MATHEMATICAL MODELING AND COMPUTING, Vol. 8, No. 1, pp. 22–34 (2021)

On the asymptotic output sensitivity problem for a discrete linear systems with an uncertain initial state

Ben Rhila S.[1], Lhous M.[2], Rachik M.[1]

[1] *Laboratory of Analysis, Modelling and Simulation, Department of Mathematics and Computer Science, Faculty of Sciences Ben M'sik, Hassan II University of Casablanca, BP 7955, Sidi Othman, Casablanca, Morocco*

[2] *Laboratory of Modelling, Analysis, Control and Statistics, Department of Mathematics and Computer Science, Faculty of Sciences Ain Chock, Hassan II University of Casablanca, BP 5366, Maarif, Casablanca, Morocco*

(Received 11 September 2020; Revised 24 November 2020; Accepted 25 November 2020)

This paper studies a finite-dimensional discrete linear system whose initial state x_0 is unknown. We assume that the system is augmented by two output equations, the first one z_i being representing measurements made on the unknown state of the system and the other y_i being representing the corresponding output. The purpose of our work is to introduce two control laws, both in closed-loop of measurements z_i and whose goal is to reduce asymptotically the effects of the unknown part of the initial state x_0. The approach that we present consists of both theoretical and algorithmic characterization of the set of such controls. To illustrate our theoretical results, we give a number of examples and numerical simulations.

Keywords: *discrete-time, relatively insensitive, linear system, observability, stability, uncertainty.*

2010 MSC: 93C55, 93B35, 93B07, 93D09 **DOI:** 10.23939/mmc2021.01.022

1. Introduction

When mathematically modeling a system, one is always confronted with disturbances mainly caused by the natural environment of the latter or due to poorly identified parameters. The problem of sensitivity, in which we are interested in this work, is a general concept that is often defined according to the considered system. But one common thing between almost all works studying the notion of sensitivity is the study of variations of an output function or a response function relative to one or more parameters acting within the model.

To give an idea of the various definitions of sensitivity, let us quote, for example, the work of R. Silvério and al [1], in which they consider an epidemic model of HRSV in Florida and study the sensitivity of the basic reproduction number with respect to a certain parameter of the model; also the work of J. Y. Semergui et al [2], where they consider an HIV/AIDS model and study the sensitivity of optimal control against some model parameters. In [3], S. A. Soldatenko and al consider an Earth's climate mathematical model where they study the sensitivity of a response function with respect to the parameter α around the unperturbed value 0. Another example is that of the work of A. Kowalewski et al [4], in which the sensitivity analysis is performed for a class of optimal control problem with a time lag parabolic equation in which delay argument appears in the state of the system and in the Neumann boundary conditions. Here again, the authors have studied the sensitivity of the optimal control compared to small variations of the delay parameter. In all the works we have quoted, the sensitivity goes through an adequate directional derivative or an appropriate gradient.

The main goal of this paper is to present a contribution to the study of the sensitivity output problem for discrete linear system. That means we consider the discrete linear system

$$\begin{cases} x_{i+1} = Ax_i + Bu_i, \\ x_0 = \sum_{j=1}^{n} \beta_j e_j, \end{cases} \tag{1}$$

where x_i is the variable state, u_i is the variable control, β_j are the components of the initial state in $\mathbb{R}^n$ with e_j being a canonical basis of $\mathbb{R}^n$. We suppose that the parameters $(\beta_1, \ldots, \beta_r)$ are unknown and $(\beta_{r+1}, \ldots, \beta_n)$ are known. As the initial state x_0 is unknown, we consider the measurement function given by

$$z_i = Mx_i, \quad \forall i \geqslant 0. \tag{2}$$

The associated output function is given by

$$y_i = Cx_i + Dv_i, \quad \forall i \geqslant 0, \tag{3}$$

where A, B, C, M and D are, respectively, (n, n), (n, m), (p, n), (n, n) and (p, l) matrices. We assume that the matrix M is not invertible, and the controls law, stabilizing the output of system, are given by

$$u_i = Lz_i \quad \text{and} \quad v_i = Kz_i \tag{4}$$

with $L \in \mathcal{M}_{mn}(\mathbb{R})$ and $K \in \mathcal{M}_{ln}(\mathbb{R})$.

Our main objective in this work is to determine the L and K gain matrices in order the impact of unknown parameters $\beta_1, \ldots, \beta_r$ on the output y_i to disappear asymptotically, to achieve this goal we take inspiration from J. L. Lions [5–7] on the notion of sentinels to determine the matrices L and K such that

$$\lim_{i \to +\infty} \frac{\partial y_i}{\partial \beta_s} = 0, \quad 1 \leqslant s \leqslant r. \tag{5}$$

To establish a more general result than (5), let us consider a real positive sequence (α_i) tending towards 0, for example $\alpha_i = \left(\frac{1}{i}\right), \left(\frac{1}{i^2}\right), e^{-i}, \ldots$, we will propose a technique to describe the controls law defined by (4), which achieves the following predefined mode of stabilization

$$\left\| \frac{\partial y_i}{\partial \beta_s} \right\| \leqslant \alpha_i, \ \forall i \geqslant 0, \ 1 \leqslant s \leqslant r. \tag{6}$$

The sequence $(\alpha_i)_i$ can be interpreted as a desired degree of stability. That means, we focus our interest on determination of the set of gain matrices K and L defined by

$$S = \left\{ K \in \mathcal{M}_{ln}(\mathbb{R}),\, L \in \mathcal{M}_{mn}(\mathbb{R}) / \left\| \tfrac{\partial y_i}{\partial \beta_s} \right\| \leqslant \alpha_i, \ \forall i \geqslant 0, \ 1 \leqslant s \leqslant r \right\}. \tag{7}$$

Inspired by the approach using for the output admissibility and the maximal output admissible sets for initial states [8–20] and under some assumption, we establish that set (7) can be described by a finite number of inequalities and an algorithmic determination of each gain matrices is presented.

