.

2. Generalized electrodiffusion equations with fractional derivatives

To describe the electrodiffusion processes of ions in heterogeneous environments with fractal structure, one of main parameters of the reduced description is the nonequilibrium density of ion numbers of corresponding sort b, $n_b(\boldsymbol{r};t) = \langle \hat{n}_b(\boldsymbol{r}) \rangle_\alpha^t$, where $\hat{n}_b(\boldsymbol{r}) = \sum_{j=1}^{N_b} \delta(\boldsymbol{r} - \boldsymbol{r}_j)$ is the microscopic ion density of the sort b. The corresponding generalized electrodiffusion equation for $n_b(\boldsymbol{r};t)$ can be obtained on base of approach [69], by using the Zubarev nonequilibrium statistical operator method within the Renyi statistics for solution of the Liouville equations with fractional derivatives,

$$\frac{\partial}{\partial t} \langle \hat{n}_a(\boldsymbol{r}) \rangle_\alpha^t = \frac{\partial^\alpha}{\partial \boldsymbol{r}^\alpha} \cdot \sum_b \int d\mu_\alpha(\boldsymbol{r}') \int\limits_{-\infty}^{t} e^{\varepsilon(t'-t)} D_q^{ab}(\boldsymbol{r}, \boldsymbol{r}'; t, t') \cdot \frac{\partial^\alpha}{\partial \boldsymbol{r}'^\alpha} \beta \nu_b^*(\boldsymbol{r}'; t') dt', \tag{1}$$

where a and b are sorts of positive and negative ions,

$$D_q^{ab}(\boldsymbol{r}, \boldsymbol{r}'; t, t') = \langle \hat{\boldsymbol{v}}_a(\boldsymbol{r}) T(t, t') \hat{\boldsymbol{v}}_b(\boldsymbol{r}') \rangle_{\alpha, rel}^t \tag{2}$$

is the generalized coefficient of mutual ion diffusion of the corresponding sorts a and b within the Renyi statistics, averaging of which is performed with a power-law Renyi distribution.

$$\rho_{rel}(t) = \frac{1}{Z_R(t)} \left(1 - \frac{q-1}{q} \beta \left(H - \sum_b \int d\mu_\alpha(\boldsymbol{r}) \nu_b^*(\boldsymbol{r}; t) \hat{n}_b(\boldsymbol{r}) \right) \right)^{\frac{1}{q-1}}, \tag{3}$$

where

$$Z_R(t) = \hat{I}^\alpha(1, \dots, N) \hat{T}(1, \dots, N) \left(1 - \frac{q-1}{q} \beta \left(H - \sum_b \int d\mu_\alpha(\boldsymbol{r}) \nu_b^*(\boldsymbol{r}; t) \hat{n}_b(\boldsymbol{r}) \right) \right)^{\frac{1}{q-1}} \tag{4}$$

is the partition function of the relevant distribution function, H is a Hamiltonian of system, $0 < q \leqslant 1$, q is the Renyi parameter.

$$\nu_b^*(\boldsymbol{r};t) = \frac{\nu_b(\boldsymbol{r};t)}{1 + \dfrac{q-1}{q} \displaystyle\sum_b \int d\mu_\alpha(\boldsymbol{r}) \nu_b(\boldsymbol{r};t) \left\langle \hat{n}_b(\boldsymbol{r}) \right\rangle_\alpha^t},$$

$T(t,t') = \exp_+ \left(-\int_{t'}^t (1 - P_{rel}(t')) iL_\alpha dt' \right)$ is the evolution operator in time containing the projection; $\exp_+$ is the ordered exponentia, $P_{rel}(t')$ is the generalized Kawasaki-Gunton projection operator depended on a structure of the relevant statistical operator (distribution function), $\rho_{rel}(x^N;t')$. iL_α is the Liouville operator for a system of ions in heterogeneous environment with fractal structure. Parameter $\nu_b(\boldsymbol{r};t) = \gamma_b(\boldsymbol{r};t) + Z_b e \varphi(\boldsymbol{r};t)$ is the electrochemical potential of ions with valence Z_b, which is determined from the self-consistency condition,

$$\left\langle \hat{n}_b(\boldsymbol{r}) \right\rangle_\alpha^t = \left\langle \hat{n}_b(\boldsymbol{r}) \right\rangle_{\alpha,rel}^t . \tag{5}$$

