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Using of partly-boundary elements as a version of the indirect near-boundary element method for potential field modeling

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In this paper, the partly-boundary elements as a version of the indirect near-boundary element method has been considered. Accuracy and effectiveness of their using for 2D problems of potential theory have been investigated. It is shown that using of partly-boundary elements for objects of canonical shape (circle, square, rectangle, ellipse) and arbitrary polygons allows us to achieve the solution accuracy, which is comparable with the accuracy of the indirect near-boundary element method, and its order of magnitude is higher than in the indirect boundary element method. In this case, the computation time is reduced by $2-2.5$ times than in the near-boundary element method case. The software of the proposed approach has been implemented in Python. Practical testing was carried out for the tasks of electrical profiling and vertical electrical sounding in the half-plane with inclusion as a polygon. The recommendations for application of the partly-boundary elements in geophysical practice have been given.

Keywords: *boundary element method, near-boundary element method, partly-boundary elements, piecewise-homogeneous medium, electrical profiling, vertical electrical sounding, two-dimensional problem of potential theory.*

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1. Introduction

In exploration of geophysics, the researchers want to obtain information about the local inclusions inside the geological environment based on the data of stationary fields of various physical nature (electrical, magnetic, thermal, gravitational, etc.) being measured on its surface. Interpretation of the obtained data involves finding solutions of inverse problems, which is often based on multiple solving of direct problems of potential theory due to the sequential search of the physical and geometric characteristics of the object components. This, in turn, requires getting the most accurate solution of the direct problem of potential theory in the shortest possible time at each step. Similar problems arise in other fields of science and technology — non-destructive testing [1], materials science, nanophysics [2, 3], technical diagnostics, medicine, etc.

Three-dimensional (3D) modeling of the geological environment produces more reliable results, closer to real ones, but requires considerable computational cost (processing time) and higher qualification of the researcher due to more complex interpretation of the obtained results for their further using. Two-dimensional (2D) modeling gives a chance to get numerical results much easier, faster and more convenient than 3D modeling, which is a significant advantage when we estimate express information under production conditions. The result of 2D modeling is a flat image, that is easily reproduced on low-power computers. In doing so, attention should be paid to the accuracy of the obtained solutions and to the limits of applicability of 2D models.

One of the modern methods of solving the direct problems of potential theory in piecewise-homogeneous media is mathematical modeling, which is based on the construction of high-precision numerical methods for finding solutions of elliptic equations systems with appropriate boundary conditions and contact conditions at the interface. To date, one of the most effective methods for solving direct problems of potential theory is the method of fictitious sources. It is based on the ideas of boundary integral equation methods developed in the works of S. H. Mikhlin [4], N. Y. Mushelishvili [5], V. D. Kupradze [6]. Its discrete analogues are the indirect boundary element (BEM) and near-boundary element (NBEM) methods. Their basics are described in [7–11] and [12].

Indirect BEM and NBEM are used in complex geometry of component boundaries at their different physical characteristics and ways of setting field perturbation sources. They make it possible to improve the accuracy of calculations by changing the number of discretization elements. Unlike BEM, NBEM has additional parameters (near-boundary domain thickness and near-boundary element shape) that can be used to improve accuracy without increasing the computation time. In [13–15], the following types of near-boundary elements have been considered: two-dimensional — polygons (which fully or non-fully covered the near-boundary domain and even have been intersected with each other), sets of one-dimensional (families of curves), and zero-dimensional (multiple points) elements. It has been shown that the shape of the elements and the near-boundary domain thickness significantly affect the accuracy of the problem solution.

In [16, 17] the theoretical foundations of partly-boundary elements (PBE) composed of a boundary section and segments located in the near-boundary domain are described. But the approbation of this approach and comparison of speed and accuracy of numerical results with known methods were not performed. Therefore, the development of effective numerical-analytical approaches to solving these problems is of considerable applied interest in engineering practice in various applications of both geophysics and other branches.

Since theoretical considerations suggest a reduction in computation time when using partly-boundary elements, in this paper the peculiarities of solving two-dimensional problems of potential theory under this approach have been investigated. The efficiency and accuracy of the suggested solutions with similar solutions obtained by the methods of boundary and near-boundary elements (in polygon shape) have been compared.

2. Mathematical model for a homogeneous isotopic medium

Let a homogeneous isotropic solid of arbitrary shape occupy a domain $\Omega \in R^2$ with a boundary $\partial\Omega$ in the Cartesian coordinate system x_1, x_2. In every point $x = (x_1, x_2)$ the vector of an outer uniquely determined normal $n = (n_1, n_2)$ to $\partial\Omega$ is known. Then the potential field is described by the Laplace equation

$$\Delta u = \lambda \left(\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} \right) = 0 \tag{1}$$

and boundary condition

$$u = u^*(x), \quad x \in \partial\Omega, \tag{2}$$

where λ is a constant describing some physical characteristics of the media (coefficient of thermal conductivity, electrical conductivity, etc.)