This paper is organised as follows: in section 2, the characterization of the gain matrices is presented. An algorithmic determination of the characterization of the tolerable set for each gain matrices will be presented in section 3. Some sufficient conditions for the characterization of the tolerable set are described in section 4 and numerical simulations are given to illustrate the obtained results. A discrete delayed system is also considered in section 5 and a conclusion is given in section 6.

2. Characterization of the tolerable sets

Consider the discrete controlled linear system described by

$$\begin{cases} x_{i+1} = Ax_i + Bu_i, \\ x_0 = \sum\limits_{j=1}^{n} \beta_j e_j, \end{cases}$$

the corresponding output is

$$y_i = Cx_i + Dv_i,$$

where $x_i \in \mathbb{R}^n$ is the state variable, $\beta_1, \beta_2, \ldots, \beta_r$ are unknown components of initial state and $\beta_{r+1}, \ldots, \beta_n$ are supposed known, e_i, $i = 1, \ldots, n$ is the canonical basis of $\mathbb{R}^n$, $u_i \in \mathbb{R}^m$ and $v_i \in \mathbb{R}^l$ are the input variables and $y_i \in \mathbb{R}^p$ is the output vector. A, B, C and D are real matrices of appropriate dimension.

Definition 1. *Output function is insensitive to the effects of the uncertainties, if the corresponding output satisfies the following condition*

$$\left\| \frac{\partial y_i}{\partial \beta_s} \right\| \leqslant \alpha_i, \ \forall i \geqslant 0, \ 1 \leqslant s \leqslant r.$$

Let $(\alpha_i)_{i \geqslant 0}$ be a positive decreasing sequence which verifies

$$\frac{\alpha_i}{\alpha_{i+1}} \leqslant \frac{\alpha_{i-1}}{\alpha_i}, \ \forall i \geqslant 1. \tag{8}$$

As examples of such sequences we cite

$$\alpha_i = \frac{1}{i+1}; \quad \alpha_i = \frac{1}{(i+1)^s}, \quad s \in [1, +\infty[; \quad \alpha_i = \rho^i, \quad \rho < 1.$$

The controls law

$$u_i = Lz_i \quad \text{and} \quad v_i = Kz_i$$

stabilizing the output of system are introduced in order to make the system insensitive to the effects of all unknown uncertainties components of initial state. To characterize the set of all control law which make the output insensitive to the effects of uncertainties, we consider the set noted the tolerable set given by

$$T(L,K) = \left\{ x \in \mathbb{R}^n / \left\| (C + DKM)(A + BLM)^i x \right\| \leqslant \alpha_i, \forall i \geqslant 0 \right\}.$$

Now for every $i \geqslant 0$, we have

$$x_{i+1} = Ax_i + Bu_i,$$
$$x_{i+1} = (A + BLM)x_i,$$

hence

$$x_i = (A + BLM)^i x_0,$$
$$y_i = (C + DKM)x_i,$$

then

$$y_i = (C + DKM)(A + BLM)^i x_0,$$
$$y_i = \sum_{j=1}^{n} \beta_j (C + DKM)(A + BLM)^i e_j,$$

and
$$\frac{\partial y_i}{\partial \beta_j} = (C + DKM)\,(A + BLM)^i\, e_j. \tag{9}$$

Using (6), we can write S as follows

$$\begin{aligned} S &= \left\{ K \in \mathbb{R}^{n\times l},\, L \in \mathbb{R}^{m\times n} \mid \left\|(C+DKM)(A+BLM)^i e_s\right\| \leqslant \alpha_i, \text{ for } \forall i \geqslant 0, 1 \leqslant s \leqslant r \right\} \\ &= \left\{ K \in \mathbb{R}^{n\times l},\, L \in \mathbb{R}^{m\times n} \mid e_s \in T\,(L,K)\,,\ 1 \leqslant s \leqslant r \right\}. \end{aligned} \tag{10}$$

We note that the set $T(L,K)$ of all gain matrices K and L is defined by an infinite number of inequalities. We will establish sufficient conditions which allow us to describe it by a finite number of inequalities.

Let rewrite the set $T(L,K)$ as

$$T(L,K) = \left\{ x \in \mathbb{R}^n / \left\|\tilde{C}\tilde{A}^i x\right\| \leqslant \alpha_i, \forall i \geqslant 0 \right\}. \tag{11}$$

where $\tilde{A} = (A + BLM)$ and $\tilde{C} = (C + DKM)$.

In order to characterize $T(L,K)$, we introduce for each integer k the set $T_k(L,K)$ defined by

$$T_k(L,K) = \left\{ x \in \mathbb{R}^n / \left\|\tilde{C}\tilde{A}^i x\right\| \leqslant \alpha_i, \forall i = 0,\dots,k \right\}. \tag{12}$$

If there are no confusion, we note $T = T(L,K)$ and $T_k = T_k(L,K)$.

Theorem 1. *i) The set T is a closed, convex and symmetric set.*

ii) If we suppose that $\lim\limits_{k\to+\infty} \sup \|\tilde{A}^k\|/\alpha_k < \varepsilon$, where $\varepsilon \geqslant 0$, then $0 \in \operatorname{int} T$.