$\gamma_b(\boldsymbol{r};t)$ is the chemical potential of ions of the sort b, $\varphi(\boldsymbol{r};t)$ is the scalar potential of electromagnetic field of ion system in a heterogeneous environment with fractal structure; $\beta = 1/k_{\mathrm{B}}T$, k_{B} is the Boltzmann constant, T is the equilibrium value of temperature; $\hat{\boldsymbol{v}}_a(\boldsymbol{r}) = \sum_{j=1}^{N_a} \boldsymbol{v}_j \delta(\boldsymbol{r} - \boldsymbol{r}_j)$ is microscopic ion flux density of the sort a. The average values in Eq. (2) are calculated by (see Ref. [69])

$$\langle (\ldots) \rangle_{\alpha,rel}^t = \hat{I}^\alpha(1,\ldots,N) \hat{T}(1,\ldots,N)(\ldots) \rho_{rel}(x^N;t),$$

where $\hat{I}^\alpha(1,\ldots,N)$ for system of N particles has the form

$$\hat{I}^\alpha(1,\ldots,N) = \hat{I}^\alpha(1),\ldots,\hat{I}^\alpha(N), \quad \hat{I}^\alpha(j) = \hat{I}^\alpha(\boldsymbol{r}_j)\hat{I}^\alpha(\boldsymbol{p}_j)$$

and defines the integration operation,

$$\hat{I}^\alpha(x) f(x) = \int_{-\infty}^{\infty} f(x) d\mu_\alpha(x), \quad d\mu_\alpha(x) = \frac{|x|^\alpha}{\Gamma(\alpha)} dx. \tag{6}$$

The operator $\hat{T}(1,\ldots,N) = \hat{T}(1),\ldots,\hat{T}(N)$ defines the operation

$$\hat{T}(x_j) f(x_j) = \frac{1}{2} \big(f(\ldots, x_j' - x_j, \ldots) + f(\ldots, x_j' + x_j, \ldots) \big).$$

In the generalized electrodiffusion equation (1), d^α is a fractional differential [74] that is defined by

$$d^\alpha f(x) = \sum_{j=1}^{2N} D_{x_j}^\alpha f(x) (dx_j)^\alpha,$$

where

$$D_x^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \int_0^x \frac{f^{(n)}(z)}{(x-z)^{\alpha+1-n}} dz \tag{7}$$

is the Caputo fractional derivative [1, 2, 75, 76], $n-1 < \alpha < n$, $f^{(n)}(z) = \frac{d^n}{dz^n} f(z)$ with the properties $D_{x_j}^\alpha 1 = 0$ and $D_{x_j}^\alpha x_l = 0$, $(j \neq l)$.

At $q = 1$, the generalized electrodiffusion equation within the Renyi statistics goes into the generalized electrodiffusion equation within the Gibbs statistics with fractional derivatives. If $q = 1$

and $\alpha = 1$, we obtain the generalized electrodiffusion equation within the Gibbs statistics. In the Markov approximation, the generalized coefficient of mutual diffusion in time and space has the form $D_q^{ab}(\boldsymbol{r},\boldsymbol{r}';t,t') \approx D_q^{ab}\delta(t-t')\delta(\boldsymbol{r}-\boldsymbol{r}')$, by excluding the parameter $\nu_a^*(\boldsymbol{r}';t')$ via the self-consistency condition, we obtain the electrodiffusion equation with fractional derivatives from Eq. (1)

$$\frac{\partial}{\partial t}\left\langle \hat{n}_a(\boldsymbol{r})\right\rangle_\alpha^t = \sum_b D_q^{ab}\frac{\partial^{2\alpha}}{\partial r^{2\alpha}}\nu_b^*(\boldsymbol{r}';t'). \tag{8}$$

The generalized electrodiffusion equation takes into account spatial fractality of system and memory effects in the generalized coefficient of mutual ion diffusion $D_q^{ab}(\boldsymbol{r},\boldsymbol{r}';t,t')$ within the Renyi statistics. Obviously, spatial fractality of system influences on ion transport processes that may manifest as multifractal time with characteristic relaxation times. It is known that the nonequilibrium correlation functions $D_q^{ab}(\boldsymbol{r},\boldsymbol{r}';t,t')$ can not be exactly calculated, therefore the some approximations based on physical reasons are used. In the time interval $-\infty \div t$, ion transport processes in spatially heterogeneous system can be characterized by a set of relaxation times that are associated with the nature of interaction between ions and particles of environment with fractal structure associated with polarizing effects and influence of electromagnetic field. In particular, in a recent Ref. [29], authors took into account polarization effects of electrode during investigation of frequency dependence of conductivity, correct behavior of which is received with taking into account fractality of transfer processes of charge carriers by modeling of memory functions. For opening of multifractal time in the generalized electrodiffusion equation, we use the following approach for the generalized coefficient of mutual ion diffusion