3. Construction of a discrete-continuous model using partly-boundary elements

For finding a solution of the given problem (1), (2) in an izotropic homogeneous medium, we use the near-boundary element method (NBEM) [12]. We consider extended domains $B \subset R^2$, which $\Omega \subset B$, $\partial B \cap \partial\Omega = \varnothing$. We discretize the boundary $\partial\Omega$ on boundary elements (BE) Γ_v and a near-boundary domain $G = \Omega \backslash B$ on near-boundary elements (NBE) G_v such that $\cup_{v=1}^{V} \Gamma_v = \partial\Omega$, $\Gamma_i \cap \Gamma_j = \varnothing$, $\cup_{v=1}^{V} G_v = G$, $G_i \cap G_j = \varnothing$, $i \neq j$, $i, j = \overline{1, V}$. In the near-boundary domain G we introduce curves or lines G_v^+, G_v^- so that the beginning of G_v^+ is the beginning of Γ_v, and the beginning of G_v^- is the ending of Γ_v. A union $G_v^- \cup \Gamma_v \cup G_v^+ = G_v^\Gamma$ we mark as a partly-boundary element [13]. Then we introduce onto each element $\gamma_v = [\Gamma_v, G_v, G_v^\Gamma]$ a fictitious source of unknown intensity $g_v^\gamma(\xi)$ and approximate it by constant d_v^γ. Using fundamental solution (FS) of Laplace equation for a half-plane

$$\boldsymbol{E}(x,\xi) = -\frac{1}{2\pi\lambda}\ln\left|\frac{r}{r_0}\right|,$$

we write integral representation of solution to determine potential

$$u^\gamma(x) = \sum_{v=1}^{V} d_v^\gamma \int_{\gamma_v} \boldsymbol{E}(x,\xi)d\gamma_v(\xi) + C. \tag{3}$$

To find intensities of the unknown sources we use the collocation technique, that is, we satisfy the boundary condition (2) in the middle of each boundary element. After substitution (3) in (2) we obtain the following SLAE:

$$\sum_{v=1}^{V} d_v^\gamma \int_{\gamma_v} \boldsymbol{E}(x,\xi)d\gamma_v(\xi) + C = u^*(x), \quad v = \overline{1,V}. \tag{4}$$

Since in (3) due to the logarithmic singularity of FS at infinity an unknown constant C appears, the system of equations (4) must be supplemented by the condition — the total power of all sources at an infinitely distant point must be equal to zero:

$$\sum_{v=1}^{V} d_v^\gamma \int_{\gamma_v} d\gamma_v(\xi) = 0, \quad v = \overline{1,V}. \tag{5}$$

If (4), (5) rewrite as $A \cdot d = b$, where $d = \{d_1, \dots, d_V, C\}$, then elements of matrix A are determined the following way:

$$\begin{aligned}
A_{wv} &= \int_{\Gamma_v} \boldsymbol{E}\left(x^w,\xi\right)\, d\Gamma_v\left(\xi\right), \text{ when we use BE,}\\
A_{wv} &= \int_{G_v} \boldsymbol{E}\left(x^w,\xi\right)\, dG_v\left(\xi\right), \text{ when we use NBE,}\\
A_{wv} &= \int_{G_v^+} \boldsymbol{E}\left(x^w,\xi\right)\, dG_v^+\left(\xi\right) + \int_{G_v^-} \boldsymbol{E}\left(x^w,\xi\right)\, dG_v^-\left(\xi\right)\\
&\quad + \int_{\Gamma_v} \boldsymbol{E}\left(x^w,\xi\right)\, d\Gamma_v\left(\xi\right), \quad w,v = \overline{1,V}, \text{ when we use PBE,}\\
A_{(V+1)v} &= \int_{\gamma_v} d\gamma_v(\xi), A_{v(V+1)} = 1, \quad A_{(V+1)(V+1)} = 0, \quad b_w = x_2^w, \quad w = \overline{1,V}, \quad b_{V+1} = 0.
\end{aligned}$$

As we can see, it is three times longer to calculate the integral on PBE than on the BE.

After finding the vector of unknown d as the solution of SLAE (4), (5), the required potential is calculated by the formula (3).