Proof. i) The results are easily checked from the definition of T. The assumption in ii) implies that there exists a constant $\gamma > 0$ such that, for all $x \in \mathbb{R}^n$ and $i \in N$, $\|\tilde{C}\tilde{A}^i x\| \leqslant \gamma\alpha_i\|x\|$. Then, $\|x\| \leqslant 1/\gamma$ implies $\|\tilde{C}\tilde{A}^i x\| \leqslant \alpha_i$ for all $i \in N$. Hence $B(0,(1/\gamma)) \in T$ where $B(0,(1/\gamma))$ is the ball with center 0 and raduis $1/\gamma$, and consequently $0 \in \operatorname{int} T$. ■

Remark 1. The condition $\lim\limits_{k\to+\infty} \sup \|\tilde{A}^k\|/\alpha_k < \varepsilon$ in the previous proposition is equivalent to $\|\tilde{C}\tilde{A}^i\| \leqslant \gamma\alpha_i$ for all $i \geqslant 0$, where γ is a positive constant.

Definition 2. *The T set is said to be finitely determined, if there exists an integer k such that $T = T_k$.*

The finite determination of T is characterized by the following proposition.

Theorem 2. *Suppose that $T_k = T_{k+1}$ for some integer k. Then the set T_k given by (12) is described by a finite number of equations; more precisely, we have $T = T_k$. Conversely, if $T = T_k$ for some integer k, then $T_k = T_{k+1} = T_j$, for all $j \geqslant k$.*

Proof. Suppose the existance of an integer k such that $T_k = T_{k+1}$ then $x \in T_k\,(K,L)$ implies that

$$\tilde{C}\tilde{A}^{k+1}x \in B\,(0,\alpha_{k+1})\,,$$

thus
$$\tilde{C}\tilde{A}^k \left(\frac{\alpha_k}{\alpha_{k+1}}\tilde{A}x\right) \in B\,(0,\alpha_k)\,, \tag{13}$$

where $B(0,\alpha_k)$ is the ball with center 0 and radius α_k, and for $i \in \{0,\dots,k-1\}$, we have

$$\tilde{C}\tilde{A}^i \left(\frac{\alpha_k}{\alpha_{k+1}}\tilde{A}x\right) = \frac{\alpha_k}{\alpha_{k+1}}\tilde{C}\tilde{A}^{i+1}x \in B\left(0,\frac{\alpha_k\alpha_{i+1}}{\alpha_{k+1}}\right)$$

since $(\alpha_j)_{j\geqslant 0}$ verfies $\frac{\alpha_j}{\alpha_{j+1}} \leqslant \frac{\alpha_{j-1}}{\alpha_j}$ for $j \geqslant 1$ then

$$\frac{\alpha_k}{\alpha_{k+1}} \leqslant \frac{\alpha_i}{\alpha_{i+1}}, \ \ \forall i \in \{0, \ldots, k-1\},$$

which implies that

$$\tilde{C}\tilde{A}^i \left(\frac{\alpha_k}{\alpha_{k+1}} \tilde{A}x \right) \in B\left(0, \alpha_i\right) \ \ \forall i \in \{0, \ldots, k-1\}. \tag{14}$$

Consequently, from (13) and (14) we deduce that

$$\frac{\alpha_k}{\alpha_{k+1}} \tilde{A}x \in T_k$$

and, by iteration, $\left(\frac{\alpha_k}{\alpha_{k+1}}\right)^j \tilde{A}^j x \in T_k$ for all $j \geqslant 0$. Then

$$\tilde{C}\tilde{A}^{j+i}x \in B\left(0, \frac{\alpha_i \alpha_{k+1}^j}{\alpha_k^j}\right) \ \ \forall i \in \{0, \ldots, k-1\}, \ \forall j \geqslant 0.$$

So, for $i = k$, we have

$$\tilde{C}\tilde{A}^{j+k}x \in B\left(0, \frac{\alpha_{k+1}^j}{\alpha_k^{j-1}}\right) \ \ \forall j \geqslant 1$$

as $(\alpha_i)_i \geqslant 0$ verifies (8), then we easily establish that

$$\frac{\alpha_{k+1}^j}{\alpha_k^{j-1}} \leqslant \alpha_{k+j}, \ \ \forall j \geqslant 1,$$

thus

$$\tilde{C}\tilde{A}^{j+k}x \in B\left(0, \alpha_{k+j}\right) \ \ \forall j \geqslant 1.$$

Therefore $x \in T$, hence $T_k \subset T$. But T is a subset of T_k, consequently $T = T_k$. Conversely, if $T = T_k$ for some integer k, then we deduce that $T_k \subset T_{k+1}$ which implies that $T_k = T_{k+1}$ (because $T \subset T_{j_1} \subset T_{j_2}$, $j_1 \geqslant j_2$). ■

3. Algorithmic determination

In order to describe the tolerable set T by a finite number of inequalities, i.e., $T = T_k$, we suggest an algorithm stated as follows: Let $\mathbb{R}^p$ be endowed with the following norm

$$\|x\| = \max_{i=1,\ldots,p} |x_i|, \quad \forall x = (x_1, \ldots, x_p) \in \mathbb{R}^p.$$

The set T_k is then described as follows

$$T_k = \left\{ x \in \mathbb{R}^n;\ h_s\left(\frac{1}{\alpha_i}\tilde{C}\tilde{A}^i x\right) \leqslant 0 \text{ for } s = 1, \ldots, 2p \text{ and } i = 0, \ldots, k \right\}$$

where $h\mathbb{R}^p \to \mathbb{R}$ are defined for every $x = (x_1, \ldots, x_p) \in \mathbb{R}^p$ by

$$\begin{cases} h_{2j-1}(x) = x_j - 1, & \text{for } j \in \{1, \ldots, p\}, \\ h_{2j}(x) = -x_j - 1, & \text{for } j \in \{1, \ldots, p\}. \end{cases}$$