$$D_q^{ab}(\boldsymbol{r},\boldsymbol{r}';t,t') = W_a(t,t')\overline{D}_q^{ab}(\boldsymbol{r},\boldsymbol{r}'), \tag{9}$$

where $W_a(t,t')$ can be defined as the time memory function. In view of this, Eq. (1) can be represented as

$$\frac{\partial}{\partial t}\left\langle \hat{n}_a(\boldsymbol{r})\right\rangle_\alpha^t = \int\limits_{-\infty}^{t} e^{\varepsilon(t'-t)}W_a(t,t')\Psi_a(\boldsymbol{r};t')dt', \tag{10}$$

where

$$\Psi_a(\boldsymbol{r};t') = \sum_b \int d\mu_\alpha(\boldsymbol{r}')\frac{\partial^\alpha}{\partial \boldsymbol{r}^\alpha}\cdot \overline{D}_q^{ab}(\boldsymbol{r},\boldsymbol{r}')\cdot\frac{\partial^\alpha}{\partial \boldsymbol{r}'^\alpha}\beta\nu_b^*(\boldsymbol{r}';t'). \tag{11}$$

Further we apply the Fourier transform to Eq. (10), and as a result we get in frequency representation

$$i\omega n_a(\boldsymbol{r};\omega) = W_a(\omega)\Psi_a(\boldsymbol{r};\omega). \tag{12}$$

We can represent frequency dependence of the memory function in the following form

$$W_a(\omega) = \frac{(i\omega)^{1-\xi}}{1+i\omega\tau_a}, \quad 0<\xi\leqslant 1, \tag{13}$$

where the introduced relaxation time τ_a characterizes ion transport processes in system. Then Eq. (12) can be represented as

$$(1+i\omega\tau_a)i\omega n_a(\boldsymbol{r};\omega) = (i\omega)^{1-\xi}\Psi_a(\boldsymbol{r};\omega). \tag{14}$$

Further we use the Fourier transform to fractional derivatives of functions,

$$L\left({}_0D_t^{1-\xi}f(t);i\omega\right) = (i\omega)^{1-\xi}L(f(t);i\omega). \tag{15}$$

By using it, the inverse transformation of Eq. (14) to time representation gives the Cattaneo-type generalized electrodiffusion equation with taking into account spatial fractality,

$$\tau_a \frac{\partial^2}{\partial t^2} n_a(\boldsymbol{r};t) + \frac{\partial}{\partial t} n_a(\boldsymbol{r};t) = {}_0D_t^{1-\xi} \Psi_a(\boldsymbol{r};t) = \frac{\partial^{1-\xi}}{\partial t^{1-\xi}} \Psi_a(\boldsymbol{r};t), \tag{16}$$

or in the expanded form

$$\tau_a \frac{\partial^2}{\partial t^2} n_a(\boldsymbol{r};t) + \frac{\partial}{\partial t} n_a(\boldsymbol{r};t) = {}_0D_t^{1-\xi} \sum_b \int d\mu_\alpha(\boldsymbol{r}') \frac{\partial^\alpha}{\partial \boldsymbol{r}^\alpha} \cdot \overline{D}_q^{ab}(\boldsymbol{r},\boldsymbol{r}') \cdot \frac{\partial^\alpha}{\partial \boldsymbol{r}'^\alpha} \beta \nu_b^*(\boldsymbol{r}';t), \tag{17}$$

is the new Cattaneo-type generalized equation within the Renyi statistics with multifractal time and spatial fractality. At $q = 1$ from Eq. (17), we get the Cattaneo-type generalized equation within the Gibbs statistics with multifractal time and spatial fractality,

$$\tau_a \frac{\partial^2}{\partial t^2} n_a(\boldsymbol{r};t) + \frac{\partial}{\partial t} n_a(\boldsymbol{r};t) = {}_0D_t^{1-\xi} \sum_b \int d\mu_\alpha(\boldsymbol{r}') \frac{\partial^\alpha}{\partial \boldsymbol{r}^\alpha} \cdot \overline{D}_q^{ab}(\boldsymbol{r},\boldsymbol{r}') \cdot \frac{\partial^\alpha}{\partial \boldsymbol{r}'^\alpha} \beta \left(\gamma_b(\boldsymbol{r}';t) + Z_b e \varphi(\boldsymbol{r}';t)\right), \tag{18}$$

