MATHEMATICAL MODELING AND COMPUTING, Vol. 8, No. 3, pp. 499–508 (2021)

Mathematical modeling of fluid flows through the piecewise homogeneous porous medium by R-function method

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(Received 1 February 2021; Revised 2 August 2021; Accepted 3 August 2021)

The stationary fluid flow through a piecewise homogeneous porous medium is considered under the assumption that Darcy's law holds. The mathematical model of this problem is defined as an elliptic equation for the stream function, supplemented by the second-type boundary conditions at the water boundaries and the first-type boundary conditions at the impervious to liquid boundaries. The problem statement also includes the conditions of conjugation at the separation line between two soils and the unknown value of fluid discharge, which can be established from the additional integral ratio. It is proposed to use the structure-variational method of R-functions in order to numerically analyze and solve the current problem. The complete solution structure for the boundary value problem of stream function regarding the R-functions method is established, moreover, the application of the Ritz method for approximating an unspecified structural formula component is substantiated. Then, the approximate value of the fluid discharge and the approximate solution of the original problem are found from the additional integral ratio. The computational experiment was carried out with different coefficients of permeability within the area, which has the shape of the lower half ring. It is established that as the number of coordinate functions increases, the value of fluid discharge becomes constant, indicating the convergence of the proposed method.

Keywords: *Darcy's law, fluid flow, porous medium, piecewise homogeneous medium, R-function method, Ritz method.*

2010 MSC: 65N99, 76S05 **DOI:** 10.23939/mmc2021.03.499

1. Introduction

Fluid flows through the porous media are widespread in nature [1]. Such flows are considered during the study of irrigation or draining processes, inflow of seawater into fresh water, flows around the hydraulic structures, etc.

The finite-difference method, majorant areas method, fictitious domain method, summative representation method and so on are the most commonly used numerical analysis methods of flows through the porous media [2–7]. The analyzed area may have a complex geometric shape, which leads to a precision loss in the numerical solutions of the mathematical physics problems by corresponding methods. The structure-variational R-functions method allows taking into account the geometric and analytical information contained in the researched problem most accurately and completely [8, 9].

Previously, the R-functions method was used for the analysis of flows under the hydraulic structures only [10–13]. The current article continues the research started in [14, 15].

Therefore, the development and improvement of the existing numerical analysis methods is an urgent scientific task. The goal of the current article is to develop and improve such kind of methods in order to analyze the flat stationary flows through a piecewise homogeneous porous medium.

2. Formulation of the problem

The stationary problem of the pressure fluid flow through a piecewise homogeneous porous medium is considered [1, 6, 7]. The equation $v = (v_x, v_y)$ describes the flow velocity vector. It is considered that Darcy's law is satisfied, according to which the pressure loss is proportional to the flow velocity through the porous medium.

The analysis of two-dimensional flow is conveniently to perform using the stream function, which defined with the following ratios:

$$v_x = \frac{\partial \psi}{\partial y}, \quad v_x = -\frac{\partial \psi}{\partial x}. \tag{1}$$

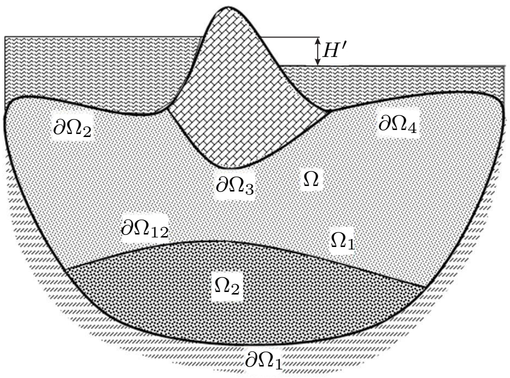


Fig. 1. The research area Ω.

The porous area Ω is surrounded by impervious boundaries $\partial\Omega_1$ and $\partial\Omega_3$, which are known as the flow contours, as well as two water boundaries $\partial\Omega_2$ and $\partial\Omega_4$, which are called the potential contours. Besides, the area Ω is filled with two types of medium that occupy the subareas Ω_1 and Ω_2 ($\bar{\Omega} = \bar{\Omega}_1 \cap \bar{\Omega}_2$ and $\operatorname{int}\Omega_1 \cap \operatorname{int}\Omega_2 = \varnothing$). The $\partial\Omega_{12}$ is the division boundary between two types of medium (Fig. 1). It is assumed that all the boundaries are piecewise, smooth, and can be described by elementary functions.