It follows from $T_{k+1} \subset T_k$ that $T_{k+1} = T_k$ if and only if $T_k \subset T_{k+1}$. So

$$\forall x \in T_k, \quad h_s\left(\frac{1}{\alpha_{k+1}}\tilde{C}\tilde{A}^{k+1}x\right) \leqslant 0 \quad \text{for } s = 1, \dots, 2p$$

or equivalently

$$\sup_{x \in T_k} h_s\left(\frac{1}{\alpha_{k+1}}\tilde{C}\tilde{A}^{k+1}x\right) \leqslant 0 \quad \text{for } s = 1, \dots, 2p.$$

Algorithm:

Step 1: Set $k = 0$.
Step 2: Solve the following optimization problems for $s = 1, \dots, 2p$.
Maximize $j_s(x) = h_s\left(\frac{1}{\alpha_{k+1}}\tilde{C}\tilde{A}^{k+1}x\right)$.
Subject to the constraints

$$h_j\left(\frac{1}{\alpha_l}\tilde{C}\tilde{A}^{l}x\right) \leqslant 0 \;\; \text{for} \;\; j = 1, \dots, 2p \;\; \text{and} \;\; \forall l = 0, \dots, k.$$

Let j_s^* be the maximum value of $j_s(x)$.
If $j_s^* \leqslant 0$, for $s = 1, \dots, 2p$ then set $k_0 = k$ and stop.
Else continue.
Step 3: Replace k by $k+1$ and return to step 2.

The optimization problem cited in step 2 is a mathematical programming and can be solved by standard methods.

4. Conditions for finite characterization

It is clear that the above algorithm converges if and only if there exists an integer k such that $T_k = T_{k+1}$. So it is desirable to establish simple condition which allows us to characterize the set T by a finite number of inequalities. Our main result in this direction is the following.

Theorem 3. *Suppose the following assumptions hold*

1. *The pair* $(\tilde{A}, \tilde{C})$ *is observable, i.e.,* $\left[\tilde{C}^\top | \tilde{A}^\top \tilde{C}^\top | \dots | (\tilde{A}^\top)^{n-1}\tilde{C}^\top\right]$ *has the rank* n.
2. $\lim\limits_{k \to +\infty} \sup \|\tilde{A}^k\| / \alpha_k < \lambda_0 (\|C\| \|H\| M)$, *where* $\lambda_0 = \inf\limits_{\lambda \in \sigma(H^T H)} \lambda > 0$ *and* $H = \begin{bmatrix} \tilde{C} \\ \tilde{C}\tilde{A} \\ \tilde{C}\tilde{A}^2 \\ \vdots \\ \tilde{C}\tilde{A}^{n-1} \end{bmatrix}$.

Then there exists an integer k *such that* $T = T_k$.

Proof. By the observability of $(\tilde{A}, \tilde{C})$, the rank of the matrix H is n, where

$$H = \begin{bmatrix} \tilde{C} \\ \tilde{C}\tilde{A} \\ \tilde{C}\tilde{A}^2 \\ \vdots \\ \tilde{C}\tilde{A}^{n-1} \end{bmatrix},$$

which implies that $H^T H$ is invertible, so there exists a constant $c = \inf\limits_{\lambda \in \sigma(H^T H)} \lambda > 0$ such that

$$c\|x^2\| \leqslant \left\langle H^T H x, x \right\rangle, \ \forall x \in \mathbb{R}^n, \tag{15}$$

which implies that

$$c\|x^2\| \leqslant \|H^\top\| \|Hx\| \|x\|, \ \forall x \in \mathbb{R}^n$$

and we have

$$Hx \in \overbrace{B(0,\alpha_0) \times B(0,\alpha_1) \times \ldots \times B(0,\alpha_{n-1})}^{n-\text{time}}, \ \forall x \in T_{n-1},$$

where $B(0,\alpha_i) = \{\forall x \in \mathbb{R}^n / \|x\| \leqslant \alpha_i\}$ since $\overbrace{B(0,\alpha_0) \times B(0,\alpha_1) \times \ldots \times B\left(0,\alpha_{n-1}\right)}^{n-\text{time}}$ is bounded, then

$$c\|x^2\| \leqslant M \|H^\top\| \|x\|, \ \forall x \in T_{n-1} \tag{16}$$

(because $\|Hx\| \leqslant \max\limits_{0<i\leqslant n-1} (\alpha_i) = \alpha_0$ and $\|x\| = \max\limits_{0<i\leqslant p} (|x_i|), \forall x \in \mathbb{R}^p$).

So

$$\|x\| \leqslant \gamma = \frac{M \left\|H^\top\right\|}{c}, \forall x \in T_{n-1}. \tag{17}$$

Hence

$$T_{n-1} \subset B(0,\gamma) = \left\{\forall x \in \mathbb{R}^n / \|x\| \leqslant \gamma\right\}.$$

The fact that $\lim\limits_{k\to+\infty} \sup \|\tilde{A}^k\| / \alpha_k = \rho$ implies that

$$\forall \beta > 0, \ \exists k, \, \forall k \geqslant k_0 \ \sup_{i\geqslant k} \frac{\|\tilde{A}^i\|}{\alpha_i} \leqslant \beta + \rho,$$

then for $\beta = 1/\gamma\|\tilde{C}\| - \rho > 0$ there exists an integer $k_0 \geqslant n-1$ such

$$\|\tilde{C}\tilde{A}^{k_0}\| \leqslant \frac{\alpha_{k_0+1}}{\gamma}.$$

For every $x \in T_{k_0}$ we have

$$\|\tilde{C}\tilde{A}^{k_0+1} x\| \leqslant \|\tilde{C}\tilde{A}^{k_0}\| \|x\|,$$

but $T_{k_0} \subset T_{n-1} \subset B(0,\gamma)$, so we deduce that

$$\|\tilde{C}\tilde{A}^{k_0+1} x\| \leqslant \varepsilon_{k_0+1}, \ \forall x \in T_{k_0},$$

consequently, $\tilde{C}\tilde{A}^{k_0+1} x \in B(0,\alpha_{k_0+1})$, for all $x \in T_{k_0}$. Thus, $T_{k_0} \subset T_{k_0+1}$, which implies that $T_{k_0} = T_{k_0+1} = T$. ■