It should be noted that the right side of Eqs. (16), (18) has fractional derivative of the scalar potential of electromagnetic field $\frac{\partial^\alpha}{\partial \boldsymbol{r}'^\alpha} \beta Z_b e \varphi(\boldsymbol{r}';t)$, that indicates the need to take into consideration the Maxwell equations with fractional derivatives for systems with spatial fractality for a full description of ion transfer processes in such environment. Eqs. (16), (18) contain significant spatial heterogeneity in $\overline{D}_q^{ab}(\boldsymbol{r},\boldsymbol{r}')$. If we neglect spatial heterogeneity,

$$\overline{D}_q^{ab}(\boldsymbol{r},\boldsymbol{r}') = \overline{D}_q^{ab} \delta(\boldsymbol{r}-\boldsymbol{r}'), \tag{19}$$

we get the Cattaneo-type diffusion equation with fractality of space-time and constant coefficients of mutual diffusion within the Renyi statistics

$$\tau_a \frac{\partial^2}{\partial t^2} n_a(\boldsymbol{r};t) + \frac{\partial}{\partial t} n_a(\boldsymbol{r};t) = {}_0D_t^{1-\xi} \sum_b \overline{D}_q^{ab} \frac{\partial^{2\alpha}}{\partial \boldsymbol{r}^{2\alpha}} \beta \nu_b^*(\boldsymbol{r};t), \tag{20}$$

or in the expanded form

$$\tau_a \frac{\partial^2}{\partial t^2} n_a(\boldsymbol{r};t) + \frac{\partial}{\partial t} n_a(\boldsymbol{r};t) = {}_0D_t^{1-\xi} \sum_b \overline{D}_q^{ab} \frac{\partial^{2\alpha}}{\partial \boldsymbol{r}^{2\alpha}} \beta \frac{\nu_a(\boldsymbol{r};t)}{1 + \frac{q-1}{q} \sum_a \int d\mu_\alpha(\boldsymbol{r}) \nu_a(\boldsymbol{r};t) \langle \hat{n}_a(\boldsymbol{r}) \rangle_\alpha^t}, \tag{21}$$

and at $q = 1$ we get the Cattaneo-type diffusion equation with fractality of space-time and constant coefficients of mutual diffusion within the Gibbs statistics,

$$\tau_a \frac{\partial^2}{\partial t^2} n_a(\boldsymbol{r};t) + \frac{\partial}{\partial t} n_a(\boldsymbol{r};t) = {}_0D_t^{1-\xi} \sum_b \overline{D}^{ab} \frac{\partial^{2\alpha}}{\partial \boldsymbol{r}^{2\alpha}} \nu_a(\boldsymbol{r};t), \tag{22}$$

It should be noted that if we put $\alpha = 1$ in Eqs. (20)–(22), i.e. we have neglected spatial fractality, we get the Cattaneo-type diffusion equations, which were obtained in Refs. [9, 18],

$$\tau_a \frac{\partial^2}{\partial t^2} n_a(\boldsymbol{r};t) + \frac{\partial}{\partial t} n_a(\boldsymbol{r};t) = {}_0D_t^{1-\xi} \sum_b \overline{D}^{ab} \frac{\partial^2}{\partial \boldsymbol{r}^2} \nu_a(\boldsymbol{r};t). \tag{23}$$

At $\tau_a = 0$, we get an important particular case — the generalized electrodiffusion equation of ions with taking into account fractality of space-time,

$$\frac{\partial}{\partial t} n_a(\boldsymbol{r};t) = {}_0D_t^{1-\xi} \sum_b \int d\mu_\alpha(\boldsymbol{r}') \frac{\partial^\alpha}{\partial \boldsymbol{r}^\alpha} \cdot \overline{D}_q^{ab}(\boldsymbol{r},\boldsymbol{r}') \cdot \frac{\partial^\alpha}{\partial \boldsymbol{r}'^\alpha} \beta \nu_b^*(\boldsymbol{r}';t), \tag{24}$$

and by neglecting spatial heterogeneity of mutual diffusion coefficients $\overline{D}_q^{ab}(\boldsymbol{r},\boldsymbol{r}')$, we also get the electrodiffusion equation with constant coefficients of mutual diffusion with fractional derivatives within the Renyi statistics,

$$\frac{\partial}{\partial t} n_a(\boldsymbol{r};t) = {}_0D_t^{1-\xi} \sum_b \overline{D}_q^{ab} \frac{\partial^{2\alpha}}{\partial \boldsymbol{r}^{2\alpha}} \beta \nu_b^*(\boldsymbol{r};t), \tag{25}$$

At $\alpha = 1$, $\tau_a = 0$, we get the electrodiffusion equation with constant coefficients of mutual diffusion without spatial fractality within the Renyi statistics

$$\frac{\partial}{\partial t} n_a(\boldsymbol{r};t) = {}_0D_t^{1-\xi} \sum_b \overline{D}_q^{ab} \frac{\partial^2}{\partial \boldsymbol{r}^2} \beta \nu_b^*(\boldsymbol{r};t), \tag{26}$$

