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Method of integral equations in the polytropic theory of stars with axial rotation. II. Polytropes with indices $n > 1$

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A new method for finding solutions of the nonlinear equilibrium equations for rotational polytropes was proposed, which is based on a self-consistent description of internal region and periphery using the integral form of equations. Dependencies of geometrical parameters, surface form, mass, moment of inertia and integration constants on angular velocity were calculated for indices $n = 2.5$ and $n = 3$.

Keywords: *polytropic stars, heterogeneous ellipsoids, axial rotation, mechanical equilibrium equation, stability of stars.*

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1. Introduction

Axial rotation of celestial bodies is an attribute of their existence and one of the formation factors of their internal structure. Stars of early spectral classes have high angular velocity, and stars of classes later than F5 rotate slowly. For example, α Eridani (B8V) with mass $4.9\,M_{\odot}$ and equatorial radius $R_e = 12\,R_{\odot}$ rotates with velocity $3 \cdot 10^{-5}\,\mathrm{s}^{-1}$, and Sun (yellow dwarf G2V) has angular velocity $3 \cdot 10^{-6}\,\mathrm{s}^{-1}$.

The polytrope theory is a kind of express method to research the influence of different factors on formation of the stars internal structures. It concerns gas stars, white dwarfs, neutron stars, circumstellar disks, as well as binary star systems. This theory based on the polytropic equation of state

$$P(\boldsymbol{r}) = K\rho^{1+1/n}(\boldsymbol{r}), \tag{1}$$

where $P(\boldsymbol{r})$ is the sum of gas and light pressure in the point with radius-vector $\boldsymbol{r}$, $\rho(\boldsymbol{r})$ is the local density of matter, K and n are constants. A specific star corresponds a model with parameter K, index n, density in center ρ_{c} and angular velocity ω.

Model with $n = 0$ corresponds to the rotation ellipsoid with constant density of matter. It is considered that the model with small index ($1 \lesssim n \lesssim 1.5$) corresponds to the stars with convective equilibrium and therefore is suitable for the description of massive stars of main sequence. Model with $n = 3$ corresponds to the stars with significant contribution of light pressure, as well as cold white dwarfs. Models with indices $4 \lesssim n \lesssim 5$ are suitable for description of stars, which are in the phase of accretion. Correct description of stars with the degenerate core, as well as hot low-massive white dwarfs demands using two-phase models, when core and periphery are described by polytropes with the different indices n [1].

Main variants of the problem solution about equilibrium of stars in the polytropic theory are described in the first part of our work [2]. Usually, it is used differential form of the star equilibrium equation, and integration constants are determined approximately according to the continuity condition of the gravitational potential on its surface. In works [3–5] was proposed a different way to solve the problem: substitution of the general analytical solution of the differential equation in the integral equilibrium equation yields the system of the linear algebraic equations for finding integration constants. Such approach allows us to describe more accurately the structure of star in the surface region, that is especially important at angular velocities, close to the critical ones causing the instability. At small angular velocities the surface of rotational polytrope is the surface of rotational ellipsoid. Therefore, in all previous works there are given values of the polar and equatorial radii as functions of angular velocity and polytropic index [6, 7]. In our work [2] it was shown that the model with index $n = 0$ is the rotational ellipsoid. In the case $n = 1$ the polytrope surface deviates from the surface of rotational ellipsoid with the same radii, and deviation is 5% near the critical value of angular velocity. It is consistent with the numerical integration, described in works [8, 9].

The aim of our work is to find the solutions of equilibrium equation for the rotational polytropes with indices $n > 1$ by generalizing the method of work [2]. There are calculated the polar and equatorial radii of rotational polytropes, deviation of the surface shape from the ellipsoidal, mass, moment of inertia relative to the rotation axis, as well as distribution of matter density along the radius as functions of polytropic index and angular velocity. The comparison with results of the other authors is presented.

2. The system of equilibrium equations

In the non-inertial reference frame in the presence of rotation the equilibrium equation takes the form [10]

$$\nabla P(\boldsymbol{r}) = -\rho(\boldsymbol{r}) \left\{\nabla\Phi_{\rm grav}(\boldsymbol{r}) + \nabla\Phi_{\rm c}(\boldsymbol{r})\right\}, \tag{2}$$

where $\Phi_{\rm grav}(\boldsymbol{r})$ is the gravitational potential inside a star, and $\Phi_{\rm c}(\boldsymbol{r})$ is the centrifugal potential. According to expression (1) relation (2) is the equation for the distribution of matter density

$$K(1+n)\Delta\rho^{1/n}(\boldsymbol{r}) = -4\pi G\rho(\boldsymbol{r}) + \frac{1}{2}\,\omega^2\Delta(r^2\sin^2\theta). \tag{3}$$

The model with the solid rotation ($\omega = \text{const}$)is considered. In equation (2), it is used the spherical coordinate system with the axis Oz coinciding to the rotation axis. A symmetrical model of star, With $\rho(\boldsymbol{r}) = \rho(r,\theta) = \rho(r, \pi-\theta)$, where θ is the polar angle, Δ is the Laplace operator is studied. In the dimensionless variables

$$\xi = r/\lambda_n, \qquad Y_n(\xi,\theta) = [\rho(r,\theta)/\rho_{\rm c}]^{1/n}, \tag{4}$$

equation (3) transforms as

$$\Delta_{\xi,\theta}\, Y_n(\xi,\theta) = \Omega^2 - [Y_n(\xi,\theta)]^n, \tag{5}$$

if the scale of length λ_n and dimensionless angular velocity Ω are determined by relations

$$K(1+n) = 4\pi G\lambda_n^2\,\rho_{\rm c}^{1-1/n}, \qquad \Omega = \omega(2\pi G\rho_{\rm c})^{-1/2}, \tag{6}$$

and the Laplace operator is written as

$$\Delta_{r,\theta} = \Delta_r + \frac{1}{r^2}\,\Delta_\theta, \qquad \Delta_r = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right), \qquad \Delta_\theta = \frac{\partial}{\partial t}(1-t^2)\,\frac{\partial}{\partial t}, \tag{7}$$

$t = \cos\theta$. Equation (5) is the second-order partial differential equation with the constant coefficients and the boundary conditions $Y_n(0,\theta) = 1$, $\partial Y_n(\xi,\theta)/\partial\xi = 0$ at $\xi = 0$, where appear two dimensionless parameters n and Ω.

Equation (5) can be rewritten into an integral form

$$Y_n(\xi,\theta) = 1 + \frac{\Omega^2\xi^2}{6}\,(1 - P_2(t)) + \frac{1}{4\pi}\int [Y_n(\xi',\theta')]^n\, Q(\boldsymbol{\xi},\boldsymbol{\xi}')\, d\boldsymbol{\xi}' \tag{8}$$

with the kernel

$$Q(\boldsymbol{\xi},\boldsymbol{\xi}') = |\boldsymbol{\xi} - \boldsymbol{\xi}'|^{-1} - (\xi')^{-1}, \tag{9}$$

where $P_{2l}(t)$ is Legendre polynomial of $2l$-th order, and integration is performed over volume of the polytrope, the surface of which is close to the surface of rotational ellipsoid.