Remark 2. The observability assumption is not really a limitation. To show this, let us suppose that $(\tilde{A},\tilde{C})$ is an unobservable pair. Let U be the nonsingular matrix which describes the change in state coordinates. Thus, systems (1) can be described by

$$\begin{cases} \hat{x}_{i+1} = \hat{A}\hat{x}_i + \hat{B}\hat{u}_i, \\ \hat{x}_0 = \sum\limits_{j=1}^{n} \beta_j e_j, \end{cases}$$

where $\hat{x_i} = U x_i$ and $\hat{A} = U\tilde{A}U^{-1}$. Consequently, y_i are given by

$$\hat{y}_i = \hat{C}\hat{x}_i + \hat{D}\hat{v}_i,$$

where $\hat{C} = \tilde{C}U^{-1}$. Now choose U in the usual way so that:

$$\hat{A} = \begin{pmatrix} E_1 & 0 \\ E_2 & E_3 \end{pmatrix}, \qquad \hat{C} = (G_1, 0)$$

and (G_1, E_1), an observable pair. Note that (G_1) is a $p \times r$ matrix, E_1 is an $r \times r$ matrix and r is an appropriate integer. Recall that $\hat{\omega} = U\omega$, if we set $\hat{\omega} = (\hat{\omega}_1, \hat{\omega}_2,)$, where $\hat{\omega}_1 \in \mathbb{R}^r$.

Then we have

$$x \in T(\tilde{A}, \tilde{C}, \alpha_i) \Leftrightarrow \|\tilde{C}\tilde{A}^i\omega\| < \alpha_i, \forall i \geqslant 0$$
$$x \in T(\tilde{A}, \tilde{C}, \alpha_i) \Leftrightarrow \|\hat{C}\hat{A}^i\hat{\omega}\| < \alpha_i, \forall i \geqslant 0$$
$$x \in T(\tilde{A}, \tilde{C}, \alpha_i) \Leftrightarrow \|G_1 E_1^i \hat{\omega}_1\| < \alpha_i, \forall i \geqslant 0.$$

Hence

$$x \in T(\tilde{A}, \tilde{C}, \alpha_i) \Leftrightarrow \hat{x} \in T(E_1, G_1, \alpha_i) \times \mathbb{R}^{n-r}$$

thus

$$T\left(\tilde{A}, \tilde{C}, \alpha_i\right) = U^{-1}\left(T\left(E_1, G_1, \alpha_i\right) \times \mathbb{R}^{n-r}\right). \tag{18}$$

Since (G_1, E_1) is observable, it is sufficient that the matrix A be asympotically stable to have $T(E_1, G_1, \alpha_i)$ finitely determined. Consequently, if A is asympltotically stable and $(\tilde{A}, \tilde{C})$ is unobservable, then (18) gives a characterization of $T(\tilde{A}, \tilde{C}, \alpha_i)$.

Example 1. Consider the following series of the RLC circuit. It is having an input voltage $v^k(t)$ and the current flowing through the circuit is $I(t)$.

There are two storage elements (inductor and capacitor) in this circuit. So, the state variables are the current flowing through the inductor $I(t)$ and the voltage across capacitor, $v^c(t)$.

From the circuit, the output voltage $v^0(t)$ is equal to the voltage across capacitor, $v^c(t)$: $v^0(t) = v^c(t)$.

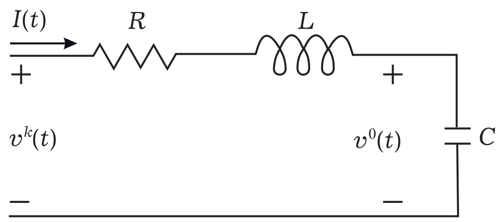

Fig. 1.

$$v^k(t) = RI(t) + L\frac{\partial I(t)}{\partial t} + v^c(t) \quad \Longrightarrow \quad \frac{\partial I(t)}{\partial t} = -\frac{RI(t)}{L} - \frac{v^c(t)}{L} + \frac{v^k(t)}{L}$$

$$\frac{\partial v^c(t)}{\partial t} = \frac{I(t)}{C}.$$

State vector, $X = \begin{bmatrix} I(t) \\ v^c(t) \end{bmatrix}$. We can arrange the differential equations and output equation into the standard form of state space model as,

$$\dot{X} = \begin{bmatrix} \frac{\partial I(t)}{\partial t} \\ \frac{\partial v^c(t)}{\partial t} \end{bmatrix} = \begin{bmatrix} -\frac{R}{L} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix} \begin{bmatrix} I(t) \\ v^c(t) \end{bmatrix} + \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix} v^k(t),$$

$$Y = [0, 1] \begin{bmatrix} I(t) \\ v^c(t) \end{bmatrix}.$$

The discrete form of the RLC circuit

$$X_{i+1} = \begin{bmatrix} I_{i+1} \\ v_i^c \end{bmatrix} = \begin{bmatrix} -\frac{R}{L} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix} \begin{bmatrix} I_i \\ v_i^c \end{bmatrix} + \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix} v_i^k,$$

$$z_i = [1, 0] \left[\begin{array}{c} I_i \\ v_i^c \end{array} \right],$$

$$Y_i = [0, 1] \left[\begin{array}{c} I_i \\ v_i^c \end{array} \right].$$

Let $L = 2.2\,\mathrm{mH}$, $C = 0.23\,\mu\mathrm{F}$, $R = 0.6\,\mathrm{w}$, $\alpha_i = 1/2^i$. With the gain matrices selected $L = [0.3\ 0.4]$, $K = 0$, we find that the pair $(\tilde{A}, \tilde{C})$ is observable and theorem 3 assure the convergence of algorithm, which gives the index of determination $k^* = 1$. Then the corresponding control laws could reduce the effects of all unknown uncertainties. Fig. 2 gives a presentation of the set T corresponding to this example.