The coefficient of permeability is a piecewise constant function for the current mathematical problem, which is defined as follows:

$$\kappa(x, y) = \begin{cases} \kappa_1, & (x, y) \in \Omega_1, \\ \kappa_2, & (x, y) \in \Omega_2. \end{cases}$$

Therefore, the stream function

$$\psi(x, y) = \begin{cases} \psi_1(x, y), & (x, y) \in \Omega_1, \\ \psi_2(x, y), & (x, y) \in \Omega_2 \end{cases}$$

is the solution of the boundary value problem

$$\frac{\partial}{\partial x}\left(\frac{1}{\kappa(x, y)}\frac{\partial \psi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{1}{\kappa(x, y)}\frac{\partial \psi}{\partial y}\right) = 0 \quad \text{in} \quad \Omega, \tag{2}$$

$$\psi|_{\partial\Omega_1} = 0, \quad \psi|_{\partial\Omega_3} = Q, \tag{3}$$

$$\left.\frac{\partial \psi}{\partial \boldsymbol{n}}\right|_{\partial\Omega_2} = 0, \quad \left.\frac{\partial \psi}{\partial \boldsymbol{n}}\right|_{\partial\Omega_4} = 0, \tag{4}$$

$$\psi_1|_{\partial\Omega_{12}} = \psi_2|_{\partial\Omega_{12}}, \quad \left.\frac{1}{\kappa_1}\frac{\partial \psi_1}{\partial \boldsymbol{n}}\right|_{\partial\Omega_{12}} = \left.\frac{1}{\kappa_2}\frac{\partial \psi_2}{\partial \boldsymbol{n}}\right|_{\partial\Omega_{12}}, \tag{5}$$

where $\boldsymbol{n}$ is the normal to the corresponding sections of the boundary.

The conditions (5) are the set at the division boundary between two types of medium and known as the conjugation conditions. The variable Q is the unknown constant that specifies the fluid discharge, and it can be found from the following ratio

$$\int_{\partial\Omega_3} \frac{1}{\kappa}\frac{\partial \psi}{\partial \boldsymbol{n}} ds = -H', \tag{6}$$

where H' is the pressure.

3. The application of the structure-variational R-function method for the numerical analysis of flow through piecewise homogeneous porous medium

The presence of conjugation conditions (5) and integral condition (6) prevents the application of classical numerical methods for solving boundary value problem (2)–(6). The structure-variational R-function method is applied instead.

The solution of the problem (2)–(6) is sought as

$$\psi(x,y) = Qu(x,y),$$

where $u(x,y) = \begin{cases} u_1(x,y), \ (x,y) \in \Omega_1, \\ u_2(x,y), \ (x,y) \in \Omega_2, \end{cases}$ is the solution of the boundary value problem

$$\frac{\partial}{\partial x}\left(\frac{1}{\kappa(x,y)}\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{1}{\kappa(x,y)}\frac{\partial u}{\partial y}\right) = 0 \quad \text{in} \quad \Omega, \tag{7}$$

$$u|_{\partial\Omega_1} = 0, \quad u|_{\partial\Omega_3} = 1, \tag{8}$$

$$\left.\frac{\partial u}{\partial \boldsymbol{n}}\right|_{\partial\Omega_2} = 0, \quad \left.\frac{\partial u}{\partial \boldsymbol{n}}\right|_{\partial\Omega_4} = 0, \tag{9}$$

$$u_1|_{\partial\Omega_{12}} = u_2|_{\partial\Omega_{12}}, \quad \left.\frac{1}{\kappa_1}\frac{\partial u_1}{\partial \boldsymbol{n}}\right|_{\partial\Omega_{12}} = \left.\frac{1}{\kappa_2}\frac{\partial u_2}{\partial \boldsymbol{n}}\right|_{\partial\Omega_{12}}. \tag{10}$$

Value Q is found from the equation (6):

$$Q = -H'\left(\int_{\partial\Omega_3}\frac{1}{\kappa}\frac{\partial u}{\partial \boldsymbol{n}}ds\right)^{-1}. \tag{11}$$

