MATHEMATICAL MODELING AND COMPUTING, Vol. 8, No. 1, pp. 106–115 (2021)

Control synthesis by full state vector in systems with fractional-order derivatives using Caputo–Fabrizio operator

Lozynskyy A. O., Lozynskyy O. Yu., Kasha L. V.

Lviv Polytechnic National University,
12 S. Bandera Str., 79013, Lviv, Ukraine

(Received 5 October 2020; Revised 11 January 2021; Accepted 15 January 2021)

In the paper, the control system synthesis by means of the full state vector is considered when using fractional derivatives in the description of this system. To conduct research in the synthesized system with fractional derivatives in the Caputo–Fabrizio representation, a fundamental matrix of the system is formed, which also allows us to analyze the influence of initial conditions on the processes within the system. In particular, the finding of the fundamental matrix of the system in the case of multiple roots of a characteristic polynomial, which are obtained by transforming the synthesized system to the binomial form, is demonstrated. The influence of the fractional derivative index and the location of the roots of the characteristic polynomial transformed to the binomial form on the system operation is analyzed.

Keywords: *fractional-order derivative, fundamental matrix, modal control.*

2010 MSC: 26A33, 93D05 **DOI:** 10.23939/mmc2021.01.106

1. Introduction

The development of control theory in recent decades has been characterized by the creation and application of new approaches to the control of nonlinear systems, in particular back-stepping, passivity base control, feedback linearization, fuzzy logic control [1–3]. In the synthesis of control actions in such approaches, the theory of control by means of the full state vector of the system is widely used. At the same time, despite the widespread use of derivatives and integrals of fractional order to describe processes in various dynamic systems [4, 5], including electromechanical systems [6–8], the main focus of researchers while creating control systems is focused on the synthesis of fractional-order PID controllers [9–12]. On the one hand, due to the increase in the number of parameters for adjusting fractional-order controllers there are greater opportunities for the formation of effective control regularities but, on the other hand, the complexity of adjusting such systems increases. As noted in [10], the use of fractional-order PID controllers increases the robustness of the system, improves the performance of systems with time delay, provides flexibility in the synthesis of systems based on various efficiency criteria. One of the most popular methods of synthesis of fractional-order PID controller is to determine its parameters by solving a system of nonlinear equations that express the relationships associated with the margin of stability of the system by phases, the gain at the cutoff frequency, the sensitivity function and robustness to change the gain within a limited range. Another popular approach being evidenced by the analysis of methods for the synthesis of systems with fractional controllers [9, 11] is the use of modern optimization algorithms to optimize the formed quality functional of the system. In [13], the problem of controller synthesis in a system with fractional derivatives is proposed to be solved using the desired characteristic forms of fractional order. In contrast to integer systems, the standard form is used for the synthesis of the direct channel controller rather than feedback coefficients for the state variables. The use of fractional-order derivatives in discontinuous control systems, as shown in [14, 15] involves the use of PD^{α} switching surfaces and the replacement of the sgn function by the PI fractional-order controller with output signal limitation.

In [16, 17], the application of fractional-order controllers in control systems synthesized by the backstepping method is shown. For the synthesis of parameters, an approach based on the Lyapunov function is used [18]. A similar approach based on the Lyapunov function and matrix inequalities is used for the synthesis of control systems by means of the full state vector [19–22]. As it is known, this procedure is complex, provides a synthesis of a stable system but does not allow forming the desired dynamic characteristics.

Thus, the analysis of publications suggests that in the case of systems described by models with derivatives of fractional order, there is no method of synthesis of systems by means of the full state vector, which provides the formation of the desired dynamic characteristics.

2. Theoretical foundations

In [23], the fractional-order derivative is proposed to be represented in the following form

$$D_t^\alpha f(t) = \frac{d^\alpha f(t)}{dt^\alpha} = \frac{1}{1-\alpha}\int_0^t \exp\left(-\frac{\alpha}{1-\alpha}(t-\tau)\right) f'(\tau)\,d\tau,$$

where $0 < \alpha \leqslant 1$. In [24], it is noted about the limited application of the Caputo–Fabrizio transformation for the study of systems, in particular the need to fulfill the condition $f(0) = 0$, which follows from the following

$${}^{\mathrm{CF}}D_0^\alpha \left[{}^{\mathrm{CF}}J_0^\alpha f(t)\right] = f(t) - \exp\left(-\frac{\alpha}{1-\alpha}t\right) f(0).$$

This remark is not critical in solving the problem of the control action synthesis, because in classical control theory for the synthesis of control systems they traditionally use models in increments of variables and do not take into account the initial conditions.

Another caveat to the use of the Caputo-Fabrizio operator and other non-singular kernel operators [24, 25] is the possibility of obtaining equivalent system models using only integer derivatives. In the synthesis of control systems, the possibility of transition to the model with integer derivatives allows us to apply the classical methods of synthesis of control actions without going into the frequency domain and without applying the approximations required when using singular kernel operators to describe the derivative of fractional order.