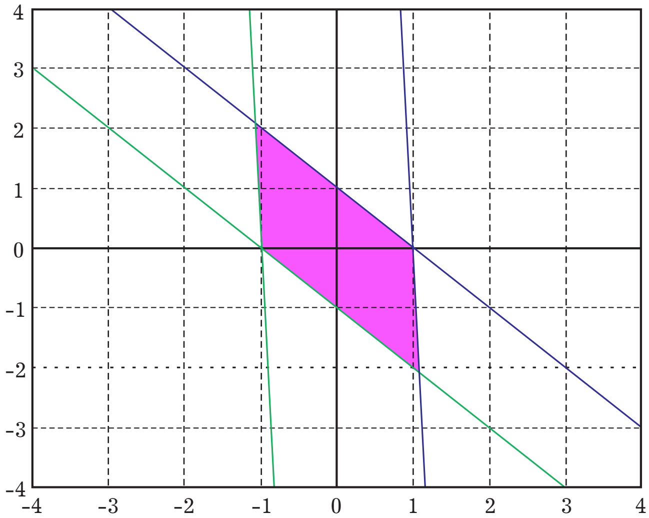

Fig. 2. The set T corresponding to Example 1.

Example 2 (Damped spring mass system). Using Hooke's law to model the spring and assuming that the damper exerts a force that is proportional to the velocity of the system, we have

$$m\ddot{q} + c\dot{q} + kq = u,$$

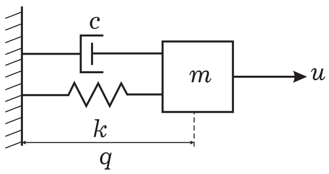

Fig. 3.

where m is the mass, q is the displacement of the mass, c is the coefficient of viscous friction, k is the spring constant and u is the applied force. In state space form, using $x = (q, \dot{q})$ as the state, u as the input, choosing $z = \dot{q}$ as measurement function and $y = q$ as the output, we have

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{pmatrix} x + \begin{pmatrix} 0 \\ \frac{1}{m} \end{pmatrix} u,$$

$$z = [0\ \ 1]\, x,$$

$$y = [1\ \ 0]\, x.$$

We consider the discrete form of system

$$x_{i+1} = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{pmatrix} x_i + \begin{pmatrix} 0 \\ \frac{1}{m} \end{pmatrix} u_i,$$

$$z_i = [0\ \ 1]\, x_i,$$

$$y_i = [1\ \ 0]\, x_i.$$

Let $m = 166.6\,\mathrm{g}$, $c = 0.070\,\mathrm{kg\,s^{-1}}$, $k = 85\,\mathrm{N\,m^{-1}}$, $\alpha_i = 1/i + 1$. With the gain matrices selected $L = [0.2 \quad -1]$ and $K = 0$, the conditions of theorem 3 are sufficient for the convergence of the algorithm, and the algorithm gives the index of determination $k^* = 3$. We conclude that the gain matrices selected K and L makes the system insensitive to all unknown uncertainties. Fig. 4 gives a presentation of the set T corresponding to this example.

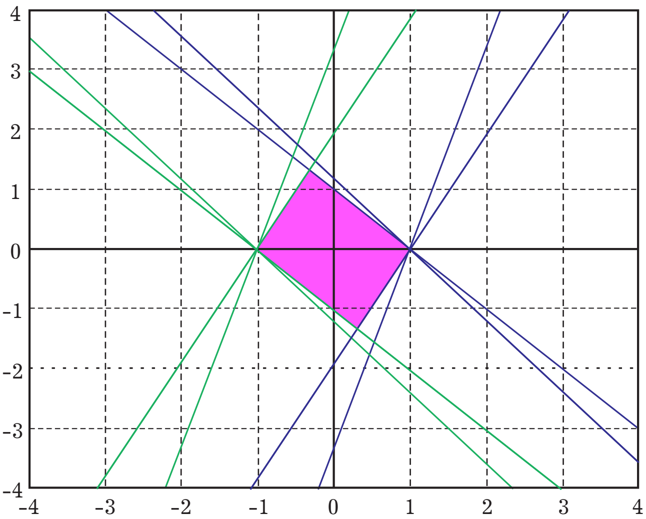

Fig. 4. The set T corresponding to Example 2.

5. Output sensitivity delayed system

In this section we consider the linear discrete delay system described by

$$\begin{cases} x_{i+1} = \sum\limits_{j=0}^{t} A_j x_{i-j} + B u_i, \\ (x_0, x_{-1}, \ldots, x_{-t}) = \sum\limits_{j=1}^{n(t+1)} \beta_j e_j \in \mathbb{R}^{n(t+1)}, \end{cases} \tag{19}$$

where $x_i \in \mathbb{R}^n$ is the state variable, r and t are the integers such $r \leqslant t$. β_j are the components of initial state in $\mathbb{R}^{n(t+1)}$ with e_j are a canonical basis of $\mathbb{R}^{n(t+1)}$. We suppose that the parameters $(\beta_1, \ldots, \beta_s)$ are unknown and $(\beta_{s+1}, \ldots, \beta_{n(t+1)})$ are known. As the initial state $(x_0, x_{-1}, \ldots, x_{-t})$ is unknown we consider the measurement function given by