At $n > 1$ equations (5) and (8) are nonlinear, however Ω^2 is small parameter, moreover $\Omega_{\max}(n) \approx 2^{1-n}\Omega_{\max}(1) \approx 0.28 \cdot 2^{1-n}$, where $\Omega_{\max}(n)$ is the maximal value of Ω, at which instability of star occurs in the vicinity of equator with index n. Because of that, in the internal region of star, which corresponds to the model without rotation, we use substitution

$$Y_n^I(\xi,\theta) = y_n(\xi) + \Omega^2 \Psi_n(\xi,\theta), \qquad 0 \leqslant \xi \leqslant \xi_1(n), \tag{10}$$

where $y_n(\xi)$ is the solution of equations (5) or (8) at $\Omega = 0$ (the Emden function for the model with index n), and $\xi_1(n)$ is the dimensionless radius of star without rotation (the Emden radius), minimal root of the equation $y_n(\xi) = 0$. In the linear approximation for Ω^2 the function $\Psi_n(\xi,\theta)$ is determined by equation

$$\Delta_{\xi,\theta}\, \Psi_n(\xi,\theta) = 1 - n\, y_n^{n-1}(\xi)\, \Psi_n(\xi,\theta). \tag{11}$$

Since Emden function satisfies the boundary conditions

$$y_n(0) = 1, \quad \frac{\partial}{\partial \xi} y_n(\xi) = 0 \quad \text{at} \quad \xi = 0, \tag{12}$$

then $\Psi_n(0,\theta) = 0$, $\partial \Psi_n(\xi,\theta)/\partial\xi = 0$ at $\xi = 0$. By analogy with the solution of equation (8) at $n = 1$ (see form. (64) of work [2]) we used expansion for the Legendre polynomials

$$\Psi_n(\xi,\theta) = \psi_{n,0}(\xi) + \sum_{l \geqslant 1} a_{2l}^{(n)}\, P_{2l}(t)\, \psi_{n,2l}(\xi), \tag{13}$$

where $a_{2l}^{(n)}$ are integration constants, $\psi_{n,0}(\xi)$ and $\psi_{n,2l}(\xi)$ are universal functions of the variable ξ for the fixed value n and determined by the system of independent linear equations

$$\begin{aligned} &\Delta_\xi\, \psi_{n,0}(\xi) = 1 - n\, y_n^{n-1}(\xi)\, \psi_{n,0}(\xi), \\ &\Delta_\xi\, \psi_{n,2l}(\xi) = \left\{ 2l\,(2l+1)\,\xi^{-2} - n\, y_n^{n-1}(\xi) \right\} \psi_{n,2l}(\xi), \end{aligned} \tag{14}$$

with zero boundary conditions

$$\psi_{n,0}(0) = \psi_{n,2l}(0) = 0, \qquad \partial\psi_{n,0}(\xi)/\partial\xi = \partial\psi_{n,2l}(\xi)/\partial\xi = 0 \tag{15}$$

at $\xi = 0$. In the region $\xi \ll 1$, where $y_n \approx 1 - \xi^2/6 + \ldots$, and asymptotics of functions $\psi_{n,2l}(\xi)$ coincide with asymptotics of the spherical Bessel functions of the first kind,

$$\psi_{n,0}(\xi) \Rightarrow 1 - j_0(\xi) + \ldots = \frac{\xi^2}{6} + \ldots,\ \psi_{n,2l}(\xi) \Rightarrow j_{2l}(\xi) + \ldots = \xi^{2l}[(4l+1)!!]^{-1} + \cdots;\ (\xi \ll 1),\ l \geqslant 1. \tag{16}$$

Therefore, asymptotics of functions $Y_n(\xi,\theta)$ coincide with asymptotics of function $Y_1(\xi,\theta)$ in the limit $\xi \ll 1$. For the first time functions $\psi_{n,0}(\xi)$ and $\psi_{n,2}(\xi)$ were calculated numerically in the region $0 \leqslant \xi \leqslant \xi_1(n)$ in [6]. These functions are sufficient for approximate calculation of the polytrope characteristics with small angular velocities. We calculated functions $\psi_{n,0}(\xi)$, $\psi_{n,2}(\xi)$ and $\psi_{n,4}(\xi)$ for the polytrope with indices $n = 1; 2; 3$ (see Figs. 1–3).

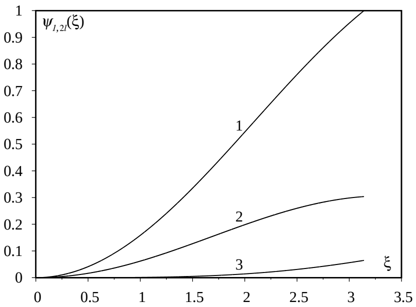

Fig. 1. Dependence of functions $\psi_{1,2l}(\xi)$ on the variable ξ. Curve 1 corresponds to $l = 0$, curve 2 — $l = 1$, curve 3 — $l = 2$.

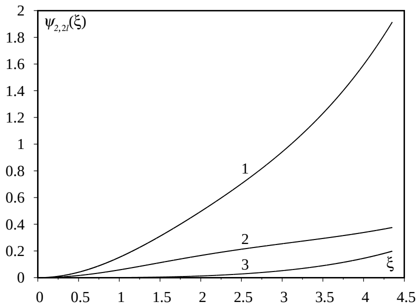

Fig. 2. Dependence of functions $\psi_{2,2l}(\xi)$ on the variable ξ. Notations are the same as in Fig. 1.

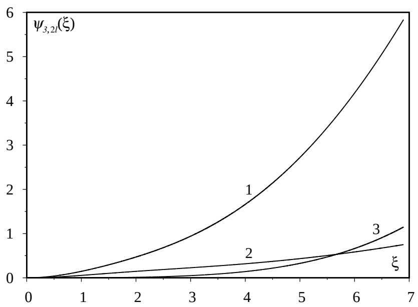

Fig. 3. Dependence of functions $\psi_{3,2l}(\xi)$ on the variable ξ. Notations are the same as in Fig. 1.

The region of variables, in which $\Omega^2 > [Y_n(\xi,\theta)]^n$, is the star periphery. In zero approximation, equation (5) in this region is replaced by equation

$$\Delta_{\xi,\theta} Y_n(\xi,\theta) = \Omega^2, \tag{17}$$

the general solution of which is

$$Y_n^{II}(\xi,\theta) = \frac{\xi^2\Omega^2}{6}\Big(1 - P_2(t)\Big) + \Omega^2 \sum_{l\geqslant 0} c_{2l}^{(n)} \xi^{2l} P_{2l}(t) + \Omega^2 \sum_{l\geqslant 0} \frac{b_{2l}^{(n)}}{\xi^{1+2l}} P_{2l}(t), \quad \xi > \xi_1(n), \tag{18}$$

where $c_{2l}^{(n)}$, $b_{2l}^{(n)}$ are integration constants. Accounting for the term $[Y_n(\xi,\theta)]^n$ on the right side of equation (5) can be realized by the method of perturbation theory. Continuous solution of equation (5) can be obtained by stitching solutions (10) and (18) on the sphere of radius $\xi_1(n)$. Substituting the function $Y_n^I(\xi,\theta)$ in equation (8), we determine integration constants $a_{2l}^{(n)}$, because this equation is inhomogeneous, and functions $y_n(\xi)$, $\psi_{n,0}(\xi)$, $\psi_{n,2l}(\xi)$ are known. Note that integration over variables ξ', θ' should be performed within that region of the polytrope, which is inside the Emden sphere $(0 \leqslant \xi \leqslant \xi_1(n))$. Determining $a_{2l}^{(n)}$ in an independent way, integration constants for the periphery can be found from the stitching conditions

$$Y_n^I(\xi,\theta) = Y_n^{II}(\xi,\theta), \quad \frac{\partial}{\partial\xi} Y_n^I(\xi,\theta) = \frac{\partial}{\partial\xi} Y_n^{II}(\xi,\theta) \quad \text{at} \quad \xi = \xi_1(n), \tag{19}$$

equating multipliers for the same Legendre polynomials.