In the case of a linear system

$${}^{\mathrm{CF}}D_t^\alpha x(t) = A\,x(t) + Bu(t),$$

after applying the Laplace transform, we obtain

$$\frac{s}{(1-\alpha)s+\alpha}X(s) = A\,X(s) + B\,U(s), \quad s\,X(s) = \alpha\left[A\,X(s) + B\,U(s)\right] + (1-\alpha)s\left[A\,X(s) + B\,U(s)\right].$$

Hence, the model in integer derivatives has the form

$$\frac{dx(t)}{dt} = \alpha\left[A\,x(t) + B\,u(t)\right] + (1-\alpha)\left[A\,\frac{dx(t)}{dt} + B\,\frac{du(t)}{dt}\right],$$
$$(I-(1-\alpha)A)\,\frac{dx(t)}{dt} = \alpha\left[A\,x(t) + B\,u(t)\right] + (1-\alpha)B\,\frac{du(t)}{dt},$$
$$\frac{dx(t)}{dt} = A^*x(t) + B^*u(t) + B^{**}\frac{du(t)}{dt},$$

where $A^* = (I-(1-\alpha)A)^{-1}\alpha\,A$, $B^* = (I-(1-\alpha)A)^{-1}\alpha\,B$, $B^{**} = (I-(1-\alpha)A)^{-1}(1-\alpha)B$. Thus, a linear system with a fractional-order derivative, using the Caputo–Fabrizio operator, can be represented by a system with integer derivatives in which there is an additional forcing action based on the control signal and the parameters of the system model change. Classical synthesis methods, including the full state vector control, can be easily applied to such a system.

For the above-mentioned linear system with fractional-order derivatives, under the full state vector control $u(t) = U_z - \sum_{i=1}^{n} k_{1i} x_i(t)$ and the representation of the fractional-order derivative using the operator Caputo–Fabrizio, after applying the Laplace transform we obtain [26]

$$\frac{1}{1-\alpha}\left(\frac{1}{s+\beta}(s\,X(s) - x_0)\right) = (A - B\,K)X(s) + B\,U_z(s), \tag{1}$$

where $\beta = \alpha/(1-\alpha)$; is a vector of feedback coefficients with respect to the system state variables; is a matrix of control actions; $U_z(s)$ is an job signal at the system input.

The expression (1) after simple transformations can be written as follows

$$(s\,(I - (1-\alpha)(A - B\,K)) - \alpha(A - B\,K))\,X(s) = x_0 + \alpha\,B\,U_z(s) + \alpha\,B\,s\,U_z(s). \tag{2}$$

Traditionally, the synthesis of control action is carried out under zero initial conditions. In the case $U_z = \text{const}$ and by putting $N = (I - (1-\alpha)(A - B\,K))^{-1}$, the transfer function of the system (2) has the form:

$$W(s) = \frac{X(s)}{U_z(s)} = (s\,I - \alpha\,N(A - B\,K))^{-1}\alpha\,N\,B. \tag{3}$$

Synthesis of feedback coefficients with respect to the state variables occurs by the method of modal control based on the use of a given arrangement of system poles, by equating the coefficients of the characteristic polynomial of the system (3) $H(p) = \det(s\,I - \alpha\,N(A - B\,K))$ to the desired characteristic polynomial of the system.

3. Application of the proposed approach

Let us demonstrate the application of the proposed approach to the synthesis of systems with fractional-order derivatives on the example of a third-order system:

$$\begin{bmatrix} \frac{d^\alpha x_1(t)}{dt^\alpha} \\ \frac{d^\alpha x_2(t)}{dt^\alpha} \\ \frac{d^\alpha x_3(t)}{dt^\alpha} \end{bmatrix} = \begin{bmatrix} 0 & \frac{-1}{J_1} & 0 \\ c & 0 & -c \\ 0 & \frac{1}{J_2} & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} \frac{1}{J_1} \\ 0 \\ 0 \end{bmatrix} M + \begin{bmatrix} 0 \\ 0 \\ \frac{-1}{J_2} \end{bmatrix} M_c. \tag{4}$$

For $\alpha{>}1$, Eqs. (4) describe a two-mass system of Figs. 1 excluding the action of external and internal friction $a_{f1} = b_{12} = a_{f2} = 0$ and, respectively, J_1 and J_2 are moments of inertia of the first and second masses, c is the coefficient of elasticity of the shaft; M_c is a moment of loading; M is a control action.

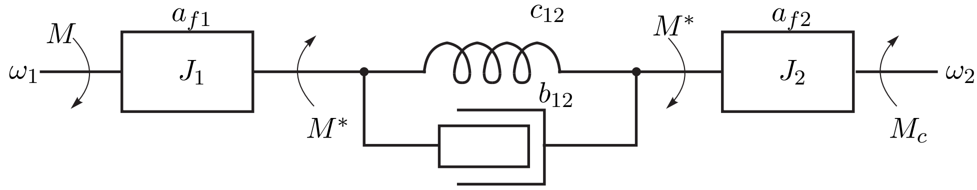

Fig. 1. Block diagram of a two-mass electromechanical system.

In the case of full state vector control, the system model will have the form

$$\begin{bmatrix} \frac{d^\alpha x_1(t)}{d^\alpha t} \\ \frac{d^\alpha x_2(t)}{d^\alpha t} \\ \frac{d^\alpha x_3(t)}{d^\alpha t} \end{bmatrix} = \begin{bmatrix} 0 & \frac{-1}{J_1} & 0 \\ c & 0 & -c \\ 0 & \frac{1}{J_2} & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} \frac{1}{J_1} \\ 0 \\ 0 \end{bmatrix} [k_{11}\ k_{12}\ k_{13}] \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} \frac{1}{J_1} \\ 0 \\ 0 \end{bmatrix} M_z + \begin{bmatrix} 0 \\ 0 \\ \frac{-1}{J_2} \end{bmatrix} M_c.$$