4. Numerical studies of the computational capabilities of the PBE

To compare the computational properties of the developed approach with BEM and NBEM the problem (1), (2) for four canonical form domains has been solved. Four canonical domains Ω with physical characteristics $\lambda = 1$ are considered, the extended domains B are chosen as:

a) a square $\Omega = \{(x_1,x_2)\colon -1 \leqslant x_1 \leqslant 1, -1 \leqslant x_2 \leqslant 1\}$,
$B = \{(x_1,x_2)\colon -1-h \leqslant x_1 \leqslant 1+h, -1-h \leqslant x_2 \leqslant 1+h\}$,

b) a circle $\Omega = \left\{(x_1,x_2)\colon x_1^2 + x_2^2 \leqslant 1\right\}$, $B = \left\{(x_1,x_2)\colon x_1^2 + x_2^2 \leqslant 1+h\right\}$,

c) a rectangle $\Omega = \{(x_1,x_2)\colon -4 \leqslant x_1 \leqslant 4, -2 \leqslant x_2 \leqslant 2\}$,
$B = \{(x_1,x_2)\colon -4-h \leqslant x_1 \leqslant 4+h, -2-h \leqslant x_2 \leqslant 2+h\}$,

d) an ellipse $\Omega = \left\{(x_1, x_2) \colon \frac{x_1^2}{16} + \frac{x_2^2}{4} = 1\right\}$, $B = \left\{(x_1, x_2) \colon \frac{x_1^2}{(4+h)^2} + \frac{x_2^2}{(2+h)^2} = 1\right\}$, where h is the thickness of the near-boundary domain.

Each of the boundary of the square and the circle were divided into 16 boundary elements of equal length, and each of the boundary of the rectangle and the ellipse — into 20 elements. On the boundary elements there were constructed near-boundary elements, which completely covered the near-boundary domain: trapezoids (for square and rectangle) and curved quadrilaterals (for circle and ellipse). At the domain boundary the condition of the first kind, that is, the value of potential $u^*(x) = x_2$, has been given.

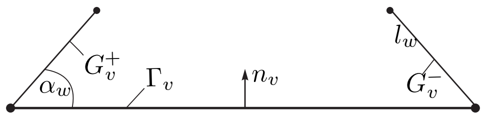

Fig. 1. Partly-boundary element geometry for the rectangle.

In the near-boundary domain G at each BE the PBE has been constructed. The G_v^-, G_v^+ were chosen as segments with a length l_w and an angle of inclination α_w to the boundary element. Fig. 1 shows the form of PBE, built on part of the rectangle boundary, and Fig. 2 illustrates the construction of partly boundary elements for four canonical domains.