3. The approximation of small velocities

Considering in expression (10) $a_{2l}^{(n)} = 0$ at $l \geqslant 2$, and in expression (18) $c_{2l}^{(n)} = 0$ at $l \geqslant 1$, $b_{2l}^{(n)} = 0$ at $l \geqslant 2$, we obtain the approximation of small velocities. From conditions (19), the coefficients $a_2^{(n)}$ and $c_0^{(n)}$ can be determined, $b_0^{(n)}$, $b_2^{(n)}$

$$\begin{aligned}
c_0^{(n)} &= \Big\{\psi_{n,0}(\xi_1) + \xi_1\psi_{n,0}'(\xi_1)\Big\} - \frac{\xi_1^2}{2} + \xi_1\frac{y_n'(\xi_1)}{\Omega^2} = \xi_1\frac{y_n'(\xi_1)}{\Omega^2},\\
b_0^{(n)} &= \xi_1\Big\{\psi_{n,0}(\xi_1) - \frac{\xi_1^2}{6} - c_0^{(n)}\Big\} = -\xi_1^2\frac{y_n'(\xi_1)}{\Omega^2},\\
b_2^{(n)} &= \xi_1^3\Big\{a_2^{(n)}\psi_{n,2}(\xi_1) + \frac{\xi_1^2}{6}\Big\},\\
a_2^{(n)} &= -\frac{5}{6}\xi_1^2\Big\{3\psi_{n,2}(\xi_1) + \xi_1\psi_{n,2}'(\xi_1)\Big\}^{-1} \quad \text{at} \quad \xi_1 \equiv \xi_1(n),\\
y_n'(\xi_1) &= \frac{d}{d\xi_1} y_n(\xi_1), \quad \psi_{n,2}'(\xi_1) = \frac{d}{d\xi_1}\psi_{n,2}(\xi_1).
\end{aligned} \tag{20}$$

Because $Y_n^{II}(\xi,\theta)$ is an extrapolation $Y_n^I(\xi,\theta)$ in the region $\xi > \xi_1(n)$, then the expression for the polytrope surface can be found from condition $Y_n^I(\xi,\theta) = 0$ or from condition $Y_n^{II}(\xi,\theta) = 0$

$$\xi_0(\theta) \cong \xi_1(n) + \frac{\Omega^2}{|y_n'(\xi_1)|}\Big\{\psi_{n,0}(\xi_1) + a_2^{(n)} P_2(t)\psi_{n,2}(\xi_1)\Big\}. \tag{21}$$

Expressions for $a_2^{(n)}$ and $\xi_0(\theta)$ were given in [6], and the solution in form (10), (18) in approximation (20) corresponds to [11]. Approximation (20) corresponds to the multipole expansion of the gravitational potential in peripheral region of the polytrope in form $\sum_{l\geqslant 0} b_{2l}^{(n)} P_{2l}(t)\xi^{-1-2l}$. In fact, such expansion is permissible only in the region $\xi > \xi_e$, where ξ_e is the equatorial radius. And the general solution (18) of equation (17) in peripheral region has more multipole terms $c_{2l}^{(n)} P_{2l}(t)\xi^{2l}$. In addition, solution (17) is approximate one. Therefore, the determination of constants $a_{2l}^{(n)}$, which appear in the function $Y_n^I(\xi,\theta)$, from the stitching conditions leads to errors at describing of the internal polytrope region. In this regard, we determine constants $a_{2l}^{(n)}$ by independent way, and the set of constants $c_{2l}^{(n)}$ and $b_{2l}^{(n)}$ — from the stitching conditions, as the result, they become functions of $a_{2l}^{(n)}$.

4. Determination of integration constants

According to relations (5), (10) and (13),

$$Y_n^I(\xi,\theta) = y_n(\xi) + \Omega^2\Big\{\psi_{n,0}(\xi) + \sum_{l\geqslant 1} a_{2l}^{(n)} P_{2l}(t)\psi_{n,2l}(\xi)\Big\}, \tag{22}$$

and $Y_n^{II}(\xi,\theta)$ we determine by expression (18).

Integration constants $a_{2l}^{(n)}$ and $c_{2l}^{(n)}$ $b_{2l}^{(n)}$ at arbitrary n,due to the division of the polytrope volume into two regions, must be determined self-consistently, generalizing the method of work [2]. In the internal region, equation (8) transforms to the form

$$\begin{aligned} y_n(\xi) + \Omega^2\left\{\psi_{n,0}(\xi) + \sum_{l=1}^{\infty} a_{2l}^{(n)} P_{2l}(t)\psi_{n,2l}(\xi)\right\} = 1 + \frac{\Omega^2\xi^2}{6}(1 - P_2(t)) \\ + \frac{1}{4\pi}\int_{V_I} d\boldsymbol{\xi}' Q(\boldsymbol{\xi},\boldsymbol{\xi}')\left\{y_n^n(\xi') + \Omega^2 n y_n^{n-1}(\xi')\left[\psi_{n,0}(\xi') + \sum_{l=1}^{\infty} a_{2l}^{(n)} P_{2l}(t')\psi_{n,2l}(\xi')\right]\right\} \\ + \frac{1}{4\pi}\int_{V_{II}} d\boldsymbol{\xi}' Q(\boldsymbol{\xi},\boldsymbol{\xi}')[Y_n^{II}(\xi',\theta')]^n. \end{aligned} \tag{23}$$

Integration regions over the variable ξ' schematically are shown in Fig. 4.

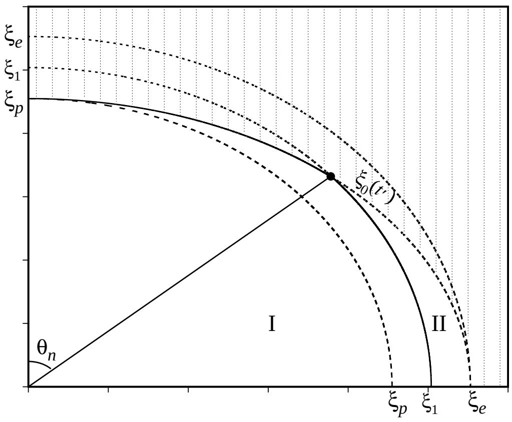

Fig. 4. Schematic representation of the integration region in formula (23).