The transfer function of the system $W(s) = \frac{X_3(s)}{M_z(s)}$ obtained on the basis of Eq. (3) will have the form:

$$W(s) = \frac{b_2 s^2 + b_1 s + b_0}{A_{den}\left(s^3 + a_2 s^2 + a_1 s + a_0\right)}, \tag{5}$$

where $b_2 = \alpha\, c(\alpha - 1)^2$, $b_1 = 2\alpha^2 c(1-\alpha)$, $b_0 = \alpha^3 c$,

$$\begin{aligned}
A_{den} &= J_1 J_2 + c(J_1 + J_2)(1-\alpha)^2 + c(k_1 + k_3)(1-\alpha)^3 + cJ_2 k_2 (1-\alpha)^2 + J_2 k_1 (1-\alpha),\\
a_2 &= \frac{\alpha}{A_{den}}\left(2c(1-\alpha)(J_1 + J_2 + J_2 k_2) + J_2 k_1 + 3c(1-\alpha)^2(k_1 + k_3)\right),\\
a_1 &= \frac{1}{A_{den}}\alpha^2 c\left(J_1 + J_2 + J_2 k_2 + 3(1-\alpha)(k_1 + k_3)\right), \quad a_0 = \frac{1}{A_{den}}\alpha^3 c(k_1 + k_3).
\end{aligned}$$

For $\alpha \to 1$, the transfer function (5) will have the form

$$W(s) = \frac{c}{J_1 J_2\left(s^3 + \frac{J_2 k_1}{J_1 J_2}s^2 + \frac{c(J_1+J_2+J_2k_2)}{J_1 J_2}s + \frac{c(k_1+k_3)}{J_1 J_2}\right)}$$

and correspond to the transfer function obtained from the model of a two-mass system without taking into account the action of external and internal friction:

$$W(s) = \frac{X(s)}{U_z(s)} = (s\,I - A + B\,K)^{-1}B.$$

Let the desired characteristic polynomial have the form $H_{des}(s) = (s + \omega_0)^3$, then the system of equations for finding the feedback coefficients with respect to the state variables will have the form:

$$\begin{cases}
\dfrac{\alpha}{A_{den}}\left(2c(1-\alpha)\left(J_1 + J_2 + J_2 k_2\right) + J_2 k_1 + 3c(1-\alpha)^2(k_1 + k_3)\right) = 3\omega_0;\\
\dfrac{1}{A_{den}}\alpha^2 c\left(J_1 + J_2 + J_2 k_2 + 3(1-\alpha)(k_1 + k_3)\right) = 3\omega_0^2;\\
\dfrac{1}{A_{den}}\alpha^3 c(k_1 + k_3) = \omega_0^3.
\end{cases} \tag{6}$$

As a result of solving the system of equations (6), we obtain:

$$\begin{aligned}
k_1 &= \frac{3J_1\omega_0}{\alpha - \omega_0 + \alpha\omega_0},\\
k_2 &= \frac{3J_1 J_2 \omega_0^2}{J_2 c(\alpha - \omega_0 + \alpha\omega_0)^2} - \frac{J_1 + J_2}{J_2},\\
k_3 &= \frac{J_1 J_2 \omega_0^3}{c(\alpha - \omega_0 + \alpha\omega_0)^3} - \frac{3J_1\omega_0}{\alpha - \omega_0 + \alpha\omega_0}.
\end{aligned} \tag{7}$$

For $\alpha \to 1$ the values of the coefficients

$$\begin{aligned}
k_1 &= 3J_1\omega_0,\\
k_2 &= \frac{3J_1 J_2\omega_0^2}{J_2 c} - \frac{J_1 + J_2}{J_2},\\
k_3 &= \frac{J_1 J_2 \omega_0^3}{c} - 3J_1\omega_0
\end{aligned}$$

correspond to the settings of the coefficients of the two-mass system obtained without taking into account the internal and external viscous friction.

As a result of the inverse Laplace transform, the expression for (t) in the case of full state vector control for $t_0 = 0$ will be written as:

$$X(t) = e^{\alpha N(A-B\,K)t} N\,X(0) + \int_0^t e^{\alpha N(A-B\,K)(t-\tau)} \alpha\,N\,B\,U(\tau)\,d\tau, \tag{8}$$

where $e^{\alpha N(A-B\,K)t}$ is the matrix exponent. The matrix $\alpha N(A - B\,K)$ has the following form:

$$\alpha N(A-B\,K) = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ \frac{c(\alpha-\omega_0+\alpha\omega_0)^3}{\alpha^2} & \frac{(3\alpha+2\alpha\omega_0-2\omega_0)(\alpha-1)\omega_0^2}{\alpha^2} & A_{23} \\ \frac{c(\alpha-1)(\alpha-\omega_0+\alpha\omega_0)^3}{\alpha^2 J_2} & \frac{(\alpha-2\alpha\omega_0+2\omega_0)(\alpha-\omega_0+\alpha\omega_0)^2}{\alpha^2 J_2} & A_{33} \end{bmatrix},$$