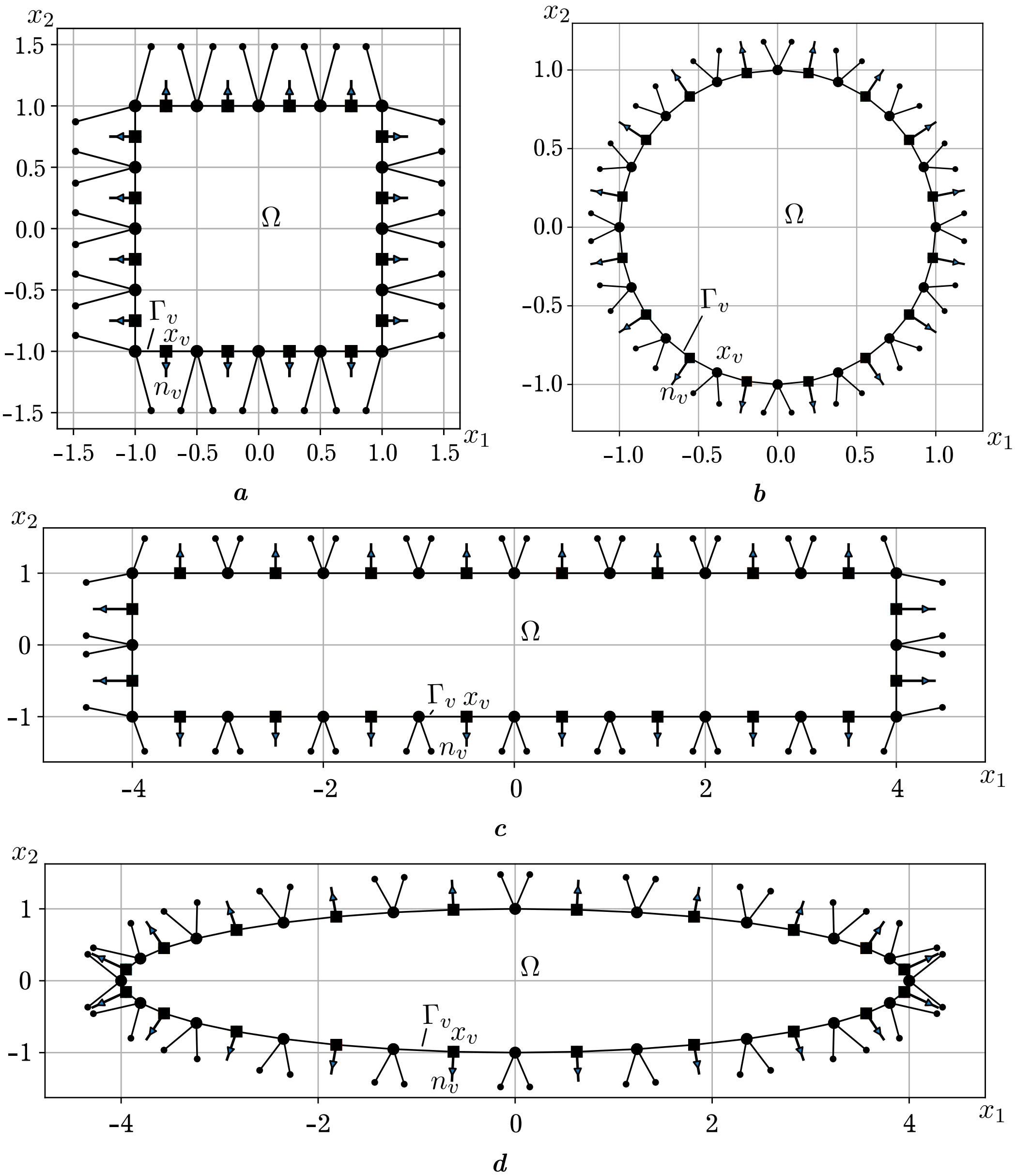

Fig. 2. PBE for four canonical domains.

The functions $u^{\gamma}(x) = \{u^{G}(x), u^{\Gamma}(x), u^{G^{\Gamma}}\}$ were calculated by formula (3) at 100 points on the first quarter of the boundary, i.e. $x_1 \in [0,1]$, $x_2 > 0$ for the circle and the square and $x_1 \in [0,4]$, $x_2 > 0$ for the rectangle and the ellipse. The values (as a percentage) of absolute errors

$$\theta^{\gamma} = 100\,|u^{*} - u^{\gamma}| \tag{6}$$

depending on the thickness h of the near-boundary domain and parameters a_w and l_w were studied. For numerical integration Gaussian formulae of four nodes for near-boundary elements and of three ones for boundary and partial boundary elements are used.

Fig. 3 shows the errors of the potential function calculated by formula (3) when problem (1), (2) has been solved for the square (Figs. 3a, 3b) and the circle (Figs. 3c, 3d). Figs. 3a, 3c show the error dependence on the angle α_w of inclination, which varied from 45 to 135 degrees for the constant length $l_w = 1$, and Figs. 3b, 3d show the error dependence when l_w changing from 1 to 5 for three constant values α_w.