Region V_I is the part of the polytrope volume, which is inside Emden sphere with radius $\xi_1(n)$ and is bounded by solid curve

$$\begin{aligned} 0 \leqslant \xi' \leqslant \xi_0(t') \quad \text{at} \quad 1 \geqslant t' \geqslant t_n, \\ 0 \leqslant \xi' \leqslant \xi_1(n) \quad \text{at} \quad 0 \leqslant t' \leqslant t_n, \end{aligned} \tag{24}$$

where $\xi_0(t')$ is the meridional section of the polytrope surface, which is close to the surface of rotational ellipsoid. Region V_{II} is the part of the polytrope volume, which is outside the Emden sphere,

$$\xi_1(n) \leqslant \xi' \leqslant \xi_0(t') \quad \text{at} \quad 0 \leqslant t' \leqslant t_n. \tag{25}$$

Herewith $t_n \equiv \cos\theta_n$, and the angle θ_n is determined by the intersection of the Emden sphere and the polytrope surface

$$\xi_p\{1 - e^2(1 - t_n^2)\}^{-1/2} \cong \xi_1(n), \tag{26}$$

where ξ_p is the polar radius of the surface.

Note that the "thickness" of region V_{II} (namely $\xi_e - \xi_1(n)$) is proportional to Ω^2, and the subintegral function in the integral for the region is proportional to Ω^{2n}. Therefore, the integral over the region V_{II} is proportional to Ω^{2n+2} and there is a small value at "large" values of the polytropic index ($n \gtrsim 1.5$). That's why we neglect the integral for the region V_{II}, which simplifies the calculation of integration constants.

Equality (23) can be given a simpler form, using the integral form of equation for Emden function (equation (8) at $\Omega = 0$, which describes the polytrope with spherically symmetry)

$$y_n(\xi) = 1 + \frac{1}{4\pi}\int Q(\boldsymbol{\xi},\boldsymbol{\xi}')y_n^n(\xi')\,d\boldsymbol{\xi}' = 1 + \int_0^{\xi} y_n^n(\xi')\left[\frac{(\xi')^2}{\xi} - \xi'\right]d\xi', \tag{27}$$

at $\xi < \xi_1(n)$ and $y_n(\xi) = 0$ at $\xi \geqslant \xi_1(n)$. Also, let us use the integral form of equation for the function $\psi_{n,0}(\xi)$, namely

$$\psi_{n,0}(\xi) = -\frac{1}{4\pi}\int Q(\boldsymbol{\xi},\boldsymbol{\xi}')\{1 - n\,y_n^{n-1}(\xi')\psi_{n,0}(\xi')\}d\boldsymbol{\xi}' = \\ = -\int_0^{\xi}\left[\frac{(\xi')^2}{\xi} - \xi'\right]\Big[1 - ny_n^{n-1}(\xi')\psi_{n,0}(\xi')\Big]d\xi'; \quad 0 \leqslant \xi \leqslant \xi_1(n). \quad (28)$$

Next, the integral calculation

$$\mathcal{J}_n(\xi,t) = \frac{1}{4\pi}\int_{V_I} Q(\boldsymbol{\xi},\boldsymbol{\xi}')y_n^n(\xi')\,d\boldsymbol{\xi}'. \quad (29)$$

Expanding the kernel $Q(\boldsymbol{\xi},\boldsymbol{\xi}')$ in series for Legendre polynomials, in the region 1 ($0 \leqslant \xi \leqslant \xi_p$) we obtain

$$\mathcal{J}_n^{(1)}(\xi,t) = \int_0^{\xi} d\xi' y_n^n(\xi')\left\{\frac{(\xi')^2}{\xi} - \xi'\right\} + \sum_{l\geqslant 1}\xi^{2l}P_{2l}(t) \\ \times\left\{\int_{t_n}^1 dt' P_{2l}(t')\int_{\xi}^{\xi_0(t')} d\xi' y_n^n(\xi')(\xi')^{1-2l} + \int_0^{t_n} dt' P_{2l}(t')\int_{\xi}^{\xi_1(n)} d\xi' y_n^n(\xi')(\xi')^{1-2l}\right\}, \quad (30)$$

Taking into account the properties of the Legendre polynomials

$$\int_0^1 P_{2l}(t)\,dt = 0 \quad \text{at} \quad l \geqslant 1, \quad (31)$$

it can be found, that

$$\mathcal{J}_n^{(1)}(\xi,t) = y_n(\xi) - 1 + A_n(\xi,t), \\ A_n(\xi,t) = -\sum_{l\geqslant 1}\xi^{2l}P_{2l}(t)\int_{t_n}^1 dt' P_{2l}(t')\int_{\xi_0(t')}^{\xi_1(n)} d\xi' y_n^n(\xi')(\xi')^{1-2l}. \quad (32)$$

According to definition (26) at the small values Ω, when ξ_p is close to $\xi_1(n)$, we find that $t_n \approx 1 - \Omega^2$. Because of that the sum over the index l in formula (32) is a small value ($\sim \Omega^4$). In the region 2 ($\xi_p \leqslant \xi \leqslant \xi_1$)

$$\mathcal{J}_n^{(2)}(\xi,t) = y_n(\xi) - 1 + A_n(\xi,t) - \int_t^1 dt'\int_{\xi_0(t')}^{\xi} y_n^n(\xi')\left\{\frac{(\xi')^2}{\xi} - \xi'\right\}d\xi' \\ - \sum_{l\geqslant 1}P_{2l}(t)\int_t^1 dt' P_{2l}(t')\int_{\xi_0(t')}^{\xi} y_n^n(\xi')\,\frac{\{(\xi')^{4l+1} - \xi^{4l+1}\}}{\xi^{2l+1}(\xi')^{2l-1}}\,d\xi'. \quad (33)$$

Herewith $t > t_n$, therefore $1 - t \sim \Omega^2$, $\xi_1(n) - \xi_0(t') \sim \Omega^2$, $\xi' - \xi_1(n) \sim \Omega^2$, so the both integral terms are proportional to Ω^6 and are small. It is reasonable to neglect them and at arbitrary values ξ in the region V_I to use the approximation

$$\mathcal{J}_n(\xi,t) \approx y_n - 1 + A_n(\xi,t). \quad (34)$$

The integral

$$\frac{1}{4\pi}\int_{V_I} Q(\boldsymbol{\xi},\boldsymbol{\xi}')\,n\,y_n^{n-1}(\xi')\psi_{n,0}(\xi')\,d\boldsymbol{\xi}', \quad (35)$$

which appears in the region (23), is represented in the form

$$\frac{1}{4\pi}\int_{V_I} Q(\boldsymbol{\xi},\boldsymbol{\xi}')\,d\boldsymbol{\xi}' - \frac{1}{4\pi}\int_{V_I} Q(\boldsymbol{\xi},\boldsymbol{\xi}')\{1-ny_n^{n-1}(\xi')\psi_{n,0}(\xi')\}\,d\boldsymbol{\xi}'$$
$$= -\frac{\xi^2}{6} - \frac{1}{2}\sum_{l\geqslant 1} P_{2l}(t)\xi^{2l} I_{2l}^{(n)} - \int_0^{\xi}\left\{\frac{(\xi')^2}{\xi}-\xi'\right\}\left\{1-ny_n^{n-1}(\xi')\psi_{n,0}(\xi')\right\}d\xi' - \sum_{l\geqslant 1} P_{2l}(t)\xi^{2l} D_{2l}^{(n)}. \quad (36)$$