At $\alpha = 1$, $\tau_a = 0$, $q = 1$, $\xi = 1$, we get the usual electrodiffusion equation for ions within the Gibbs statistics,

$$\frac{\partial}{\partial t} n_a(\boldsymbol{r};t) = \sum_b \overline{D}^{ab} \frac{\partial^2}{\partial \boldsymbol{r}^2} \beta \nu_b(\boldsymbol{r};t). \tag{27}$$

Let us consider another model of the memory function

$$W_a(\omega) = \frac{(i\omega)^{1-\xi}}{1 + (i\omega\tau_a)^{\gamma-1}}, \tag{28}$$

then in frequency representation we get

$$\left(1 + (i\omega\tau_a)^{\gamma-1}\right) i\omega n_a(\boldsymbol{r};\omega) = (i\omega)^{1-\xi} \Psi_a(\boldsymbol{r};\omega). \tag{29}$$

By using Eq. (15) and inverse transformation of Eq. (29) to the time t, we get the generalized Cattaneo-type electrodiffusion equation with taking into account multifractal time and spatial fractality,

$$\tau_a^{\gamma-1} \frac{\partial^\gamma}{\partial t^\gamma} n_a(\boldsymbol{r};t) + \frac{\partial}{\partial t} n_a(\boldsymbol{r};t) = {}_0D_t^{1-\xi} \Psi_a(\boldsymbol{r};t) = \frac{\partial^{1-\xi}}{\partial t^{1-\xi}} \Psi_a(\boldsymbol{r};t) \tag{30}$$

which has a similar structure to the Cattaneo equation of Ref. [77].

In the case of such model for the memory function

$$W_a(\omega) = \frac{(i\omega)^{1-\xi}}{(i\omega\tau_a)^{\gamma-1}}, \tag{31}$$

we get the generalized electrodiffusion equation of ions with fractality of space-time within the Renyi statistics,

$$\tau_a^{\gamma-1} \frac{\partial^\gamma}{\partial t^\gamma} n_a(\boldsymbol{r};t) = {}_0D_t^{1-\xi} \Psi_a(\boldsymbol{r};t) = \frac{\partial^{1-\xi}}{\partial t^{1-\xi}} \Psi_a(\boldsymbol{r};t). \tag{32}$$

At $\xi = 1$, this equation has the form

$$\tau_a^{\gamma-1} \frac{\partial^\gamma}{\partial t^\gamma} n_a(\boldsymbol{r};t) = \Psi_a(\boldsymbol{r};t), \tag{33}$$

and in the case of neglecting the spatial dependence of mutual diffusion coefficient, we get

$$\tau_a^{\gamma-1}\frac{\partial^\gamma}{\partial t^\gamma}n_a(\boldsymbol{r};t)=\sum_b \overline{D}_q^{ab}\frac{\partial^{2\alpha}}{\partial \boldsymbol{r}^{2\alpha}}\beta\nu_b^*(\boldsymbol{r};t). \tag{34}$$

Solutions of Eq. (34) are studied in Ref. [78].

3. Conclusions

By using approach of Ref. [69], the new non-Markovian electrodiffusion equations of ions in spatially heterogeneous environment with fractal structure are obtained. By using approaches for the memory functions and fractional calculus [1–5], the generalized Cattaneo-type diffusion equations with taking into account fractality of space-time are obtained. It is considered the different models for the frequency dependence of the memory functions, which lead to the known diffusion equations with the fractality of space-time [9, 18, 24, 77, 78] and their generalizations.

[1] Oldham K. B., Spanier J. *The Fractional Calculus: Theory and Applications of Differentiation and Integration to Arbitrary Order.* Dover Books on Mathematics (Dover Publications, 2006).

[2] Samko S. G., Kilbas A. A., Marichev O. I. *Fractional Integrals and Derivatives: Theory and Applications* (Gordon and Breach Science Publishers, 1993).

[3] Podlubny I. *Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution and Some of Their Applications.* Mathematics in Science and Engineering 198 (Academic Press, 1998).

[4] Mandelbrot B. B. *The fractal geometry of nature* (W. H. Freeman and Company, 1982).

[5] Uchaikin V. V. *Fractional Derivatives Method* (Artishock-Press, Uljanovsk, 2008).

[6] Sahimi M. Non-linear and non-local transport processes in heterogeneous media: from long-range correlated percolation to fracture and materials breakdown. Physics Reports, **306**, 213–395 (1998).