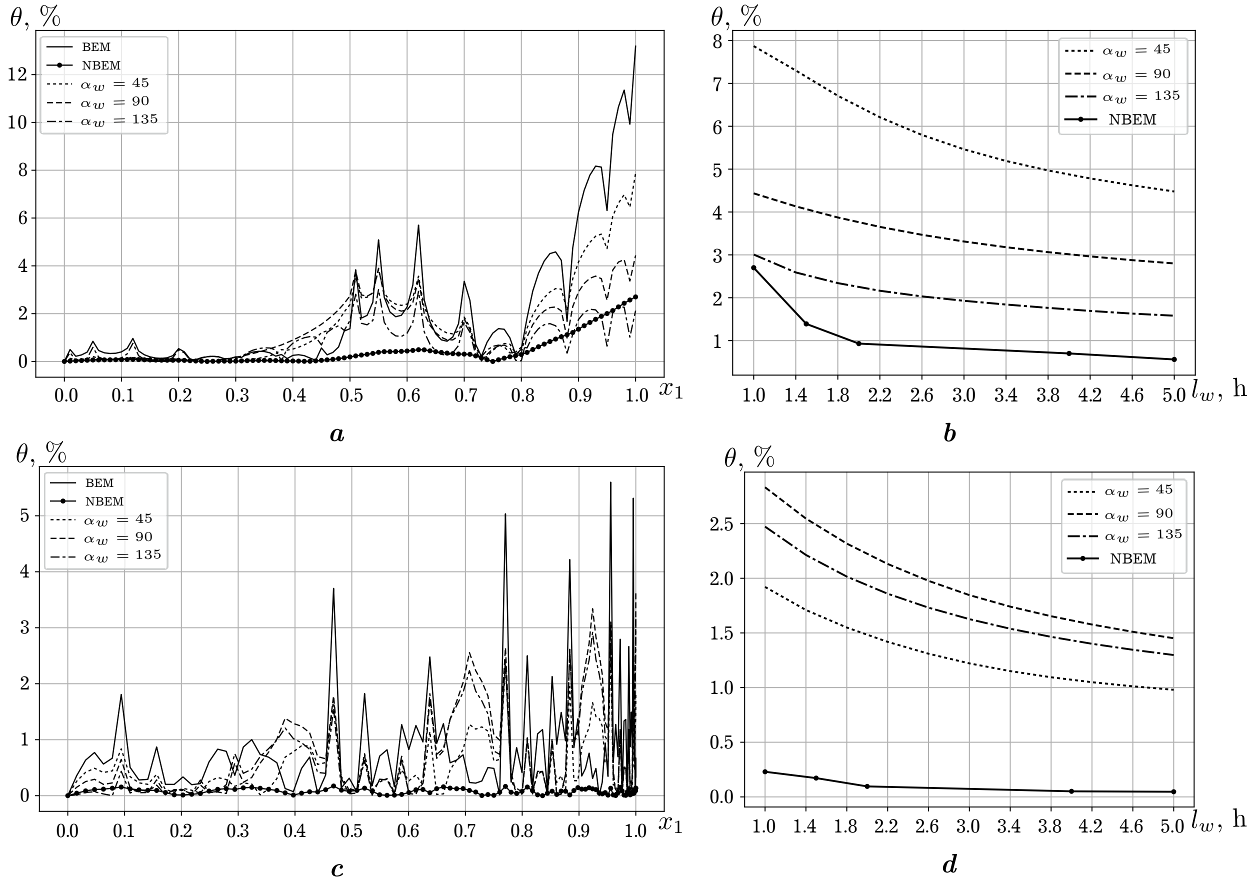

Fig. 3. Comparison of absolute errors of potential function for circle and square using boundary (solid curves), two-dimensional near-boundary (solid curves with symbols), and partly-boundary (dashed, dotted, dashed and dotted curves) elements.

As it can be seen from the above graphs, the maximum error of calculation (with the same number of elements) in BEM reaches 12% for a square and 8% for a circle, in NBEM it is 2.5% for a square and 0.3% for a circle. When using partly-boundary elements it is within the range from 1% to 3.5%.

Similar dependencies have been obtained for rectangle and ellipse. They allow us to conclude: as a result of increasing the size of element along one axis, the maximum error in all approaches increases by $2-3$ times. The highest error in NBEM and BEM is observed when approaching the angles of a square and a rectangle. For the PBE it is more evenly distributed over the study area, that is, the proposed approach avoids the smoothing of angles. Table 1 provides a comparison of the computation time for different variants of the domain shape when using BEM, NBEM, PBE.

Table 1.

	Square	Circle	Rectangle	Ellipse
BE	0.28 s	0.30 s	0.31 s	0.39 s
NBE	3.45 s	4.04 s	3.38 s	4.48 s
PBE	0.85 s	0.88 s	1.08 s	1.37 s

The graphs and the table above show that the accuracy of the NBEM calculation is the highest, but the calculation time is 3 times longer than when using the PBE. However, in the case of PBE, the accuracy can be adjusted with two parameters (α_w and l_w), which allows you (with the help of a software module) to select automatically a couple of parameters with a predetermined accuracy (for example, such as in NBEM) and thus significantly reduce (by $2-2.5$ times) computation time.

Software implementation of the proposed approaches is implemented by the modern powerful Python programming language, since it is distributed free of charge, has a large number of additional libraries, such as NumPy, SciPy, Matplotlib and others. Their use greatly accelerates and facilitates the writing of programs that allow you to control visually the processes of formation of the geometric area of study, its discrete model, and the computational process as a whole. Based on the proposed approach, Python has created an automated computer module that builds optimally accurate partly-boundary elements in the form of sections of the given length with an arbitrary angle of inclination for polygons and ellipses.

5. Mathematical model for a piecewise homogeneous isotropic half-plane

Let a boundary $\partial\Omega$ of a half-plane $\Omega = R^{2-}$ is electrically insulated. Current sources — supply electrodes A and B of intensity g_A and g_B are located at the points $A(x_A, 0)$ and $\mathrm{B}(x_B, 0)$. The half-plane contains an inclusion Ω_1 that is in ideal contact with the domain $\Omega_0 = \Omega\backslash\Omega_1$. To find electrical potentials u_0, u_1 in the domains Ω_0, Ω_1, we obtain the following boundary value problem:

$$\Delta u_0 = \lambda_0 \left(\frac{\partial^2 u_0}{\partial x_1^2} + \frac{\partial^2 u_0}{\partial x_2^2}\right) = 0, \quad x \in \Omega_0, \tag{7}$$

$$\lambda_1 \left(\frac{\partial^2 u_1}{\partial x_1^2} + \frac{\partial^2 u_1}{\partial x_2^2}\right) = 0, \quad x \in \Omega_1 \tag{8}$$

$$-\lambda_0 \frac{\partial^2 u_0}{\partial \boldsymbol{n}} = -g_{AB}(x), \quad x \in \partial\Omega, \tag{9}$$

$$u_0(x) = u_1(x), \quad \lambda_0 \frac{\partial^2 u_0}{\partial \boldsymbol{n}} = \lambda_1 \frac{\partial^2 u_1}{\partial \boldsymbol{n}}, \quad x \in \partial\Omega_1. \tag{10}$$