where

$$\begin{aligned}
A_{11} &= -\frac{c(\alpha-1)(\alpha-\omega_0+\alpha\,\omega_0)^3}{\alpha^2 J_2} - \frac{3\alpha\,\omega_0(\alpha-\omega_0+\alpha\,\omega_0)+(\alpha-1)^2\omega_0^3}{\alpha^2},\\
A_{12} &= \frac{(\alpha-2\alpha\,\omega_0+2\omega_0)(\alpha-\omega_0+\alpha\,\omega_0)^2}{\alpha^2 J_2} - \frac{(3\alpha+2\alpha\,\omega_0-2\omega_0)\omega_0^2}{\alpha^2 c},\\
A_{13} &= \frac{c^2(\alpha-1)(\alpha-\omega_0+\alpha\,\omega_0)^3+3c\,\alpha\,\omega_0(\alpha-\omega_0+\alpha\,\omega_0)-J_2{}^2\omega_0^3}{\alpha^2 J_2 c},\\
A_{23} &= -\frac{c(\alpha-\omega_0+\alpha\,\omega_0)^3-(\alpha-1)J_2\omega_0^3}{\alpha^2},\\
A_{33} &= \frac{(\alpha-1)c(\alpha-\omega_0+\alpha\,\omega_0)^3-(\alpha-1)J_2\omega_0^3}{\alpha^2 J_2}.
\end{aligned}$$

To analyze the processes in the synthesized system (8), we use the approach described in [27], which provides the possibility to take into account the influence of initial conditions on the system. The matrix exponent is defined according to the expression:

$$e^{\alpha N(A-B\,K)t} = \Phi(t)\,\Phi(0)^{-1},$$

where $\Phi(t)$ is the fundamental matrix of the system, which is defined as follows:

$$\Phi(t) = [x_1(t)\ \ x_2(t)\ \ x_3(t)]\,.$$

In the case of multiple roots (the system is configured for the binomial form):

$$x_1(t) = v_1 e^{\lambda t}, \quad x_2(t) = (v_1\,t + v_2)e^{\lambda t}, \quad x_3(t) = \left(v_1\frac{t^2}{2} + v_2\,t + v_3\right)e^{\lambda t},$$

where λ is the eigenvalue of the matrix $\alpha N(A - B\,K)$; v_1, v_2, v_3 are eigenvectors of the matrix $\alpha N(A - B\,K)$, which, as shown in [28], are determined from the system of equations:

$$\begin{aligned}
(\alpha\,N\,(A-B\,K) - \lambda\,I)^3 v_3 &= 0,\\
(\alpha\,N\,(A-B\,K) - \lambda\,I)\,v_3 &= v_2,\\
(\alpha\,N\,(A-B\,K) - \lambda\,I)\,v_2 &= v_1.
\end{aligned}$$

Since $(\alpha\,N(A - B\,K) - \lambda\,I)^3 = 0$, then any nonzero vector v_3 will be the solution of the first equation of the system. Then, accordingly, for $v_3 = [1\ 0\ 0]^T$ the eigenvectors v_1 and v_2 will be equal:

$$v_1 = \begin{bmatrix} \frac{(\alpha-\omega_0+\alpha\omega_0)^2\left[J_2\omega_0^2+c(\alpha-\omega_0+\alpha\,\omega_0)^2\right]}{J_2\cdot\alpha^2} \\ -\frac{c\,\omega_0(\alpha-\omega_0+\alpha\,\omega_0)^3}{\alpha^2} \\ \frac{c(\alpha-\omega_0+\alpha\,\omega_0)^4}{J_2\alpha^2} \end{bmatrix},$$

$$v_2 = \begin{bmatrix} \frac{-(\alpha-\omega_0+\alpha\,\omega_0)\left[J_2\,\omega_0(2\alpha-\omega_0+\alpha\,\omega_0)+c(\alpha-1)(\alpha-\omega_0+\alpha\,\omega_0)^2\right]}{J_2\,\alpha^2} \\ \frac{c(\alpha-\omega_0+\alpha\,\omega_0)^3}{\alpha^2} \\ -\frac{c(\alpha-1)(\alpha-\omega_0+\alpha\,\omega_0)^3}{J_2\,\alpha^2} \end{bmatrix}.$$

The matrix $\Phi(0) = [v_1 \;\; v_2 \;\; v_3]$, and the matrix $\Phi(0)^{-1}$

$$\Phi(0)^{-1} = \begin{bmatrix} 0 & \frac{-\alpha(\alpha-1)}{c(\alpha-\omega_0+\alpha\,\omega_0)^3} & \frac{J_2\,\alpha}{c(\alpha-\omega_0+\alpha\omega_0)^3} \\ 0 & \frac{\alpha}{c(\alpha-\omega_0+\,\omega_0)^2} & \frac{J_2\,\alpha\omega_0}{c(\alpha-\omega_0+\alpha\,\omega_0)^3} \\ 1 & \frac{2\,\omega_0}{c(\alpha-\omega_0+\alpha\,\omega_0)} & \frac{c(\alpha-\omega_0+\alpha\,\omega_0)^2-J_2\,\omega_0^2}{c(\alpha-\omega_0+\alpha\,\omega_0)^2} \end{bmatrix}.$$

The expression (8) can be rewritten as follows:

$$X(t) = \Phi(t)\,\Phi(0)^{-1}N\,X(0) + \int_0^t \Phi(t-\tau)\,\Phi(0)^{-1}\alpha\,NB\,U(\tau)\,d\tau.$$