The solution structure of the boundary value problem (7)–(10) according to the R-function method is constructed. It was proved [14] that the boundary conditions (8), (9) are satisfied by a sheaf of functions

$$u = f - \frac{\omega_{1-3}\omega_{2-4}}{\omega_{1-3}+\omega_{2-4}}D_1^{(2-4)}f + \omega_{1-3}\Phi - \frac{\omega_{1-3}\omega_{2-4}}{\omega_{1-3}+\omega_{2-4}}D_1^{(2-4)}(\omega_{1-3}\Phi), \tag{12}$$

where $\Phi = \Phi(x,y)$ is indefinite structure component, and

$$f(x,y) = \frac{\omega_1(x,y)}{\omega_1(x,y)+\omega_3(x,y)},$$

$$D_1^{(2-4)}g = \frac{\partial\omega_{2-4}}{\partial x}\frac{\partial g}{\partial x} + \frac{\partial\omega_{2-4}}{\partial y}\frac{\partial g}{\partial y},$$

$$\omega_{2-4}(x,y) = \omega_2(x,y) \wedge_\alpha \omega_4(x,y),$$

$$\omega_{1-3}(x,y) = \omega_1(x,y) \wedge_\alpha \omega_3(x,y).$$

Functions $\omega(x,y)$, $\omega_i(x,y)$, $i = 1,2,3,4$, developed using the constructive R-function theory apparatus [9], are defined as follows:

$$\omega(x,y) = 0 \quad \text{at} \quad \partial\Omega; \qquad \omega(x,y) > 0 \quad \text{in} \quad \Omega; \qquad \left.\frac{\partial\omega}{\partial \boldsymbol{n}}\right|_{\partial\Omega} = -1,$$

$$\omega_i(x,y) = 0 \quad \text{at} \quad \partial\Omega_i; \qquad \omega(x,y) > 0 \quad \text{in} \quad \Omega \cup (\partial\Omega \backslash \partial\Omega_i);$$

$$\left.\frac{\partial\omega_i}{\partial \boldsymbol{n}}\right|_{\partial\Omega_i} = -1, \quad i = 1,2,3,4.$$

The normalized equation of the division boundary $\partial\Omega_{12}$ is defined as $\omega_{12}=0$. The replacement of variables in the structure (12) [9] is done in order to satisfy the conjugation conditions (10):

$$x \mapsto x + \frac{\kappa_1-\kappa_2}{\kappa_1+\kappa_2}\cdot\frac{\omega^2\,|\omega_{12}|}{\omega^2+\omega_{12}^2}\cdot\frac{\partial\omega_{12}}{\partial x}, \tag{13}$$

$$y \mapsto y + \frac{\kappa_1-\kappa_2}{\kappa_1+\kappa_2}\cdot\frac{\omega^2\,|\omega_{12}|}{\omega^2+\omega_{12}^2}\cdot\frac{\partial\omega_{12}}{\partial y}. \tag{14}$$

As a result, a complete structure solution of the (7)–(10) boundary value problem, which satisfies all the boundary conditions (8)–(10) for any choice of indefinite component Φ, is obtained. Thus, the following theorem holds.

Theorem 1. *The sheaf of functions* (12)*, where the substitutions* (13), (14) *are made, satisfies the boundary conditions* (8), (9) *and the conjugation condition* (10) *for any choice of the sufficiently smooth indefinite component* Φ.

Thus, it remains to solve the problem of choosing an indefinite component Φ so as to best satisfy the differential equation (7). In order to achieve this goal the Ritz method is used.

A variable replacement in the problem (7)–(10) is made in form

$$u = \varphi + v,$$

where $\varphi = f - \frac{\omega_{1-3}\omega_{2-4}}{\omega_{1-3}+\omega_{2-4}} D_1^{(2-4)} f$, and v is a new unknown function. It's important to replace the arguments in φ function using formulas (13), (14).