Here $g_{AB}(x) = g_A\chi_A + g_B\chi_B$, χ_A, χ_B are characteristic functions of points A and B; λ_0,λ_1 are physical characteristics of the domains Ω_0, Ω_1.

To solve the problem (7)–(10) in an isotropic piecewise-homogeneous half-plane, in addition to the FS for inclusion, we use the special FS of the Laplace equation for Ω_0 which exactly satisfies the electrically insulation condition (9) at the half-plane boundary, and a derivative $\boldsymbol{F}^+(x,\xi)$ of it along a normal $\boldsymbol{n}$ to $\partial\Omega$:

$$\begin{aligned} \boldsymbol{E}^+(x,\xi) &= -\frac{1}{2\pi\lambda_0}\left(\ln\left|\frac{r}{r_0}\right| + \ln\left|\frac{r'}{r_0}\right|\right), \\ \boldsymbol{F}^+(x,\xi) &= -\frac{\partial E^+(x,\xi)}{\partial \boldsymbol{n}} = \frac{(y_1 n_1 + y_2 n_2)}{2\pi r^2} + \frac{(y_1 n_1 + y_2' n_2)}{2\pi r'^2}. \end{aligned} \tag{11}$$

Here $r'^2 = y_1^2 + y_2'^2$, $y_i = x_i - \xi_1$, $i = 1, 2$, $y_2' = x_2 + \xi_2$.

Due to the use of SFS (11), only the boundary $\partial\Omega_1$ is discretized into boundary elements. Then near-boundary G_{0v} and partly boundary G_{0v}^{Γ}, $v = \overline{1,V}$ elements for the domain Ω_0 are introduced in the domain $G_0 \subset B_0 \subset R_0^2$, and for the domain Ω_1 in the $G_1 \subset B_1 \subset R_1^2$. Here $G_s = B_s \backslash \Omega_s$, $s = 1, 2$, $\partial B_s \cap \partial \Omega_s = \varnothing$; R_0^2, R_1^2 are planes which have such properties as: $R_1^2 \cap R_0^2 = \partial\Omega_1$, $R_s^2 \cap R^2 = \Omega_s \cup \partial\Omega_s$, $(R_0^2 \cup R_0^2) \cap R^2 = \Omega \cup \partial\Omega$ [12].

After entering the unknown sources and approximating them by constants, the integral representations of the solutions will look like:

$$u_0^\gamma(x) = \sum_{v=1}^{V} d_{0v}^\gamma \int_{\gamma_{0v}} \boldsymbol{E}^+(x,\xi)\, d\gamma_{0v}(\xi) + g_A(x)\, \boldsymbol{E}^+(x, x_A) + g_B(x)\, \boldsymbol{E}^+(x, x_B), \tag{12}$$

$$u_1^\gamma(x) = \sum_{v=1}^{V} d_{1v}^\gamma \int_{\gamma_{1v}} \boldsymbol{E}(x,\xi)\, d\gamma_{1v}(\xi) + C_1, \tag{13}$$

$$\frac{\partial u_0^\gamma(x)}{\partial \boldsymbol{n}} = -0.5\chi_v^\gamma d_{0v}^\gamma + \sum_{v=1}^{V} d_{0v}^\gamma \int_{\gamma_{0v}} \boldsymbol{F}^+(x,\xi)\, d\gamma_{0v}(\xi) + g_A(x)\, \boldsymbol{F}^+(x, x_A) + g_B(x)\, \boldsymbol{F}^+(x, x_B), \tag{14}$$

$$\frac{\partial u_1^\gamma(x)}{\partial \boldsymbol{n}} = -0.5\chi_v^\gamma d_{1v}^\gamma + \sum_{v=1}^{V} d_{1v}^\gamma \int_{\gamma_{1v}} \boldsymbol{F}^+(x,\xi)\, d\gamma_{1v}(\xi), \tag{15}$$

where $\chi_v^\gamma(x) = 1$ for $x \in \Gamma_v$, $\chi_v^\gamma(x) = 0$ for $x \notin \Gamma_v$, i.e. they are characteristic functions of Γ_v.