Given that

$$\alpha\,NB = \begin{bmatrix} \frac{\left(c(\alpha-1)^2+J_2\right)(\alpha-\omega_0+\alpha\,\omega_0)^3}{\alpha^2\,J_1\cdot J_2} \\ \frac{c(\alpha-1)(\alpha-\omega_0+\alpha\,\omega_0)^3}{\alpha^2\,J_1} \\ \frac{c(\alpha-1)^2(\alpha-\omega_0+\alpha\,\omega_0)^3}{\alpha^2\,J_1\,J_2} \end{bmatrix},$$

$$\Phi(0)^{-1}\,\alpha\,NB = \begin{bmatrix} 0 & -\frac{\alpha-1}{J_1} & \frac{\alpha-\omega_0+\alpha\cdot\omega_0}{J_1} \end{bmatrix}^T,$$

for $(0) = 0$ we obtain:

$$X(t) = \begin{bmatrix} \int_0^t \frac{(\alpha-\omega_0+\alpha\,\omega_0)^3 \begin{pmatrix} \left(c\cdot(\alpha-\omega_0+\alpha\,\omega_0)^2+J_2\,\omega_0^2\right)(t-\tau)^2 \\ -4(\alpha-1)(\alpha-\omega_0+\alpha\,\omega_0)c(t-\tau) \\ -4J_2\,\omega_0(t-\tau)+2c(\alpha-1)^2+2J_2 \end{pmatrix}}{2\alpha^2 J_1 J_2} e^{-\omega_0(t-\tau)}U(\tau)\,d\tau \\ \int_0^t \frac{c(\alpha-\omega_0+\alpha\,\omega_0)^3 \begin{pmatrix} (\alpha-\omega_0+\alpha\,\omega_0)\omega_0(t-\tau)^2 \\ +(4\omega_0-2\alpha-4\alpha\,\omega_0)(t-\tau) \\ +2\alpha-2 \end{pmatrix}}{2\alpha^2\,J_1 J_2} e^{-\omega_0(t-\tau)}U(\tau)\,d\tau \\ \int_0^t \frac{c(\alpha-\omega_0+\alpha\,\omega_0)^3 \begin{pmatrix} (\alpha-\omega_0+\alpha\,\omega_0)^2(t-\tau)^2 \\ -4(\alpha-1)(\alpha-\omega_0+\alpha\,\omega_0)(t-\tau) \\ +2(\alpha-1)^2 \end{pmatrix}}{2\alpha^2 J_1 J_2} e^{-\omega_0(t-\tau)}U(\tau)\,d\tau \end{bmatrix}.$$

After performing the integration, the expression for the base point $_3$ will have the form:

$$x_3(t) = -\frac{c(\alpha-\omega_0+\alpha\,\omega_0)^3 \begin{pmatrix} (\alpha-\omega_0+\alpha\,\omega_0)^2\omega_0^2\,\frac{t^2}{2}e^{-\omega_0 t} \\ +\omega_0\left(\alpha^2-\omega_0^2(\alpha-1)^2\right)t\,e^{-\omega_0 t} \\ +\alpha^2 e^{-\omega_0 t}-\alpha^2 \end{pmatrix}}{\alpha^2 J_1 J_2\,\omega_0^3}U.$$

The analysis of graphs in Fig. 2 shows that the dynamics of processes in the system changes significantly depending on the index of the fractional order α and on the location of the roots, which is determined by ω_0. Therefore, the control action synthesized by the method of modal control does not provide the desired dynamics of the system. We analyze the causes of this situation using the transfer function of the system $W(s) = \frac{X_3(s)}{U(s)}$.

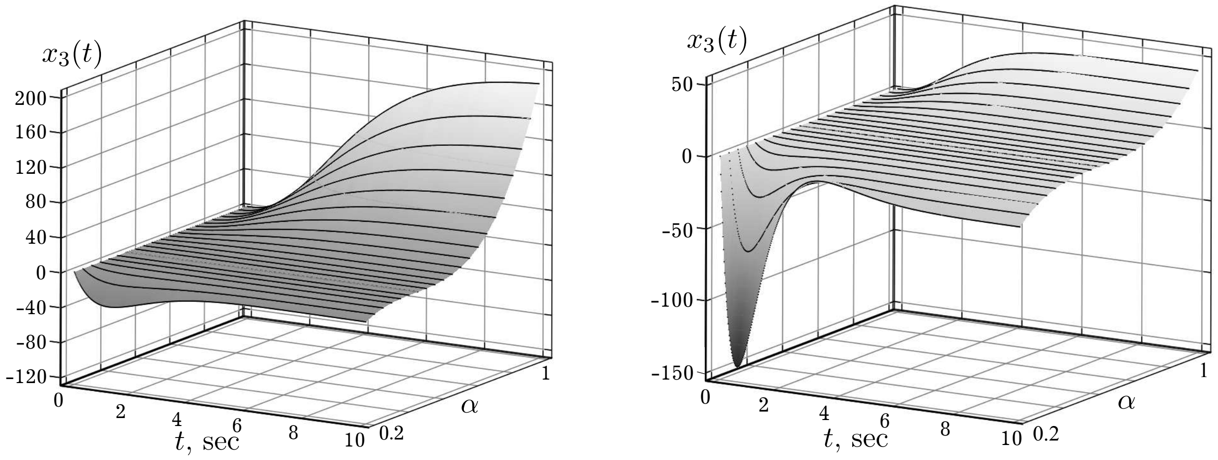

Fig. 2. Dependence of change of base point of the system when changing α for $\omega_0 = 0.75$ and $\omega_0 = 1.25$.