$$z_i = \sum_{j=0}^{t} M_j x_{i-j}, \quad \forall i \geqslant 0, \tag{20}$$

where M_j are $(n \times n)$ real matrices, the corresponding output is

$$y_i = \sum_{j=0}^{r} C_j x_{i-j} + D v_i. \tag{21}$$

And the out variable $y_i \in \mathbb{R}^p$ satisfies

$$\left\| \frac{\partial y_i}{\partial \beta_s} \right\| \leqslant \alpha_i, \quad \text{for } \forall i \geqslant 0,\ 1 \leqslant s \leqslant r, \tag{22}$$

where C_j are $(p \times n)$ real matrices. As previously, the output admissible if the resulting output (21) satisfies (22). In order to characterize the set of all possible gain matrix $T(L,K)$, we define the new state variable $X_i \in \mathbb{R}^n$ for $i \geqslant 0$ such that

$$X_i = (x_i, x_{i-1}, \dots, x_{i-t})^\top .$$

And the matrices $A \in \mathcal{M}_{n(t+1)}(\mathbb{R})$, $C \in \mathcal{M}_{p,n(t+1)}(\mathbb{R})$, $\overline{B} \in \mathcal{M}_{n(t+1),m}(\mathbb{R})$, and $M \in \mathcal{M}_{n,n(t+1)}(\mathbb{R})$ by

$$A = \begin{pmatrix} A_0 & A_1 & \dots & A_t \\ I & 0_n & & 0_n \\ \vdots & \ddots & \ddots & \vdots \\ 0_n & \dots & I & 0_n \end{pmatrix}, \quad \overline{B} = \begin{pmatrix} B \\ 0_m \\ \vdots \\ \vdots \\ 0_m \end{pmatrix},$$

$$M = (M_0, M_1, \dots, M_t), \quad C = (C_0, C_1, \dots, C_r, \underbrace{0_{p,n}, \dots, 0_{p,n}}_{(t-r)-\text{times}}),$$

where I_n is the $(n \times n)$-unit matrix, 0_n is the $(n \times n)$-zero matrix and 0_m is the $(m \times m)$-zero matrix, $0_{p,l}$ is the $(p \times l)$-zero matrix. Then the system (19) can be equivalently rewritten in the form

$$\begin{cases} X_{i+1} = AX_i + \overline{B}u_i, \\ X_0 = (x_0, x_{-1}, \dots, x_{-t}). \end{cases} \tag{23}$$

The measurement function and the output delay function can be expressed as follows

$$z_i = MX_i,$$

$$y_i = CX_i + Dv_i,$$

where the controls law

$$u_i = Lz_i \ \text{ and } \ v_i = Kz_i.$$

Thus, the admissible set is given by

$$T(L,K) = \left\{ X \in \mathbb{R}^n / \|(C + DKM)(A + \overline{B}LM)^i X\| \leqslant \alpha_i, \forall i \geqslant 0 \right\}.$$

Thus,it is obvious that Theorem 3 gives sufficient conditions to characterize the set $T(L,K,\varepsilon)$ by a finite number of functional inequalities.

Example 3. Consider the discrete delayed system described by

$$\begin{cases} x_{i+1} = 1.2x_i + 0.9x_{i-1} + u_i, \\ x_0 = x_0, \end{cases} \tag{24}$$

the corresponding output is

$$y_i = x_i + v_i.$$

We take

$$u_i = L(x_{i,}, x_{i-1})^\top \ \text{ and } \ v_i = K(x_{i,}, x_{i-1})^\top,$$

where $L = \begin{bmatrix} -0.75 & -0.1 \\ -0.5 & -0.5 \end{bmatrix}$ and $K = \begin{bmatrix} 0.1 & -1 \\ -1 & -1.2 \end{bmatrix}$, then we use the algorithm described in the previous section, to establish that $k^* = 5$. Fig. 5 gives the representation of the set of outputs corresponding to this example.

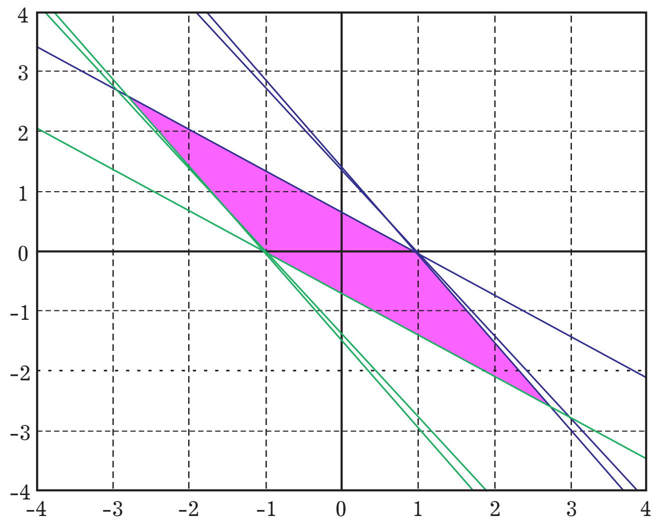

Fig. 5. The set T corresponding to Example 3.

6. Conclusion

In this paper, we have studied the asymptotic output sensitivity problem of discrete-time linear systems with perturbed initial state, and we focus our interest in this work on determination of the set of possible gain matrices whose role is not only to make the system insensitive to all disturbances but to achieve a predefined stabilization mode. The necessary conditions have been obtained. The case of delayed system is also considered and numerical simulations have proven the effectiveness of our results.

[1] Silvério R., Delfim F. M. T. Parameter Estimation, Sansitivity Analysis and Optimal Control of a Periodic Epidemic Model With Application to HRSV in Florida. Statistics, Optimisation and Information Computing. **6** (1), 137–147 (2018).