Then, the function $v(x,y)=\begin{cases} v_1(x,y), & (x,y)\in\Omega_1,\\ v_2(x,y), & (x,y)\in\Omega_2\end{cases}$ becomes the solution of the boundary value problem

$$-\frac{\partial}{\partial x}\left(\frac{1}{\kappa(x,y)}\frac{\partial v}{\partial x}\right)-\frac{\partial}{\partial y}\left(\frac{1}{\kappa(x,y)}\frac{\partial v}{\partial y}\right)=F \quad \text{at} \quad \Omega, \tag{15}$$

$$v|_{\partial\Omega_1\cup\partial\Omega_3}=0, \qquad \left.\frac{\partial v}{\partial \boldsymbol{n}}\right|_{\partial\Omega_2\cup\partial\Omega_4}=0, \tag{16}$$

$$v_1|_{\partial\Omega_{12}} = v_2|_{\partial\Omega_{12}}, \qquad \left.\frac{1}{\kappa_1}\frac{\partial v_1}{\partial \boldsymbol{n}}\right|_{\partial\Omega_{12}} = \left.\frac{1}{\kappa_2}\frac{\partial v_2}{\partial \boldsymbol{n}}\right|_{\partial\Omega_{12}}, \tag{17}$$

where $F=\frac{\partial}{\partial x}\left(\frac{1}{\kappa}\frac{\partial\varphi}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{1}{\kappa}\frac{\partial\varphi}{\partial y}\right)$.

Let us connect the operator A, which operates in $L_2(\Omega)$ space by the rule

$$Av = -\frac{\partial}{\partial x}\left(\frac{1}{\kappa(x,y)}\frac{\partial v}{\partial x}\right)-\frac{\partial}{\partial y}\left(\frac{1}{\kappa(x,y)}\frac{\partial v}{\partial y}\right) \tag{18}$$

with the boundary problem (15)–(17).

It is assumed that the domain D_A of the operator A (18) consists of those functions from $L_2(\Omega)$ which belong to the set $C^2(\Omega)\cap C^1(\bar{\Omega}\backslash\partial\Omega_{12})$ and satisfy boundary and conjugation conditions (16), (17). It is clear that D_A is linear.

The properties of the operator A are set forth in the following lemma.

Lemma 1. *Operator A, which operates in $L_2(\Omega)$ space by rule* (18) *and is defined on the set D_A, has properties as follows:*

a) linearity;

b) symmetry;

c) it is positive;

d) it is positively defined.

Proof. Linearity of the operator A is obvious. Let us consider the scalar product $(Av,\ w)$, where $v, w \in D_A$. Here

$$v(x,y) = \begin{cases} v_1(x,y), & (x,y) \in \Omega_1, \\ v_2(x,y), & (x,y) \in \Omega_2, \end{cases}$$

$$w(x,y) = \begin{cases} w_1(x,y), & (x,y) \in \Omega_1, \\ w_2(x,y), & (x,y) \in \Omega_2. \end{cases}$$

The following equation is obtained after applying first Green's formula [16, 17]:

$$\begin{aligned}(Av, w) &= \iint_\Omega \left[-\frac{\partial}{\partial x}\left(\frac{1}{\kappa}\frac{\partial v}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{1}{\kappa}\frac{\partial v}{\partial y}\right)\right] w\, dx\, dy \\ &= \iint_{\Omega_1} \frac{1}{\kappa_1}\left(\frac{\partial v_1}{\partial x}\frac{\partial w_1}{\partial x} + \frac{\partial v_1}{\partial y}\frac{\partial w_1}{\partial y}\right) dx\, dy + \iint_{\Omega_2} \frac{1}{\kappa_2}\left(\frac{\partial v_2}{\partial x}\frac{\partial w_2}{\partial x} + \frac{\partial v_2}{\partial y}\frac{\partial w_2}{\partial y}\right) dx\, dy \\ &\quad - \int_{\partial\Omega_{12}} \frac{1}{\kappa_1}\frac{\partial v_1}{\partial \boldsymbol{n}_{12}} w_1 ds - \int_{\partial\Omega_{12}} \frac{1}{\kappa_2}\frac{\partial v_2}{\partial \boldsymbol{n}_{21}} w_2\, ds - \int_{\partial\Omega} \frac{1}{\kappa}\frac{\partial v}{\partial \boldsymbol{n}} w\, ds,\end{aligned}$$

where $\boldsymbol{n}_{12}$ is normal to the boundary $\partial\Omega_{12}$ and it is outward to Ω_1; n_{21} is normal to the boundary $\partial\Omega_{12}$, outward to Ω_2. Integral for $\partial\Omega$ equals zero since $v, w \in D_A$, and, therefore, $w|_{\partial\Omega_1 \cup \partial\Omega_3} = 0$, $\frac{\partial v}{\partial \boldsymbol{n}}\big|_{\partial\Omega_2 \cup \partial\Omega_4} = 0$, and $\int_{\partial\Omega} \frac{1}{\kappa}\frac{\partial v}{\partial \boldsymbol{n}} w\, ds = \int_{\partial\Omega_1 \cup \partial\Omega_3} \frac{1}{\kappa}\frac{\partial v}{\partial \boldsymbol{n}} w\, ds + \int_{\partial\Omega_2 \cup \partial\Omega_4} \frac{1}{\kappa}\frac{\partial v}{\partial \boldsymbol{n}} w\, ds$.