The transition matrix of the system is determined as follows:

$$(s\,I - \alpha\,N\,(A - B\,K))^{-1}\alpha\,NB = \begin{bmatrix} \frac{(\alpha-\omega_0+\alpha\,\omega_0)^3\left(J_2\,s^2+c(\alpha+(1-\alpha)s)^2\right)}{\alpha^2 J_1\,J_2(s+\omega_0)^3} \\ \frac{c(\alpha-\omega_0+\alpha\,\omega_0)^3 s(\alpha+(1-\alpha)s)}{\alpha^2 J_1(s+\omega_0)^3} \\ \frac{c(\alpha-\omega_0+\alpha\,\omega_0)^3(\alpha+(1-\alpha)s)^2}{\alpha^2 J_1 J_2(s+\omega_0)^3} \end{bmatrix},$$

hence the transfer function

$$W(s) = \frac{X_3(s)}{U(s)} = \frac{c(\alpha - \omega_0 + \alpha\,\omega_0)^3(\alpha + (1-\alpha)s)^2}{\alpha^2 J_1\,J_2(s+\omega_0)^3}.$$

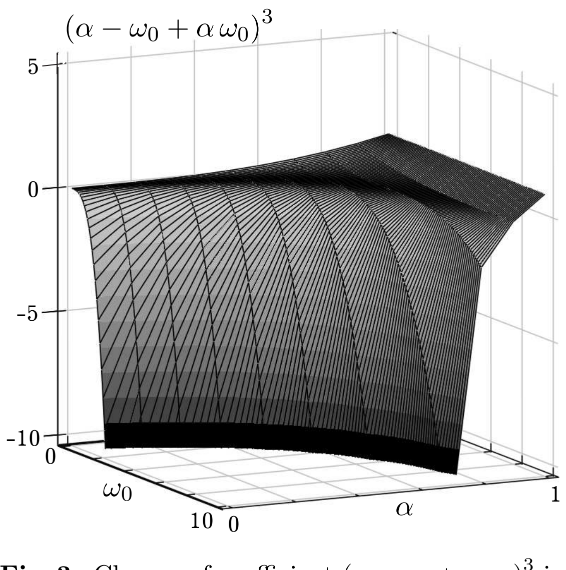

Fig. 3. Change of coefficient $(\alpha - \omega_0 + \alpha\,\omega_0)^3$ in change of α and ω_0.

From the obtained transfer function of the system it follows that the dynamics of the system is significantly influenced by the polynomial of the numerator $(\alpha + (1-\alpha)s)^2$. Multiple zeros of the transfer function $z = -\alpha/(1-\alpha)$ for small α are close to the imaginary axis and cause significant fluctuations in the initial coordinate of the system. Traditionally, the influence of zeros of the transfer function is compensated by the use of an appropriate filter at the input of the system, which leads to a deterioration of its dynamic characteristics. Another possible approach, as shown in [29], is the formation of the desired characteristic polynomial taking into account the compensation of zeros of the transfer function. On the other hand, the statics of the system depends on the coefficient $(\alpha - \omega_0 + \alpha\,\omega_0)^3$, the influence of which can be theoretically eliminated by appropriate correction of the job signal.

Shown in Figs. 4–7 dependences demonstrate that the more distant zeros and poles of the system are, the greater fluctuations are observed when the order of the fractional derivative decreases. Along with this, due to the forcing action of the zeros of the transfer function, it is possible (as shown in Figs. 4 and 5) to improve the dynamic characteristics of the system.

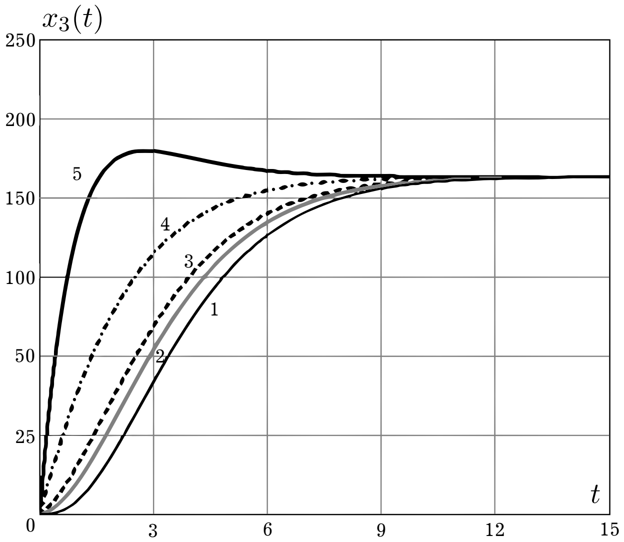

Fig. 4. Dependence of change of base point of the system with the correction of input signal for $\omega_0 = 0.75$ and with change of α: 1, $\alpha \to 1$; 2, $\alpha = 0.85$; 3, $\alpha = 0.7$; 4, $\alpha = 0.5$, and 5, $\alpha = 0.35$.

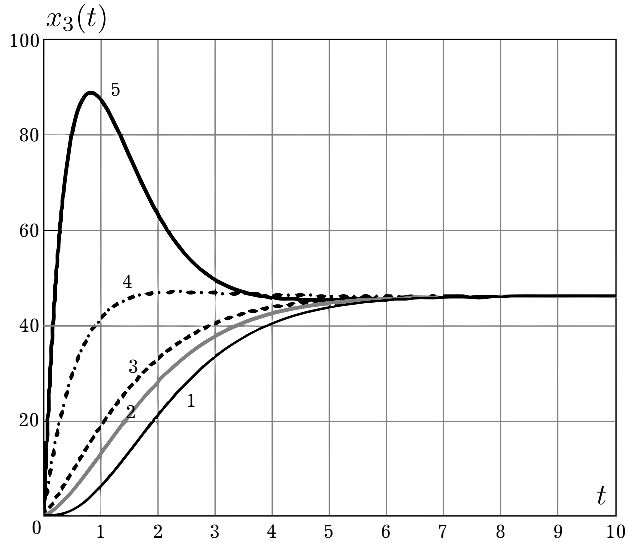

Fig. 5. Dependence of change of base point of system with correction of input signal for $\omega_0 = 1.25$ and with change of α: 1, $\alpha \to 1$; 2, $\alpha = 0.85$; 3, $\alpha = 0.7$; 4, $\alpha = 0.5$, and 5, $\alpha = 0.35$.