[7] Korošak D., Cvikl B., Kramer J., Jecl R., Prapotnik A. Fractional calculus applied to the analysis of spectral electrical conductivity of clay–water system. Journal of Contaminant Hydrology, **92**, 1–9 (2007).

[8] Hobbie R. K., Roth B. J. *Intermediate Physics for Medicine and Biology* (Springer, New York, 2007).

[9] Compte A., Metzler R. The generalized Cattaneo equation for the description of anomalous transport processes. Journal of Physics A: Mathematical and General, **30**, 7277–7289 (1997).

[10] Metzler, R. Klafter, J. The random walk's guide to anomalous diffusion: a fractional dynamics approach. Physics Reports **339**, 1–77 (2000).

[11] Hilfer R. Fractional dynamics, irreversibility and ergodicity breaking. Chaos, Solitons Fractals, **5**, 1475–1484 (1995).

[12] Hilfer R. Fractional diffusion based on Riemann-Liouville fractional derivatives. The Journal of Physical Chemistry B, **104**, 3914–3917 (2000).

[13] Hilfer R. *Fractional Time Evolution*, chap. II, 87–130 (World Scientific, Singapore, New Jersey, London, Hong Kong, 2000).

[14] Kosztołowicz T., Dworecki K., Mrówczyński S. Measuring subdiffusion parameters. Phys. Rev. E **71**, 041105 (2005).

[15] Kosztołowicz T. Subdiffusive random walk in a membrane system: the generalized method of images approach. Journal of Statistical Mechanics: Theory and Experiment, **2015**, P10021 (2015).

[16] Bisquert J., Garcia-Belmonte G., Fabregat-Santiago F., Ferriols N. S., Bogdanoff P., Pereira E. C. Doubling exponent models for the analysis of porous film electrodes by impedance. Relaxation of TiO_2 nanoporous in aqueous solution. The Journal of Physical Chemistry B, **104**, 2287–2298 (2000).

[17] Bisquert J., Compte A. Theory of the electrochemical impedance of anomalous diffusion. Journal of Electroanalytical Chemistry, **499**, 112–120 (2001).

[18] Kosztołowicz T., Lewandowska K. D. Hyperbolic subdiffusive impedance. Journal of Physics A: Mathematical and Theoretical, **42**, 055004 (2009).

[19] Pyanylo Y. D., Prytula M. G., Prytula N. M., Lopuh N. B. Models of mass transfer in gas transmission systems. Mathematical Modeling and Computing, **1**, 84–96 (2014).

[20] Berkowitz B., Scher H. Theory of anomalous chemical transport in random fracture networks. Phys. Rev. E, **57**, 5858–5869 (1998).

[21] Bouchaud J. P., Georges A. Anomalous diffusion in disordered media: Statistical mechanisms, models and physical applications. Physics Reports, **195**, 127–293 (1990).

[22] Nigmatullin R. R. To the Theoretical Explanation of the "Universal Response". Physica Status Solidi (b), **123**, 739–745 (1984).

[23] Nigmatullin R. R. On the Theory of Relaxation for Systems with "Remnant" Memory. Physica Status Solidi (b), **124**, 389–393 (1984).

[24] Nigmatullin R. R. The realization of the generalized transfer equation in a medium with fractal geometry. Physica Status Solidi (b), **133**, 425–430 (1986).

[25] Nigmatullin R. R. Fractional integral and its physical interpretation. Theoretical and Mathematical Physics, **90**, 242–251 (1992).

[26] Nigmatullin R. R., Ryabov Y. E. Cole-Davidson dielectric relaxation as a self-similar relaxation process. Physics of the Solid State, **39**, 87–90 (1997).

[27] Nigmatullin R. R. Dielectric relaxation phenomenon based on the fractional kinetics: theory and its experimental confirmation. Physica Scripta, **2009**, 014001 (2009).

[28] Khamzin A. A., Nigmatullin R. R., Popov I. I. Microscopic model of a non-Debye dielectric relaxation: The Cole-Cole law and its generalization. Theoretical and Mathematical Physics, **173**, 1604–1619 (2012).

[29] Popov I. I., Nigmatullin R. R., Koroleva E. Y., Nabereznov A. A. The generalized Jonscher's relationship for conductivity and its confirmation for porous structures. Journal of Non-Crystalline Solids, **358**, 1–7 (2012).

[30] Gryhorchak I. I., Kostrobiy P. P., Stasyuk I. V., Tokarchuk M. V., Velychko O. V., Ivashchyshyn F. O., Markovych B. M. *Fizychni protsesy ta yikh mikroskopichni modeli v periodychnykh neorhanichno/ orhanichnykh klatratakh* (Rastr-7, L'viv, 2015), (in Ukrainian).