The fact that $\boldsymbol{n}_{12} = -\boldsymbol{n}_{21}$ and functions v and w satisfy the conjugation condition (17), is used to simplify the integrals by $\partial\Omega_{12}$. In other words,

$$v_1|_{\partial\Omega_{12}} = v_2|_{\partial\Omega_{12}}, \qquad \left.\frac{1}{\kappa_1}\frac{\partial v_1}{\partial \boldsymbol{n}}\right|_{\partial\Omega_{12}} = \left.\frac{1}{\kappa_2}\frac{\partial v_2}{\partial \boldsymbol{n}}\right|_{\partial\Omega_{12}},$$

$$w_1|_{\partial\Omega_{12}} = w_2|_{\partial\Omega_{12}}, \qquad \left.\frac{1}{\kappa_1}\frac{\partial w_1}{\partial \boldsymbol{n}}\right|_{\partial\Omega_{12}} = \left.\frac{1}{\kappa_2}\frac{\partial w_2}{\partial \boldsymbol{n}}\right|_{\partial\Omega_{12}}.$$

Then

$$\int_{\partial\Omega_{12}} \frac{1}{\kappa_1}\frac{\partial v_1}{\partial \boldsymbol{n}_{12}} w_1 ds + \int_{\partial\Omega_{12}} \frac{1}{\kappa_2}\frac{\partial v_2}{\partial \boldsymbol{n}_{21}} w_2\, ds = \int_{\partial\Omega_{12}} \left(\frac{1}{\kappa_1}\frac{\partial v_1}{\partial \boldsymbol{n}_{12}} w_1 - \frac{1}{\kappa_2}\frac{\partial v_2}{\partial \boldsymbol{n}_{12}} w_2\right) ds = 0.$$

Consequently,

$$(Av, w) = \iint_\Omega \frac{1}{\kappa}\left(\frac{\partial v}{\partial x}\frac{\partial w}{\partial x} + \frac{\partial v}{\partial y}\frac{\partial w}{\partial y}\right) dx\, dy \tag{19}$$

and A is a symmetric operator.

The positivity of the operator$A$ follows from the fact that for any $v \in D_A$

$$(Av, v) = \iint_\Omega \frac{1}{\kappa}\left[\left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right] dx\, dy \geqslant 0.$$

Moreover, the equality $(Av, v) = 0$ (11) is possible if only $v = 0$ due to condition (11).

Positive definedness of the operator$A$ is proved similarly as it was done in [15]. At the same time, the following inequality is obtained for any $v \in D_A$:

$$(Av, v) \geqslant (c\mu)^{-1} \|v\|^2_{L_2(\Omega)}.$$

Here $\mu = \max\{\kappa_1, \kappa_2\}$, and the constant $c > 0$ is determined by Friedrichs's inequality [17]:

$$\|u\|^2_{W_2^1(\Omega)} \leqslant c\left\{\iint_\Omega \left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right] dx\, dy + \int_{\Gamma_1} u^2 ds\right\},$$

which makes sense for functions u from the Sobolev space $W_2^1(\Omega)$. Here Ω is area with the Lipschitz boundary $\partial\Omega$, Γ_1 is opened part of the boundary $\partial\Omega$ in the area Ω of the positive measure of Lebesgue, $c > 0$ is constant, which depends on Ω and Γ_1. Lemma is proved. ∎

The energy product $[v,w]$ is introduced in D_A according to (19) for any $v, w \in D_A$

$$[v,w] = \iint_\Omega \frac{1}{\kappa}\left(\frac{\partial v}{\partial x}\frac{\partial w}{\partial x} + \frac{\partial v}{\partial y}\frac{\partial w}{\partial y}\right) dx\, dy.$$