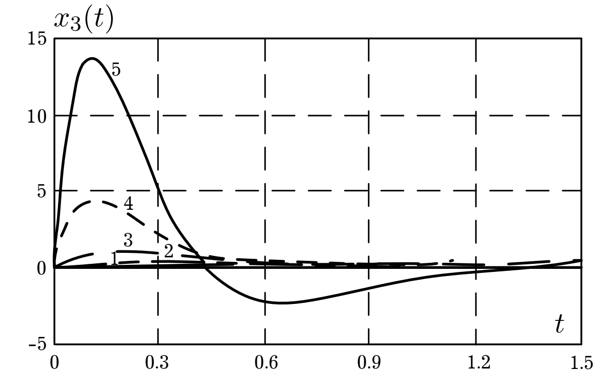

Fig. 6. Dependence of change of base point of system with correction of input signal for $\omega_0 = 5.75$ and with change of α: 1, $\alpha \to 1$; 2, $\alpha = 0.85$; 3, $\alpha = 0.7$; 4, $\alpha = 0.5$, and 5, $\alpha = 0.35$.

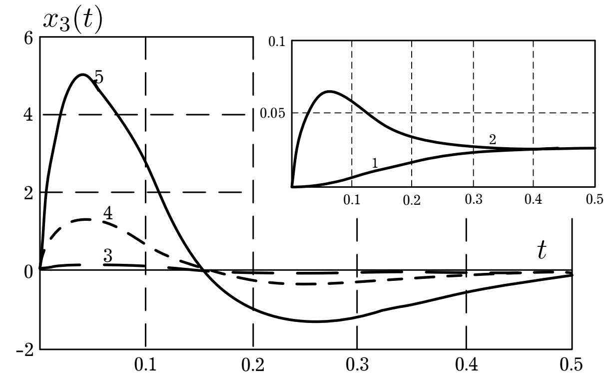

Fig. 7. Dependence of change of base point of system with correction of input signal for $\omega_0 = 15$ and with change of α: 1, $\alpha \to 1$; 2, $\alpha = 0.85$; 3, $\alpha = 0.7$; 4, $\alpha = 0.5$, and 5, $\alpha = 0.35$.

4. Conclusions

The proposed approach to the synthesis of systems with fractional derivatives using the correction factor $1/(\alpha - \omega_0 + \alpha\,\omega_0)^3$ allows forming the desired dynamic characteristics of the system according to the job signal.

Multiple zeros of the transfer function $z = -\alpha/(1-\alpha)$ for small α and significant spacing with the poles of the transfer function cause significant fluctuations of the base point of the system.

[1] Khalil H. K. Nonlinear Systems. Prentice Hall, New York (2002).

[2] Isidori A. Nonlinear control systems. Springer–Verlag (1995).

[3] Lozynskyy A., Demkiv L., Vantsevich V. Enhancement of dynamical characteristics of a fuzzy control system by using unstable subsystem. 2018 IEEE International Conference on Fuzzy Systems (FUZZ–IEEE). Rio de Janeiro, Brazil, July 08-13. 1240–1247 (2018).

[4] Petráš I. Fractional-Order Nonlinear Systems: Modeling, Analysis and Simulation. Springer–Verlag, Berlin, Heidelberg (2011).

[5] Monje C. A., Chen Y., Vinagre B. M., Xue D., Feliu-Batlle V. Fractional-order Systems and Controls. Fundamentals and Applications. Springer–Verlag, London (2010).

[6] Asadollahi M., Rikhtegar ghiasi A., Dehghani H. Excitation control of a synchronous generator using a novel fractional-order controller. IET Generation, Transmission & Distribution. **9** (15), 2255–2260 (2015).

[7] Tytiuk V., Chornyi O., Baranovskaya M., Serhiienko S., Zachepa I., Tsvirkun L., Kuznetsov V., Tryputen N. Synthesis of a fractional-order $\mathrm{PI}^{\lambda}\mathrm{D}^{\mu}$-controller for a closed system of switched reluctance motor control. Eastern-European Journal of Enterprise Technologies. **2** (2), 35–42 (2019).

[8] Busher V., Melnikova L., Horoshko V. Synthesis and implementation of fractional-order controllers in a current curcuit of the motor with series excitation. Eastern-European Journal of Enterprise Technologies. **2** (2), 63–72 (2019).

[9] Soukkou A., Belhour M. C., Leulmi S. Review design optimization and stability analysis of fractional-order PID controller. Intelligent Systems and Applications. **8** (7), 73–96 (2016).

[10] Birs I., Muresan C., Nascu I., Ionescu C. A Survey of Recent Advances in Fractional Order Control for Time Delay Systems. IEEE Access. **7**, 30951–30965 (2019).