The following notations are introduced

$$\begin{aligned}
I_2^{(n)} &= -2\int_{t_n}^1 P_2(t')\big\{\ln\xi_0(t') - \ln\xi_1(n)\big\}dt',\\
I_{2l}^{(n)} &= (l-1)^{-1}\int_{t_n}^1 P_{2l}(t')\big\{[\xi_0(t')]^{2-2l} - [\xi_1(n)]^{2-2l}\big\}dt' \quad \text{at} \quad l\geqslant 2,\\
D_{2l}^{(n)} &= \int_{t_n}^1 P_{2l}(t')\,dt'\int_{\xi_0(t')}^{\xi_1(n)} (\xi')^{1-2l}\{\Delta_{\xi'}\psi_{n,0}(\xi')\}d\xi' =\\
&= \int_{t_n}^1 P_{2l}(t')\,dt'\left\{\left((\xi')^{1-2l}\,\frac{d\psi_{n,0}(\xi')}{d\xi'}\right)\Bigg|_{\xi_0(t')}^{\xi_1(n)} - \int_{\xi_0(t')}^{\xi_1(n)}(2l+1)(\xi')^{-2l}\,\frac{d\psi_{n,0}(\xi')}{d\xi'}\,d\xi'\right\}.
\end{aligned} \quad (37)$$

Expressions (27)–(37) allow us to simplify equality (23) to the next form

$$\sum_{l\geqslant 1} a_{2l}^{(n)} P_{2l}(t)\psi_{n,2l}(\xi) = -\frac{\xi^2}{6}\,P_2(t) - \sum_{l\geqslant 1} P_{2l}(t)\,\xi^{2l}\left\{\frac{1}{2}I_{2l}^{(n)} + L_{2l}^{(n)} + D_{2l}^{(n)}\right\}$$
$$+\frac{1}{4\pi}\int_{V_I} Q(\boldsymbol{\xi},\boldsymbol{\xi}')\,n\,y_n^{n-1}(\xi')\sum_{l\geqslant 1} a_{2l}^{(n)} P_{2l}(t')\psi_{n,2l}(\xi')\,d\boldsymbol{\xi}', \quad (38)$$

where

$$L_{2l}^{(n)} = \Omega^{-2}\int_{t_n}^1 dt' P_{2l}(t')\int_{\xi_0(t)}^{\xi_1(n)} y_n^n(\xi')(\xi')^{1-2l}d\xi'. \quad (39)$$

To calculate the integral for the vector $\boldsymbol{\xi}'$ in equation (38), we expand the kernel $Q(\boldsymbol{\xi},\boldsymbol{\xi}')$ in series for Legendre polynomials,

$$\frac{1}{4\pi}\sum_{l\geqslant 1} a_{2l}^{(n)}\int P_{2l}(t')\,n\,y_n^{n-1}(\xi')\psi_{n,2l}(\xi')Q(\boldsymbol{\xi},\boldsymbol{\xi}')\,d\boldsymbol{\xi}'$$
$$= \sum_{l\geqslant 1} a_{2l}^{(n)} P_{2l}(t)\frac{1}{4l+1}\left\{\frac{1}{\xi^{2l+1}}\int_0^{\xi}(\xi')^{2l+2}F_{2l}^{(n)}(\xi')\,d\xi' + \xi^{2l}\int_{\xi}^{\xi_1(n)}(\xi')^{1-2l}F_{2l}^{(n)}(\xi')\,d\xi'\right\}$$
$$-\sum_{l\geqslant 1} a_{2l}^{(n)}\xi^{2l}P_{2l}(t)\int_{t_n}^1 P_{2l}^2(t')\,dt'\int_{\xi_0(t')}^{\xi_1(n)}(\xi')^{1-2l}F_{2l}^{(n)}(\xi')\,d\xi' - \sum_{l,m\geqslant 1}(1-\delta_{l,m})P_{2m}(t)\xi^{2m}a_{2l}^{(n)}S_{2l,2m}^{(n)}, \quad (40)$$

where

$$F_{2l}^{(n)}(\xi) = n\,y_n^{n-1}(\xi)\psi_{n,2l}(\xi) = 2l\,(2l+1)\xi^{-2}\psi_{n,2l}(\xi) - \Delta_{\xi}\psi_{n,2l}(\xi) \quad (41)$$

according to equations (14),

$$S_{2l,2m}^{(n)} = \int_{t_n}^1 dt'\,P_{2l}(t')P_{2m}(t')\int_{\xi_0(t')}^{\xi_1(n)}(\xi')^{1-2m}F_{2l}^{(n)}(\xi')\,d\xi'. \quad (42)$$

The sum of the first two series in formula (40) equals

$$\sum_{l\geqslant 1} a_{2l}^{(n)} P_{2l}(t)\psi_{n,2l}(\xi) - \sum_{l\geqslant 1} a_{2l}^{(n)} P_{2l}(t)\xi^{2l} S_{2l,2l}^{(n)},$$

$$\begin{aligned} S_{2l,2l}^{(n)} &= (4l+1)^{-1}\xi_1^{-2l}\left[(2l+1)\psi_{n,2l}(\xi_1) + \xi_1 \frac{d\psi_{n,2l}(\xi_1)}{d\xi_1}\right] \\ &+ \int_{t_n}^{1} dt\, P_{2l}^2(t)\left\{\xi_0^{-2l}(t)\left[(2l+1)\psi_{n,2l}(\xi_0(t)) + \xi_0(t)\frac{d\psi_{n,2l}(\xi_0(t))}{d\xi_0(t)}\right]\right. \\ &- \left.\xi_1^{-2l}\left[(2l+1)\psi_{n,2l}(\xi_1) + \xi_1\frac{d\psi_{n,2l}(\xi_1)}{d\xi_1}\right]\right\}. \end{aligned} \tag{43}$$

Because of orthogonality of the Legendre polynomials in the integral over ξ' in formula (42), it is important only to consider the upper vicinity of boundary. Expanding subintegral function in the vicinity $\xi_0(t)$ and integrating by parts, we find, that

$$\begin{aligned} S_{2l,2m}^{(n)} \simeq \int_{t_n}^{1} & P_{2l}(t)P_{2m}(t)\xi_0^{-2l}(t)\left\{\left[(2l+1)\psi_{n,2m}(\xi_0(t)) + \xi_0(t)\frac{d\psi_{n,2m}(\xi_0(t))}{d\xi_0(t)}\right]\right. \\ &- [2l(2l-1)]^{-1}[2l(2l+1) - 2m(2m+1)]\left.\left[(2l-1)\psi_{n,2m}(\xi_0(t)) + \xi_0(t)\frac{d\psi_{n,2m}(\xi_0(t))}{d\xi_0(t)}\right]\right\}dt. \end{aligned} \tag{44}$$