As a result, the energy space H_A of the operator A of form (18), replenishing D_A by means of convergence by an energy norm, is obtained,

$$\|v\|_A = \sqrt{[v,v]} = \sqrt{\iint_\Omega \frac{1}{\kappa}\left[\left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right] dx\, dy}.$$

Then, according to the theorem about the energy functional [16], problem (15)–(17) has a unique (generalized) solution v^* in H_A provided by $F \in L_2(\Omega)$. This solution is the minimum point in H_A of the following energy functional:

$$J[v] = \|v\|_A^2 - 2(F,v) = \int_\Omega \left[\frac{1}{\kappa}\left[\left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right] - 2Fv\right] dx\, dy.$$

The approximate solution $J[v] \to \inf_{v\in H_A}$ of the variational problem in the following form is found using the Ritz method:

$$v_n = \sum_{k=1}^{n} c_k \varphi_k.$$

According to the structural formula (12), the coordinate sequence $\{\varphi_k\}$ consists of

$$\varphi_k = \omega_{1-3}\tau_k - \frac{\omega_{1-3}\omega_{2-4}}{\omega_{1-3}+\omega_{2-4}} D_1^{(2-4)}(\omega_{1-3}\tau_k),$$

where variables x and y are replaced according to formulas (13), (14). Therefore, the usage of the R-function method gives an opportunity to construct a coordinate sequence while implementing the Ritz method, i.e. a system of functions which accurately satisfies all the boundary conditions of the problem. Here $\{\tau_k\}$ is any complete system of functions in $L_2(\Omega)$ (power or trigonometric polynomials, splines, etc.).

Then, the following system of linear algebraic equations (Ritz system) is resolved in order to determine constants c_k, $k = 1, 2, \ldots, n$:

$$\sum_{k=1}^{n} [\varphi_k, \varphi_j] c_k = (F, \varphi_j), \quad j = 1, 2, \ldots, n,$$

where

$$[\varphi_k, \varphi_j] = \int_\Omega \frac{1}{\kappa}\left[\frac{\partial \varphi_k}{\partial x}\frac{\partial \varphi_j}{\partial x} + \frac{\partial \varphi_k}{\partial y}\frac{\partial \varphi_j}{\partial y}\right] dx\, dy,$$

$$(F, \varphi_j) = \int_\Omega F \cdot \varphi_j \, dx\, dy, \qquad k, j = 1, 2, \ldots, n.$$

The following theorem follows from the general convergence theorems of the Ritz method [16].

Theorem 2. *The solutions sequence $\{v_n\}$ of the problem* (15)–(17)*, approximated by Ritz, converges to the exact (generalized) solution of this problem both in the energy and the $L_2(\Omega)$ norms.*

Then, function $u^* = \varphi + v^*$ is considered as a generalized solution of problem (7)–(10), to which the sequence of approximate solutions $\{u_n\}$ converges in norm $L_2(\Omega)$ (which is formed by $u_n = \varphi + v_n$ rule).

Therefore, the following theorem holds.

Theorem 3. *Let $F \in L_2(\Omega)$. Then the sequence*

$$\psi_n = Q_n u_n,$$

where

$$Q_n = -H' \cdot \left(\int_{\partial\Omega_3} \frac{1}{\kappa} \cdot \frac{\partial u_n}{\partial \boldsymbol{n}}\, ds \right)^{-1}, \quad u_n = \varphi + v_n,$$

converges in $L_2(\Omega)$ to a unique generalized solution (2)–(6).

4. Results of computational experiment

Figure 2 represents the area Ω, where the computational experiment performed for the problem (2)–(6). The boundary $\partial\Omega$ of the mentioned area consists of the external circle with R radius, internal circle with r ($r < R$) radius, two horizontal segments of $y = 0$ line and the division boundary $y_0 = -1.5$, which separates two types of medium.