[11] Shah P., Agashe S. Review of fractional PID controller. Mechatronics. **38**, 29–41 (2016).

[12] Badri V., Tavazoei M. S. Some Analytical Results on Tuning Fractional-Order [Proportional–Integral] Controllers for Fractional-Order Systems. IEEE Transactions on Control Systems Technology. **24** (3), 1059–1066 (2016).

[13] Lozynskyy O., Lozynskyy A., Marushchak Y., Kopchak B., Kalenyuk P., Paranchuk Y. Synthesis and research of electromechanical systems described by fractional order transfer functions. 2017 International Conference on Modern Electrical and Energy Systems (MEES). Kremenchuk, Ukraine. 16–19 (2017).

[14] Zeng F., Shu H., Zhu T., Swe T., Yang B. Fractional-order Feedback Linearization Sliding-mode Control Design for Grid-connected PV Inverters. 2019 IEEE 3rd International Electrical and Energy Conference (CIEEC). Beijing, China. 874–878 (2019).

[15] Jiacai H., Hongsheng L., Fulin T., Di L. Fractional order sliding mode controller for the speed control of a permanent magnet synchronous motor. 2012 24th Chinese Control and Decision Conference (CCDC). Taiyuan, China. 1203–1208 (2012).

[16] Liu H., Pan Y., Li S., Chen Y. Adaptive Fuzzy Backstepping Control of Fractional-Order Nonlinear Systems. IEEE Transactions on Systems, Man, and Cybernetics: Systems. **47** (8), 2209–2217 (2017).

[17] Nikdel N., Badamchizadeh M., Azimirad V., Nazari M. A. Fractional-Order Adaptive Backstepping Control of Robotic Manipulators in the Presence of Model Uncertainties and External Disturbances. IEEE Transactions on Industrial Electronics. **63** (10), 6249–6256 (2016).

[18] Aguila-Camacho N., Duarte-Mermoud M. A., Gallegos J. A. Lyapunov functions for fractional order systems. Communications in Nonlinear Science and Numerical Simulation. **19** (9), 2951–2957 (2014).

[19] Zhao Y., Wang Y., Li H. State feedback control for a class of fractional order nonlinear systems. IEEE/CAA Journal of Automatica Sinica. **3** (4), 483–488 (2016).

[20] Zhao Z., Zhang X., Wang Q. Output Feedback Stabilization of Uncertain Rectangular Descriptor Fractional Order Systems With $0 < \alpha < 1$. IEEE Access. **7**, 108948–108956 (2019).

[21] Zhang X., Zhao Z., Wang Q. Static and dynamic output feedback stabilisation of descriptor fractional order systems. IET Control Theory & Applications. **14** (2), 324–333 (2020).

[22] Luo Junhai. State-Feedback Control for Fractional-Order Nonlinear Systems Subject to Input Saturation. Mathematical Problems in Engineering. Special Issue: Chaos-Fractals Theories and Applications. **2014**, Article ID 891639, 8 pages (2014).

[23] Caputo M., Fabrizio M. A new Definition of Fractional Derivative without Singular Kernel. Progress in Fractional Differentiation and Applications. An International Journal. **1** (2), 73–85 (2015).

[24] Diethelm K., Garrappa R., Giusti A., Stynes M. Why fractional derivatives with nonsingular kernels should not be used. Fractional Calculus and Applied Analysis. **23** (3), 610–634 (2020).

[25] Li H., Cheng J., Li H.-B., Zhong S.-M. Stability Analysis of a Fractional-Order Linear System Described by the Caputo-Fabrizio Derivative. Mathematics. **7** (2), 200 (2019).

[26] Ortigueira M. D., Machado J. T. A critical analysis of the Caputo–Fabrizio operator. Communications in Nonlinear Science and Numerical Simulation. **59**, 608–611 (2018).

[27] Lozynsky A. O., Biletsky Yu. O., Lozynsky O. Yu., Moroz V. I. Formation of the fundamental matrix of an open electromechanical system and its application for the calculation of time processes of state variables. Energy Engineering and Control Systems. **6** (2), (2020), (under review), (in Ukrainian).

[28] Edwards C. H., Penney D. E. Differential Equations and Linear Algebra. Pearson (2008).

[29] Marushchak Ya. Yu., Lozynsky A. O., Kushnir A. P. Dynamics of two-mass mode stabilization systems in electric arc furnaces. Lviv, Lviv Polytechnic Publishing House (2011), (in Ukrainian).

Синтез керування за повним вектором стану в системах з похідними дробового порядку при застосуванні оператора Капуто–Фабріціо

Лозинський А. О., Лозинський О. Ю., Каша Л. В.

Національний університет "Львівська політехніка", вул. С. Бандери, 12, 79013, Львів, Україна

У роботі розглянуто синтез системи керування за повним вектором стану у випадку використання в описі системи похідних дробового порядку. Для проведення досліджень в синтезованій системі з дробовими похідними у представленні Капуто–Фабріціо сформовано фундаментальну матрицю системи, що дозволяє аналізувати також і вплив початкових умов на процеси в системі. Зокрема, продемонстровано знаходження фундаментальної матриці системи у випадку кратних коренів характеристичного полінома, які отримуються при налаштуванні синтезованої системи на біноміальну форму. Проаналізовано вплив показника дробової похідної та розміщення коренів характеристичного полінома, налаштованого на біноміальну форму, на роботу системи.

Ключові слова: *похідна дробового порядку, фундаментальна матриця, модальне керування.*