[2] Semergui J. Y., Yambwera S. V., Marcus N., Okosun K. O., Witbooi P. J., Abidun G. J. Sensitivity and Optimal Control Analysis of HIV/AIDS Model. Applied Mathematics E-Notes. **19**, 606–620, (2019).

[3] Soldatenko S. A., Yusupov K. M. Sensitivity Analysis in Optimal Control of the Earth's Climate System. Recent Advances in Environmental and Earth Science and Economics. 40–46 (2016).

[4] Kowalewski A., Emirsajłow Z., Sokołowski J., Krakowiak A. Sensitivity Analysis of Optimal Control Parabolic System with Retardations. In: Mitkowski W., Kacprzyk J., Oprzędkiewicz K., Skruch P. (eds) Trends in Advanced Intelligent Control, Optimization and Automation. KKA 2017. Advances in Intelligent Systems and Computing. **577**, 98–107 (2017).

[5] Lions J. L. Sur les sentinelles des systèmes distribués. Le cas des conditions initiales incomplètes. C. R. Acad. Sci. Paris. **307** (16), 819–823 (1988).

[6] Lions J. L. Sur les sentinelles des systèmes distribués. Le cas des conditions frontières, termes sources, coefficients incomplètement connus. C. R. Acad. Sci. Paris. **307** (17), 865–870 (1988).

[7] Lions J. L. Sentinels with special sensitivity. Fifth Symposiym on Control of Distributed Parameter Systems. Perpignan, France (1989).

[8] Darup M. S., Mönnigmann M. Computation of the largest constraint admissible set for linear continuous-time systems with state and input constraints. IFAC Proceedings Volumes. **47** (3), 5574–5579 (2014).

[9] Dorea C. E. T., Hennet J. C. Computation of maximal admissible sets of constrained Linear Systems. Proc. 4th IEEE Mediterranean Symposium on New Directions in Control and Automation. 286–291 (1996).

[10] Gilbert E. G., Tan K. T. Linear systems with state and control constraints. IEEE Transactions on Automatic Control. **36** (9), 1008–1020 (1991).

[11] Kolmanovsky I., Gilbert E. G. Maximal output admissible sets for discrete-time systems with disturbance inputs. Proceedings of 1995 American Control Conference - ACC'95. 1995–1999 (1995).

[12] Hirata K., Ohta Y. Exact determinations of the maximal output admissible set for a class of nonlinear systems. Automatica. **44** (2), 526–533 (2008).

[13] Larrache A., Lhous M., Ben Rhila S., Rachik M., Tridane A. An output sensitivity problem for a class of linear distributed systems with uncertain initial state. Archives of Control Sciences. **30** (1), 139–155 (2020).

[14] Limpiyamitr A., Ohta Y. On the approximation of maximal output admissible set and reachable set via forward Euler discretization. IFAC Proceedings Volumes. **37** (11), 395–400 (2004).

[15] Lombardi W., Luca A., Olaru S., Niculescu S.-I. State admissible sets for discrete systems under delay constraints. Proceedings of the 2010 American Control Conference. 5185–5190 (2010).

[16] Osorio J., Ossareh H. R. A Stochastic Approach to Maximal Output Admissible Sets and Reference Governors. 2018 IEEE Conference on Control Technology and Applications (CCTA). 704–709 (2018).

[17] Rachik M., Lhous M., Tridane A. On the maximal output admissible set for a class of nonlinear discrete systems. Systems Analysis Modelling Simulation. **42** (11), 1639–1658 (2002).

[18] Rachik M., Tridane A., Lhous M., Idrissi Kacemi O., Tridane Z. Maximal Output Admissible Set and Admissible Perturbations Set For Nonlinear Discrete Systems. Applied Mathematical Sciences. **1** (32), 1581–1598 (2007).

[19] Tarbouriech S., Castelan E. B. Maximal admissible polyhedral sets for discrete-time singular systems with additive disturbances. Proceedings of the 36th IEEE Conference on Decision and Control. **4**, 3164–3169 (1997).

[20] Zakary O., Rachik M., Tridane A., Abdelhak A. Identifying the set of all admissible disturbances: discrete-time systems with perturbed gain matrix. Mathematical Modeling and Computing. **7** (2), 293–309 (2020).

Про задачу асимптотичної чутливості за виходом для дискретних лінійних систем з невизначеним початковим станом

Бен Ріла С.[1], Лоус М.[2], Рачик М.[1]

[1] *Лабораторія аналізу, моделювання та симулювання, кафедра математики та обчислювальних наук, факультет наук Бен М'сік, Університет Хасана II Касабланки, BP 7955, Сіді Отман, Касабланка, Марокко*

[2] *Лабораторія моделювання, аналізу, контролю та статистики, кафедра математики та інформатики, факультет наук Айн Чок, Університет Хасана II Касабланки, BP 5366, Маріф, Касабланка, Марокко*

У цій роботі досліджується скінченновимірна дискретна лінійна система, початковий стан x_0 якої невідомий. Припускається, що система доповнена двома вихідними рівняннями, перше з яких z_i зображає вимірювання, які зроблені в невідомому стані системи, а інше y_i — відповідний вихід. Метою роботи є введення двох законів керування, як у замкненому циклі вимірювань z_i, так і для асимптотичного зменшення впливу невідомої частини початкового стану x_0. Запропонований підхід полягає у теоретичній та алгоритмічній характеристиці множини таких елементів керування. Для ілюстрації теоретичних результатів наведено декілька прикладів та чисельне моделювання.

Ключові слова: *дискретний час, відносна нечутливість, лінійна система, спостережуваність, стабільність, невизначеність.*