The coefficients of permeability κ_1 and κ_2 are valid in the subareas Ω_1 and Ω_2, respectively. Functions $\omega_1(x,y)$, $\omega_2(x,y)$, $\omega_3(x,y)$, $\omega_4(x,y)$, $\omega_{12}(x,y)$ for the selected area are defined as follows:

$$\omega_1(x,y) = \frac{1}{2R}(R^2 - x^2 - y^2), \quad \omega_2(x,y) = -y,$$

$$\omega_3(x,y) = \frac{1}{2r}(r^2 - x^2 - y^2), \quad \omega_4(x,y) = -y,$$

$$\omega_{12}(x,y) = y - y_0.$$

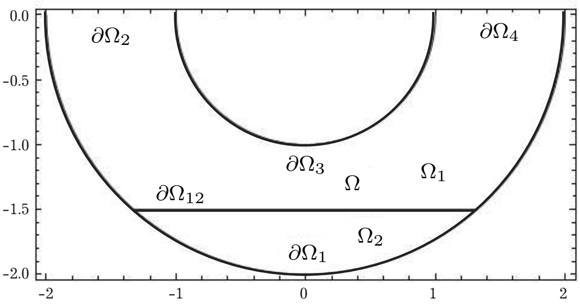


Fig. 2. The computational experiment area Ω.

The computational experiment was carried out for the κ_1, κ_2 coefficients and different number of coordinate functions. The coordinate functions were constructed on the basis of the Legendre polynomials that establish an orthogonal system of function in $[-1, 1]$. The dependences of Q_n value on the number of coordinate functions n for combinations of coefficients of permeability κ_1 and κ_2 are represented in Table 1.

Table 1. Q_n values for combinations of κ_1 and κ_2 coefficients (dependent on the number of the coordinate functions n.)

κ_1	κ_2	6	10	15	21	28
0.391	1, 591	0.249	0.221	0.219	0.205	0.194
$1.593\, e^{2y}$	0.391	0.419	0.442	0.372	0.379	0.339
$1.593\, e^{2y}$	$0.811 \cosh^{-2} y$	0.359	0.335	0.304	0.298	0.274

It is established that as the number of coordinate functions n increases, the value Q_n becomes constant, indicating the convergence of the proposed method. The contour lines of the problem solution are represented in Figs. 3*a*, 4*a*, 5*a*, and the flow velocity vectors are shown in Figs. 3*b*, 4*b* and 5*b*, respectively.

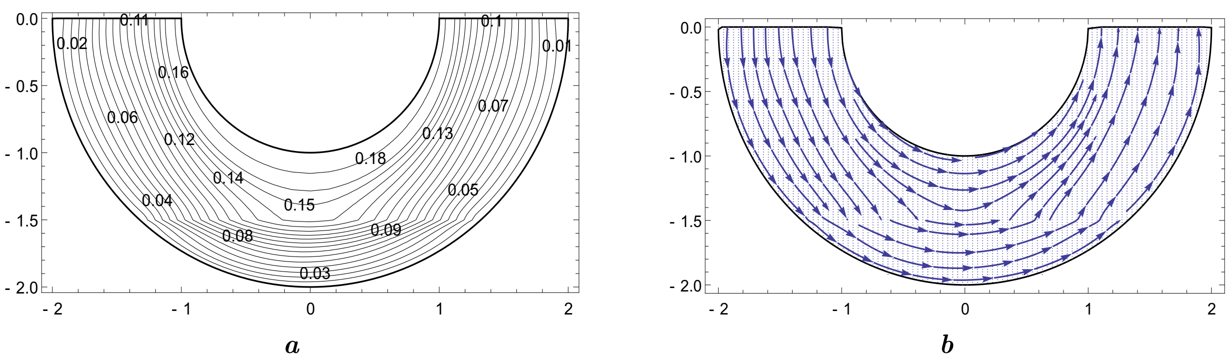
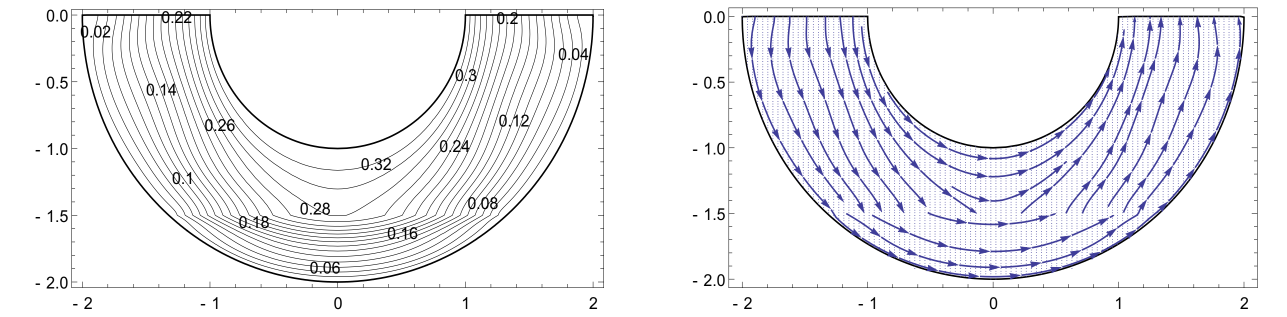
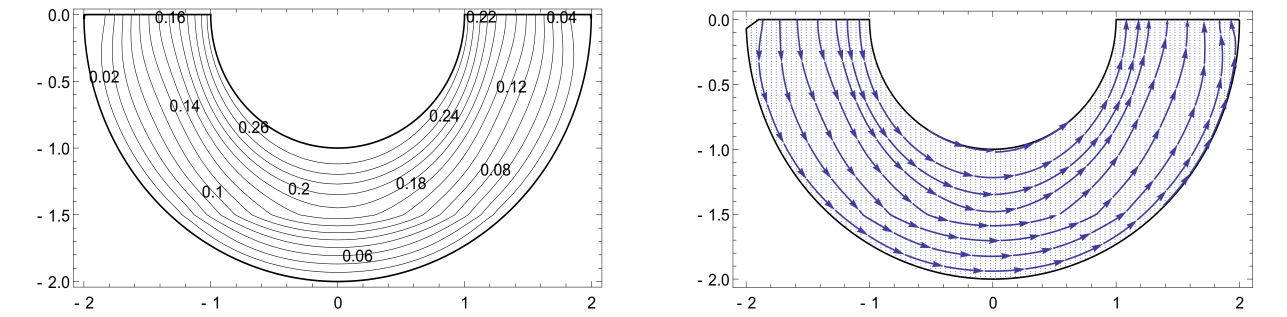