Substituting expressions (40), (43) in equality (38) and equating coefficients of the products $P_{2l}(t)\xi^{2l}$, it is the system of the linear inhomogeneous algebraic equations for the constants $a_{2l}^{(n)}$

$$\begin{aligned} a_2^{(n)} S_{2,2}^{(n)} + \sum_{m\geqslant 2} a_{2m}^{(n)} S_{2,2m} &= -\frac{1}{6} - \frac{1}{2} I_2^{(n)} - L_2^{(n)} - D_2^{(n)}, \\ a_{2l}^{(n)} S_{2l,2l}^{(n)} + \sum_{m\geqslant 1}(1-\delta_{l,m}) a_{2m}^{(n)} S_{2l,2m} &= -\frac{1}{2} I_{2l}^{(n)} - L_{2l}^{(n)} - D_{2l}^{(n)},\ l \geqslant 2. \end{aligned} \tag{45}$$

Stitching conditions (19) the relations between the integration constants $a_{2l}^{(n)}$ and the constants $c_0^{(n)}$, $b_0^{(n)}$, $c_2^{(n)}$ are established, $b_2^{(n)}$, $c_4^{(n)}$, $b_4^{(n)}, \ldots$

$$\begin{aligned} c_0^{(n)} &= \left\{\psi_{n,0}(\xi_1) + \xi_1\psi'_{n,0}(\xi_1)\right\} - \frac{\xi_1^2}{2} + \xi_1\,\frac{y'_n(\xi_1)}{\Omega^2}, \\ b_0^{(n)} &= \xi_1\left\{\psi_{n,0}(\xi_1) - \frac{\xi_1^2}{6} - c_0^{(n)}\right\}, \\ c_2^{(n)} &= \frac{3}{5}\left\{\frac{5}{18} + \frac{a_2^{(n)}}{\xi_1^2}\left[\psi_{n,2}(\xi_1) + \frac{\xi_1}{3}\psi'_{n,2}(\xi_1)\right]\right\}, \\ b_2^{(n)} &= \xi_1^3\left\{a_2^{(n)}\psi_{n,2}(\xi_1) + \frac{\xi_1^2}{6} - \xi_1^2 c_2^{(n)}\right\}, \\ c_4^{(n)} &= \frac{5}{9}\frac{a_4^{(n)}}{\xi_1^4}\left\{\psi_{n,4}(\xi_1) + \frac{\xi_1}{5}\psi'_{n,4}(\xi_1)\right\}, \\ b_4^{(n)} &= \xi_1^5\{a_4^{(n)}\psi_{n,4}(\xi_1) - c_4^{(n)}\xi_1^4\}, \quad \ldots, \quad \xi_1 \equiv \xi_1(n). \end{aligned} \tag{46}$$

As one can see from the definitions $S_{2l,2l}^{(n)}$ and $S_{2m,2l}^{(n)}$, values of these quantities depend on the polytrope surface $\xi_0(t)$, which is determined by the condition $Y_n(\xi,\theta) = 0$. Therefore, the problem of calculating integration constants is self-consistent, and can be performed via the iterative method. The surface of rotational polytrope is close to the surface of rotational ellipsoid, determined by values

(e, ξ_p), or (e, ξ_e), where ξ_p i ξ_e are the polar and equatorial radii, $e = \{1 - \xi_p^2/\xi_e^2\}^{1/2}$ is its eccentricity. Polar radius can be found from the condition $Y_n^I(\xi, 0) = 0$, and the equatorial one from the condition $Y_n^{II}(\xi, \pi/2) = 0$. Consequently, it is necessary to find constants $a_{2l}^{(n)}$ at $l \geqslant 1$ and constants $c_{2l}^{(n)}$ in each iteration, $b_{2l}^{(n)}$ at $l \geqslant 0$. As zero approximation, There are used expressions (20), being good approximation for the small values of angular velocities. We restricted ourselves with approach $a_{2l}^{(n)} = c_{2l}^{(n)} = b_{2l}^{(n)} = 0$ at $l \geqslant 3$. The iterative process is fast converging and requires no more than 5 iterations even at the maximum value of angular velocity $\Omega_{\max}(n)$.

The calculating results of the dependence of geometrical elements on the polytrope surface (ξ_p, ξ_e, e) on angular velocity for the case $n = 2.5$ are shown in Table 1. There is also shown the dependence of

Table 1. Dependence of the model characteristics with index $n = 2.5$ on angular velocity in approximation $a_{2l} = 0$ at $l \geqslant 3$.

Ω	$e(\Omega)$	$\xi_p(\Omega)$	$\xi_e(\Omega)$	$a_2(\Omega)$	$a_4(\Omega)$	$\eta(n,\Omega)$	$\zeta(n,\Omega)$
0.01000	0.06066	5.35266	5.36254	−10.27730	−0.00115019	1.00046	1.00157
0.02000	0.12251	5.34499	5.38556	−10.28090	−0.00506429	1.00185	1.00632
0.03000	0.18449	5.33222	5.42536	−10.28690	−0.011586	1.00418	1.01440
0.04000	0.24705	5.31437	5.48437	−10.29500	−0.0206999	1.00752	1.02603
0.05000	0.31060	5.29147	5.56680	−10.30530	−0.0323615	1.01192	1.04159
0.06000	0.37578	5.26359	5.67989	−10.31740	−0.0464533	1.01748	1.06165
0.07000	0.44375	5.23085	5.83701	−10.33070	−0.0626984	1.02434	1.08701
0.08000	0.51687	5.19342	6.06661	−10.34430	−0.0804401	1.03269	1.11896
0.09000	0.60184	5.15169	6.45080	−10.35610	−0.0978105	1.04282	1.15961
0.09100	0.61171	5.14731	6.50666	−10.35700	−0.0993382	1.04395	1.16429
0.09200	0.62201	5.14290	6.56812	−10.35770	−0.100777	1.0451	1.16909
0.09300	0.63285	5.13845	6.63651	−10.35840	−0.102109	1.04627	1.17404
0.09400	0.64439	5.13398	6.71372	−10.35890	−0.103295	1.04747	1.17913
0.09500	0.65683	5.12949	6.80268	−10.35920	−0.104284	1.04869	1.18437

the relative increase of the polytrope mass $\eta(n,\Omega) = M(n,\Omega)/M(n,0)$, where $M(n,0)$ is the polytrope mass without rotation, as well as the relative increase of the inertia moment relative to the rotation axis $\zeta(n,\Omega) = I(n,\Omega)/I(n,0)$, where $I(n,0)$ is the inertia moment of the polytrope with index n at $\Omega = 0$. In Table 2 was shown dependence of integration constants on angular velocity for the polytrope $n = 2.5$. The results of analogues calculations for the polytrope $n = 3$ are shown in Tables. 3 and 4.

Table 2. Dependence of integration constants for the polytrope with index $n = 2.5$ on angular velocity according to expressions (46).