SLAE for finding the vector of unknowns ($s = 1, 2$) is obtained by substituting (11)–(15) in (10) and adding a condition similar to (5):

$$\sum_{v=1}^{V} d_{1v}^\gamma \int_{\gamma_{1v}} d\gamma_{1v}(\xi) = 0.$$

6. Practical implementation of the proposed approaches in modeling direct current prospecting problems for the piecewise-homogeneous half-plane

The methods of electrical profiling and vertical electric sounding (VES) [18] at the half-plane boundary have been modeled. An inclusion in the form of a heptagon was chosen, the supply electrodes had the intensity $g_A = -0.5$ and $g_B = 0.5$, the current force was chosen equal to 1. The measuring electrodes are located at points M and N within the segment AB (Fig. 4).

The inclusion boundary was divided into 17 elements of approximately the same length. The automatically determined parameters of the partly-boundary elements at which the highest accuracy (3%) is achieved were as follows: $\alpha_w = 102$, $l_w = 2$.

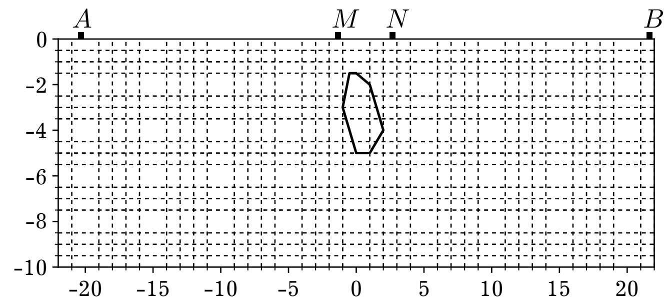

Fig. 4. Geometry of a heptagonal inclusion.

In the case of the electrical profiling method, the supply electrodes were positioned at points $A(-21.0)$ and $B(21.0)$, and the distance between the measuring electrodes N and M was chosen $MN = 0.1AB$. N and M are moved along the line $(-21, 21)$ with 0.1 step. In the case of VES, the distance between the measuring electrodes was the same one, but they were stationary and are located above the center of the inclusion, the length AB was varied according to the law of geometric progression $AB = 0.4 \cdot p^k$, where, $p = 1.6$, $k = [1, \ldots, 13]$.

The time of finding the solution and its accuracy were investigated.

Initially, for verification of the software complex, the conductivities in the domains Ω_0 and Ω_1 were chosen the same ones and we estimated the absolute error of the apparent resistivity (a value is in inverse relation to the coefficient of electrical conductivity):

$$\theta = 100 \cdot |\rho^* - \rho^\gamma| , \tag{16}$$

where

$$\rho^\gamma = \frac{k_u}{I} \left| u^\gamma(x^M) - u^\gamma(x^N) \right| ,$$

$$k_u = 2\pi \left(\frac{1}{\ln r_{AM}} - \frac{1}{\ln r_{AN}} - \frac{1}{\ln r_{BM}} + \frac{1}{\ln r_{BN}} \right)^{-1}$$

is the installation coefficient [18]; $r_{CD} = \sqrt{(x_1^C - x_1^D)^2 + (x_2^C - x_2^D)^2}$; $x^C = (x_1^C, x_2^C)$ are coordinates of a point C. It is clear that the apparent resistivity ρ^* of a homogeneous half-plane is equal to one at each point.

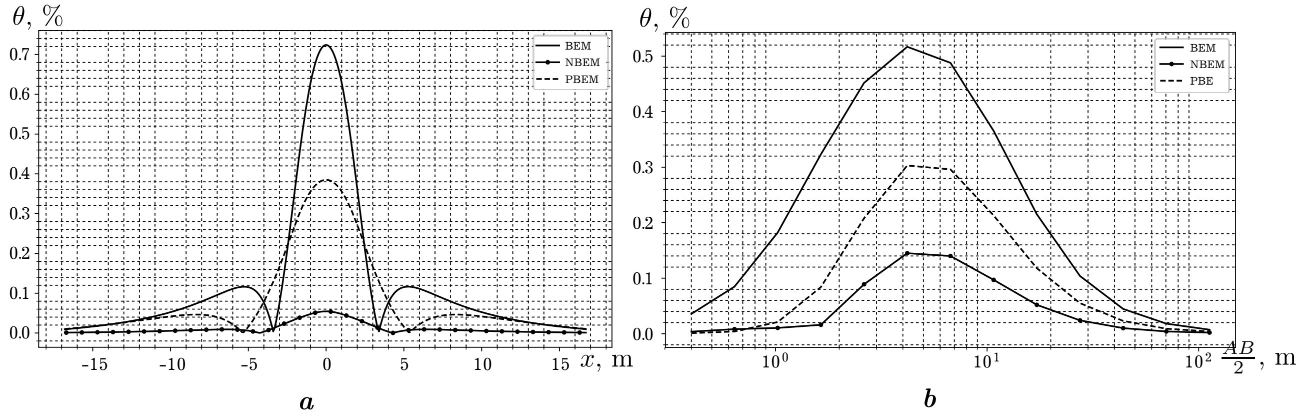

Fig. 5. Comparison of the calculation accuracy of the apparent resistivity in a homogeneous plane for the methods of electrical-profiling (***a***) and VES (***b***).