[31] Kostrobij P. P., Grygorchak I. I., Ivaschyshyn F. O., Markovych B. M., Viznovych O., Tokarchuk M. V. Mathematical modeling of subdiffusion impedance in multilayer nanostructures. Mathematical Modeling and Computing, **2**, 154–159 (2015).

[32] Balescu, R. Anomalous transport in turbulent plasmas and continuous time random walks. *Phys. Rev. E* **51**, 4807–4822 (1995).

[33] Tribeche M., Shukla P. K. Charging of a dust particle in a plasma with a non extensive electron distribution function. Physics of Plasmas, **18**, 103702 (2011).

[34] Gong J., Du J. Dust charging processes in the nonequilibrium dusty plasma with nonextensive power-law distribution. Physics of Plasmas, **19**, 023704 (2012).

[35] Carreras B. A., Lynch V. E., Zaslavsky G. M. Anomalous diffusion and exit time distribution of particle tracers in plasma turbulence model. Physics of Plasmas, **8**, 5096–5103 (2001).

[36] Tarasov V. E. Electromagnetic field of fractal distribution of charged particles. Physics of Plasmas, **12**, 082106 (2005).

[37] Tarasov V. E. Magnetohydrodynamics of fractal media. Physics of Plasmas, **13**, 052107 (2006).

[38] Monin A. S. Uravnenija turbulentnoj difuzii. DAN SSSR, ser. geofiz. **2**, 256–259 (1955).

[39] Klimontovich J. L. *Vvedenie v fiziku otkrytyh sistem* (Moskva Janus, 2002), (in Russian).

[40] Zaslavsky G. Chaos, fractional kinetics, and anomalous transport. Physics Reports, **371**, 461–580 (2002).

[41] Zaslavsky G. Fractional kinetic equation for hamiltonian chaos. Physica D: Nonlinear Phenomena, **76**, 110–122 (1994).

[42] Saichev A. I., Zaslavsky G. M. Fractional kinetic equations: solutions and applications. Chaos, **7**, 753–764 (1997).

[43] Zaslavsky G., Edelman M. Fractional kinetics: from pseudochaotic dynamics to Maxwell's demon. Physica D: Nonlinear Phenomena, **193**, 128–147 (2004).

[44] Nigmatullin R. 'Fractional' kinetic equations and 'universal' decoupling of a memory function in mesoscale region. Physica A: Statistical Mechanics and its Applications, **363**, 282–298 (2006).

[45] Chechkin A. V., Gonchar V. Y., Szydłowski M. Fractional kinetics for relaxation and superdiffusion in a magnetic field. Physics of Plasmas, **9**, 78–88 (2002).

[46] Gafiychuk V. V., Datsko B. Y. Stability analysis and oscillatory structures in time-fractional reaction-diffusion systems. Phys. Rev. E, **75**, 055201 (2007).

[47] Kosztołowicz T., Lewandowska K. D. Time evolution of the reaction front in a subdiffusive system. Phys. Rev. E, **78**, 066103 (2008).

[48] Shkilev V. P. Subdiffusion of mixed origin with chemical reactions. Journal of Experimental and Theoretical Physics, **117**, 1066–1070 (2013).

[49] Uchaikin V. V. Fractional phenomenology of cosmic ray anomalous diffusion. Physics-Uspekhi, **56**, 1074–1119 (2013).

[50] Stanislavsky A. A. Probability Interpretation of the Integral of Fractional Order. Theoretical and Mathematical Physics, **138**, 418–431 (2004).

[51] Tarasov V. E. Fractional generalization of Liouville equations. Chaos, **14**, 123–127 (2004).

[52] Tarasov V. E. Fractional Liouville and BBGKI equations. Journal of Physics: Conference Series, **7**, 17 (2005).

[53] Tarasov V. E. Fractional systems and fractional Bogoliubov hierarchy equations. Phys. Rev. E, **71**, 011102 (2005).

[54] Tarasov V. E. Fractional statistical mechanics. Chaos, **16**, 033108 (2006).

[55] Tarasov V. E. Transport equations from Liouville equations for fractional systems. International Journal of Modern Physics B, **20**, 341–353 (2006).

[56] Tarasov V. E. Fractional diffusion equations for open quantum system. Nonlinear Dynamics, **71**, 663–670 (2013).

[57] Tarasov V. E. Fractional Heisenberg equation. Physics Letters A, **372**, 2984 – 2988 (2008).