Ω	$c_0^{(n)} \cdot \Omega^2$	$b_0^{(n)} \cdot \Omega^2$	$c_2^{(n)}$	$b_2^{(n)} \cdot \Omega^2$	$c_4^{(n)}$	$b_4^{(n)}$
0.01000	−0.408791	2.18821	−0.000217194	−0.00512369	$-6.00647 \cdot 10^{-7}$	−0.0188939
0.02000	−0.409836	2.19122	−0.000275652	−0.0205019	$-2.64465 \cdot 10^{-6}$	−0.0831899
0.03000	−0.411581	2.19623	−0.00037308	−0.0461562	$-6.05039 \cdot 10^{-6}$	−0.19032
0.04000	−0.414021	2.20326	−0.000504609	−0.0821202	$-1.08098 \cdot 10^{-5}$	−0.340032
0.05000	−0.41716	2.21229	−0.000671861	−0.128441	$-1.68997 \cdot 10^{-5}$	−0.531595
0.06000	−0.420995	2.22333	−0.000868342	−0.185172	$-2.42586 \cdot 10^{-5}$	−0.763077
0.07000	−0.425529	2.23638	−0.00108431	−0.252365	$-3.27421 \cdot 10^{-5}$	−1.02993
0.08000	−0.43076	2.25143	−0.00130515	−0.330054	$-4.20071 \cdot 10^{-5}$	−1.32137
0.09000	−0.436688	2.26849	−0.00149676	−0.418201	$-5.10782 \cdot 10^{-5}$	−1.60671
0.09100	−0.43732	2.2703	−0.00151137	−0.427583	$-5.1876 \cdot 10^{-5}$	−1.6318
0.09200	−0.437958	2.27214	−0.00152274	−0.437061	$-5.26273 \cdot 10^{-5}$	−1.65544
0.09300	−0.438603	2.274	−0.0015341	−0.446645	$-5.33229 \cdot 10^{-5}$	−1.67732
0.09400	−0.439254	2.27587	−0.00154222	−0.456324	$-5.39423 \cdot 10^{-5}$	−1.6968
0.09500	−0.439914	2.27777	−0.0015471	−0.466098	$-5.44588 \cdot 10^{-5}$	−1.71305

The polytrope characteristics without rotation are shown in our work [2].

Table 3. Dependence of the model characteristics with index $n = 3$ on angular velocity in approximation $a_{2l} = 0$ at $l \geqslant 3$.

Ω	$e(\Omega)$	$\xi_p(\Omega)$	$\xi_e(\Omega)$	$a_2(\Omega)$	$a_4(\Omega)$	$\eta(n,\Omega)$	$\zeta(n,\Omega)$
0.01000	0.09220	6.89116	6.92064	−10.87250	−0.00125638	1.00063	1.00242
0.02000	0.18451	6.87473	6.99482	−10.87740	−0.00465916	1.00249	1.00981
0.03000	0.27926	6.84737	7.13108	−10.88530	−0.0102429	1.00566	1.02260
0.04000	0.37855	6.80919	7.35666	−10.89590	−0.0177874	1.01024	1.04158
0.05000	0.48751	6.76042	7.74285	−10.90790	−0.0266846	1.01642	1.06821
0.05100	0.49938	6.75498	7.79677	−10.90910	−0.0275959	1.01714	1.07138
0.05200	0.51152	6.74945	7.85487	−10.91030	−0.0285006	1.01787	1.07466
0.05300	0.52398	6.74381	7.91776	−10.91140	−0.0293932	1.01863	1.07805
0.05400	0.53679	6.73808	7.98620	−10.91250	−0.0302697	1.01941	1.08155
0.05500	0.55003	6.73226	8.06117	−10.91360	−0.0311239	1.0202	1.08517
0.05600	0.56377	6.72634	8.14394	−10.91470	−0.0319476	1.02102	1.08892
0.05700	0.57813	6.72034	8.23623	−10.91560	−0.0327273	1.02186	1.09280
0.05800	0.59325	6.71425	8.34048	−10.91650	−0.0334517	1.02272	1.09681
0.05900	0.60936	6.70808	8.46027	−10.91720	−0.0340975	1.0236	1.10097
0.06000	0.62682	6.70183	8.60132	−10.91780	−0.0346355	1.02451	1.10529
0.06100	0.64624	6.69552	8.77373	−10.91820	−0.0350125	1.02544	1.10977

Table 4. Dependence of integration constants for the polytrope with index $n = 3$ on angular velocity according to expressions (46).

Ω	$c_0^{(n)} \cdot \Omega^2$	$b_0^{(n)} \cdot \Omega^2$	$c_2^{(n)}$	$b_2^{(n)} \cdot \Omega^2$	$c_4^{(n)}$	$b_4^{(n)}$
0.01000	−0.293028	2.01957	−0.000201124	−0.00752171	$-6.35822 \cdot 10^{-7}$	−0.0632967
0.02000	−0.294192	2.02328	−0.000276327	−0.0301004	$-2.35788 \cdot 10^{-6}$	−0.234729
0.03000	−0.296133	2.02946	−0.000397574	−0.0677751	$-5.18367 \cdot 10^{-6}$	−0.516039
0.04000	−0.298851	2.03811	−0.00056026	−0.120606	$-9.00175 \cdot 10^{-6}$	−0.896132
0.05000	−0.302345	2.04924	−0.000744432	−0.188655	$-1.35044 \cdot 10^{-5}$	−1.34438
0.05100	−0.302738	2.05049	−0.000762849	−0.196298	$-1.39656 \cdot 10^{-5}$	−1.39029
0.05200	−0.303137	2.05176	−0.000781266	−0.204094	$-1.44234 \cdot 10^{-5}$	−1.43587
0.05300	−0.303543	2.05306	−0.000798149	−0.212041	$-1.48752 \cdot 10^{-5}$	−1.48083
0.05400	−0.303961	2.05438	−0.000815031	−0.22014	$-1.53187 \cdot 10^{-5}$	−1.52499
0.05500	−0.304385	2.05573	−0.000831914	−0.228392	$-1.5751 \cdot 10^{-5}$	−1.56803
0.05600	−0.304814	2.0571	−0.000848796	−0.236797	$-1.61679 \cdot 10^{-5}$	−1.60953
0.05700	−0.305253	2.05849	−0.000862609	−0.245349	$-1.65625 \cdot 10^{-5}$	−1.64881
0.05800	−0.305699	2.05992	−0.000876422	−0.254054	$-1.69291 \cdot 10^{-5}$	−1.6853
0.05900	−0.306154	2.06136	−0.000887166	−0.262907	$-1.72559 \cdot 10^{-5}$	−1.71784
0.06000	−0.306616	2.06284	−0.000896374	−0.27191	$-1.75281 \cdot 10^{-5}$	−1.74494
0.06100	−0.307085	2.06433	−0.000902513	−0.281059	$-1.77189 \cdot 10^{-5}$	−1.76394

As was shown in Tables. 2 and 4, integration constants are weakly dependent on angular velocity and this dependence decreases with increasing index n. Obtained results for polar and equatorial radii in all region of change of angular velocity with high accuracy coincide with the results of numerical integration of equilibrium equation in [7], which is illustrated in Table 5 for the polytrope $n = 3$. Note that in work cited above [11], the deviations from results of work [7] are several percent (at

Table 5. Comparison of the polytrope characteristics calculated in this study with the values obtained in the case $n = 3$ in [7]. Here $M(\Omega)$ is the polytrope mass in units $4\pi\rho_c\lambda^3$, $V(\Omega)$ is the dimensionless volume in units $10^3\lambda^3$, $g_e(\Omega)$ is the equatorial gravity of star in units $4\pi G\rho_c\lambda$ ([∗] corresponds to our results).