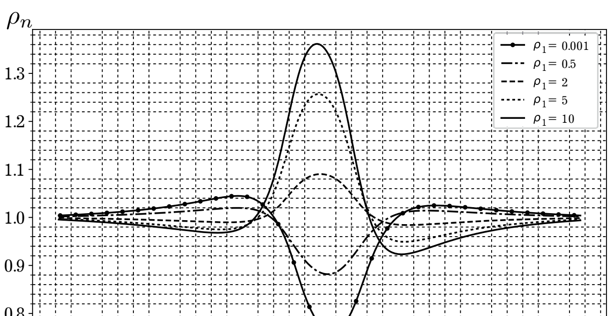

Fig. 6. Investigation of the influence of the resistance of the heptagonal inclusion by the method of electrical profiling.

Fig. 5 presents the errors of the apparent resistivity calculated by the formula (16) for problem (7)–(10) for electrical profiling and VES over the homogeneous half-plane.

Fig. 6 shows the similar results for the method of electrical profiling over the heptagonal inclusion with different resistivity $\rho_1 = 0.001$, 0.5, 2, 5, 10.

As it can be seen from Figs. 5a, 5b, the accuracy of the calculations using the PBE is higher than BEM and lower than NBEM. The calculation time of one curve of the apparent resistivity by the method of electrical profiling in BEM is 2.73 s, in NBEM is 26.64 s, by using PBE is 7.47 s, and by method of VES is 3.09 s, 33.62 s, 8.69 s, respectively.

7. Conclusions

Conducted numerical experiments have indicated the feasibility and effectiveness of the PBE for solving the problems of the potential theory, because they automatically adjust the parameters to achieve high accuracy $(0.3 - 5\%)$ much faster than in the NBEM with the same number of elements (which defines the dimension of the matrix) and the degree of approximation of the unknown functions. In particular, the accuracy of the solution obtained by using the PBE is commensurate with the accuracy of the NBEM, and is $8-10$ times higher than in the BEM. However, the time for solving the problem with the PBE is $2-2.5$ times shorter than in the NBEM.

At the same time, the use of PBE requires, as in BEM, a higher qualification of the researcher due to the appearance of integrals in the Cauchy sense in determining the derivative of the potential. In BEM, to achieve the given accuracy, it is necessary to increase the number of elements, which leads to an increase in computation time. In NBEM, you can change the thickness of the near-boundary domain, which does not affect the solution time.

The results of the studies indicate the feasibility of using PBE in the following cases:

— when constructing solutions of inverse problems, both to find the initial approximation and in algorithms that use multiple repetitions of the solution of the problems;
— in express methods for solving inverse geophysical problems, when the solution time is more important than the high precision;
— when constructing mathematical models and computer programs for the prompt search of anomalous geological objects with sufficient precision for practice;
— for obtaining express information about the state of the environment in the field conditions and to evaluate the possibility of applying certain geophysical methods;
— for minimizing the risk of making erroneous decisions, for determining model settings and a strategy choice for further field research.

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Використання частково-граничних елементів як варіанту непрямого методу приграничних елементів при моделюванні потенціальних полів

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У статті розглянуто частково-граничні елементи як варіант непрямого методу приграничних елементів. На прикладі двовимірних задач теорії потенціалу досліджено точність та ефективність їх використання. Для об'єктів канонічної форми (круг, квадрат, прямокутник, еліпс) та довільних багатокутників показано, що використання частково-граничних елементів дозволяє досягнути точності розв'язку, співвимірної з точністю методу приграничних елементів, і на порядок вищої, ніж у методі граничних елементів. При цьому зменшено у $2-2.5$ рази час обчислень, ніж у методі приграничних елементів. Програмну реалізацію запропонованого підходу здійснено на Python. Здійснено практичну апробацію для задач електропрофілювання та вертикального електричного зондування у півплощині з багатокутним включенням. Наведено рекомендації щодо застосування частково-граничних елементів у геофізичній практиці.

Ключові слова: *метод граничних елементів, метод приграничних елементів, частко-граничні елементи, кусково-однорідне середовище, електричне профілювання, вертикальне електричне зондування, двовимірна задача теорії потенціалу.*