[58] Tarasov V. E. Fractional hydrodynamic equations for fractal media. Annals of Physics, **318**, 286 – 307 (2005).

[59] Tarasov V. E. Liouville and Bogoliubov equations with fractional derivatives. Modern Physics Letters B, **21**, 237–248 (2007).

[60] Tarasov V. E. The fractional Chapman–Kolmogorov equation. Modern Physics Letters B, **21**, 163–174 (2007).

[61] Tarasov V. E. Fractional generalization of the quantum Markovian master equation. Theoretical and Mathematical Physics, **158**, 179–195 (2009).

[62] Tarasov V. E. Quantum dissipation from power-law memory. Annals of Physics, **327**, 1719–1729 (2012).

[63] Tarasov V. E. Power-law spatial dispersion from fractional Liouville equation. Physics of Plasmas, **20**, 102110 (2013).

[64] Tarasov V. E. *Fractional Dynamics: Applications of Fractional Calculus to Dynamics of Particles, Fields and Media*. Nonlinear Physical Science (Springer Berlin Heidelberg, 2010), 1st edition.

[65] Kobelev L. Y. The Multifractal Time and Irreversibility in Dynamic Systems (2000). `arXiv:physics/0002002`.

[66] Kobelev Y. L., Kobelev L. Y., Romanov E. P. Kinetic equations for large systems with fractal structures. Doklady Physics, **45**, 194–197 (2000).

[67] Kobelev Y. L., Kobelev L. Y., Kobelev V. L., Romanov E. P. Description of diffusion in fractal media on the basis of the klimontovich kinetic equation in fractal space. Doklady Physics, **47**, 580–582 (2002).

[68] Lukashchuk S. Y. Time-fractional extensions of the Liouville and Zwanzig equations. Central European Journal of Physics, **11**, 740–749 (2013).

[69] Kostrobij P., Markovych B., Viznovych O., Tokarchuk M. Generalized diffusion equation with fractional derivatives within renyi statistics. Journal of Mathematical Physics, **57**, 093301 (2016).

[70] Zubarev D. N. Modern methods of the statistical theory of nonequilibrium processes. Journal of Soviet Mathematics, **16**, 1509–1571 (1981).

[71] Zubarev D. N., Morozov V. G., Röpke G. *Statistical mechanics of nonequilibrium processes*, vol. 1 (Fizmatlit, 2002).

[72] Zubarev D. N., Morozov V. G., Röpke G. *Statistical mechanics of nonequilibrium processes*, vol. 2 (Fizmatlit, 2002).

[73] Markiv B. B., Tokarchuk R. M., Kostrobij P. P., Tokarchuk M. V. Nonequilibrium statistical operator method in Renyi statistics. Physica A: Statistical Mechanics and its Applications, **390**, 785–791 (2011).

[74] Cottrill-Shepherd K., Naber M. Fractional differential forms. Journal of Mathematical Physics, **42**, 2203–2212 (2001).

[75] Mainardi F. *Fractional Calculus*, 291–348 (Springer Vienna, Vienna, 1997).

[76] Caputo M., Mainardi F. A new dissipation model based on memory mechanism. Pure and Applied Geophysics, **91**, 134–147 (1971).

[77] Qi H., Jiang X. Solutions of the space-time fractional Cattaneo diffusion equation. Physica A: Statistical Mechanics and its Applications, **390**, 1876–1883 (2011).

[78] Sun H., Chen W., Li C., Chen Y. Fractional differential models for anomalous diffusion. Physica A: Statistical Mechanics and its Applications, **389**, 2719–2724 (2010).

Узагальнені рівняння електродифузії з просторово-часовою фрактальністю

Костробій П. П.[1], Маркович Б. М.[1], Візнович О. В.[1], Токарчук М. В.[1,2]

[1] *Національний університет “Львівська політехніка”*
вул. С. Бандери, 12, 79013, Львів, Україна
[2] *Інститут фізики конденсованих систем НАН України*
вул. Свєнціцького, 1, 79011, Львів, Україна

Отримано нові немарковські рівняння електродифузії іонів у просторово неоднорідному середовищі з фрактальною структурою та узагальнені рівняння дифузії типу Каттано з врахуванням просторово-часової фрактальності. Розглянуто різні моделі частотної залежності для функцій пам'яті, які приводять до відомих рівнянь дифузії з просторово-часовою фрактальністю, а також їх узагальнень.

Ключові слова: *узагальнене рівняння дифузії, нерівноважний статистичний оператор, статистика Рені, часова мультифрактальність, просторова фрактальність, просторово-часова фрактальність.*

2000 MSC: 82C05, 82C70

УДК: 538.93