Ω	$\xi_p(\Omega)$[∗]	$\xi_e(\Omega)$[∗]	$g_e(\Omega) \cdot 10^2$[∗]	$M(\Omega)$[∗]	$V(\Omega) \cdot 10^{-3}$[∗]	$\xi_p(\Omega)$ [7]	$\xi_e(\Omega)$ [7]	$g_e(\Omega) \cdot 10^2$ [7]	$M(\Omega)$ [7]	$V(\Omega) \cdot 10^{-3}$ [7]
0.02	6.87473	6.99482	3.99525	2.02323	1.40880	6.87504	6.99456	3.9975	2.02323	1.40867
0.04	6.80919	7.35666	3.17874	2.03887	1.53848	6.80949	7.35492	3.1874	2.03882	1.53745
0.06	6.70183	8.60132	1.24656	2.06767	1.97737	6.70013	8.57635	1.2763	2.06738	1.96685

$n = 3$ it is 4% in the vicinity $\Omega_{\max}(3)$). As was shown in Table 5, our results for all characteristics of the polytrope only slightly deviate from results of [7] obtained by numerically integration of the equilibrium equation.

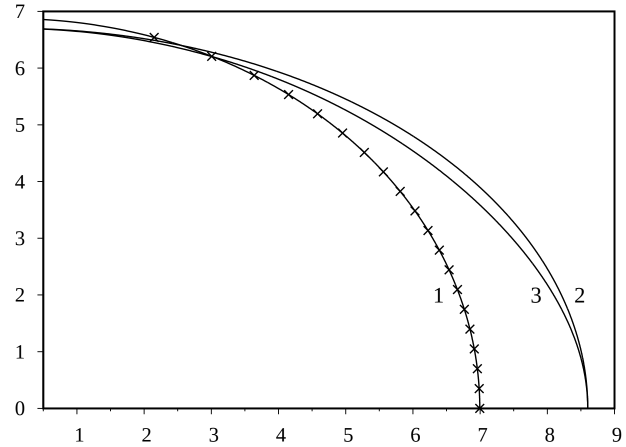

Fig. 5. The meridional section of the polytropes surface with index $n = 3$. Curve 1 corresponds to the surface of an ideal rotational ellipsoid at $\Omega_1 = 0.02$. Curve 2 — the same, but the angular velocity $\Omega_2 = 0.06$. Curve 3 represents the true surface of the polytrope at $\Omega_2 = 0.06$, which is determined by the smallest root of equation $Y_n(\xi, \theta) = 0$ relative to ξ. Crosses corresponds to this condition at $\Omega_1 = 0.02$.

In our research work [2] it was shown, that the surface of rotational polytrope $n = 1$ deviates from the surface of rotational polytrope with the same values ξ_p and ξ_e, and the deviation value is a monotonically increasing function of Ω. Our calculations show that this effect is also observed for polytropes with $n > 1$, as evidenced by Fig. 5. Obviously, this could be discovered by the author of [7], if he would have calculated $\xi_0(t)$, and not only ξ_p, ξ_e.

5. Conclusions

Method of separate description of the internal region and periphery, proposed in [11], summarizes Milne–Chandrasekhar approach [6,12] and is a very rational in finding the nonlinear equations solutions ($n > 1$). However, the method of the equal determination of integration constants from the stitching condition of solutions at the border of these regions, used in [11] does not provide sufficient accuracy and uses the fitting parameter and suitable for the small angular velocities.

Our approach is based on the unequal description: integration constants for the important internal region are determined by the integral form of the equilibrium equation, and integration constants for periphery are found from the stitching condition of solutions without using fitting parameters. At the same time, our approach is self-consistent, and integration constants for the periphery obtained in the i-th iteration, are used in determination of integration constants for the inner region in the $(i+1)$-th iteration. The rationale in favor of our approach is shown in Fig. 6, which depicts the normalized density of matter $\rho(\boldsymbol{r})/\rho_c = [Y_n(\xi, \theta)]^n$ as function of variable ξ for polar (curve 1) and equatorial (curve 2) directions in the case $n = 2.5$ at $\Omega \approx \Omega_{\max} = 0.095$. Dashed curve corresponds to the polytrope without rotation. As was shown in Figure, in the region of periphery (at $\xi > \xi_1(n)$), the density drops to zero very quickly, which justifies the approximate description in this region.

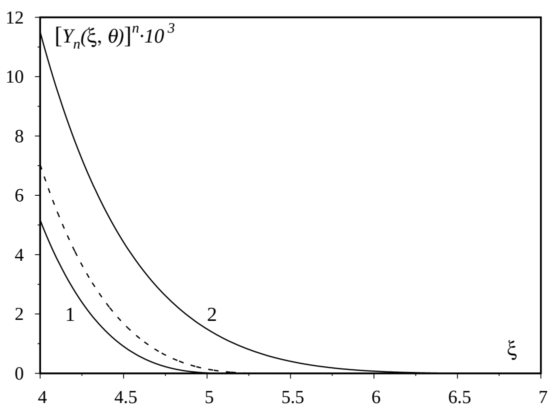

Fig. 6. The distribution of dimensionless density for the polytrope with index $n = 2.5$ at $\Omega = 0.095$ for polar (curve 1) and equatorial (curve 2) directions. Dashed curve corresponds to the model without rotation.

Unlike [11], calculated in this work polar and equatorial radii with high accuracy coincide with the results of [7], and all integration constants are the functions of angular velocity. The surface of rotational polytropes calculated at $n = 3$ deviates from the ideal rotational polytrope. This effect marked for the first time in our work [2], is obviously natural.

The deviation of the polytrope surface from the surface of ideal rotational polytrope with the same values ξ_e and ξ_p confirmed by the calculation of the polytrope volume,

$$V(\Omega) = \frac{4}{3}\pi\lambda^3 \int_0^1 \xi_0^3(t)dt. \tag{47}$$

The dimensionless volume $v = V\lambda^{-3}$ for the ideal rotational ellipsoid equals $4/3\pi\xi_e^2\xi_p$, that at $n = 3$ and $\Omega = 0.06$ is $2.07688 \cdot 10^3$. However, volume calculated in this study by formula (47) at these values n and Ω equals to $1.97737 \cdot 10^3$, which deviates from the result of [7] at 0.5% (see Table 5).

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Метод інтегральних рівнянь у політропній теорії зір з осьовим обертанням. II. Політропи з індексами $n > 1$

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Запропоновано новий спосіб знаходження розв'язків нелінійних рівнянь рівноваги для обертових політроп, що грунтується на самоузгодженому описі внутрішньої області та периферії при використанні інтегральної форми рівнянь. Розраховано залежність геометричних параметрів, форми поверхні, маси, моменту інерції і сталих інтегрування від кутової швидкості для індексів $n = 2.5$ і $n = 3$.

Ключові слова: *зорі-політропи, неоднорідні еліпсоїди, осьове обертання, рівняння механічної рівноваги, стабільність зір.*