Fig. 3. The contour lines (***a***) and the flow velocity vectors (***b***) recreated by function ψ_{28} for the coefficients of permeability $\kappa_1 = 0.391$, $\kappa_2 = 1.591$.



a ***b***

Fig. 4. The contour lines (***a***) and the flow velocity vectors (***b***) recreated by function ψ_{28} using the coefficients of permeability $\kappa_1 = 1.593\, e^{2y}$, $\kappa_2 = 0.391$.



a ***b***

Fig. 5. The contour lines (***a***) and the flow velocity vectors (***b***) recreated by function ψ_{28} using the coefficients of permeability $\kappa_1 = 1.593\, e^{2y}$, $\kappa_2 = 0.811 \cosh^{-2} y$.

The obtained numerical results are in good agreement with the results of both physical experiments and numerical results obtained by other authors [1, 6, 7].

5. Conclusions

The article represents the problem of mathematical modeling of flows through a piecewise homogeneous porous medium, the computational experiment for the mentioned problem as well as the experiment results. Based on the R-function method, a solution structure that satisfies all the boundary conditions is developed. The application of the Ritz method for approximation of the indeterminate component is substantiated as well.

The computational experiment performed for the described problem demonstrates the efficiency and precision of the proposed modified method. The results can be extended to other models of the fluid-flow theory as well as can be used to solve the application problems related to the calculation and modeling of the fluid flows through porous media.

This fact proves the scientific novelty and practical relevance of the obtained results.

[1] Polubarinova-Kochina P. Ja. Teorija dvizheniya gruntovyh vod. Moskva, Nauka, (1977), (in Russian).

[2] Bomba A. Ja., Bulavackij V. M., Skopeckij V. V. Nelinijni matematichni modeli procesiv geogidrodinamiki. Kyiv, Naukova dumka (2007), (in Ukrainian).

[3] Vabishevich P. N. Metod fiktivnyh oblastej v matematicheskoj fizike. Moskva, Izd-vo MGU (1991), (in Russian).

[4] Vengerskij P. Pro zadachu sumisnogo ruhu poverhnevih i gruntovih potokiv na teritoriyi vodozboru. Visnik Lviv. un-tu. Ser. prikl. matem. ta inf. Vip. **22**, 41–53 (2014), (in Ukrainian).

[5] Connor J. J., Brebbia C. A. Finite Element Techniques for Fluid Flow. London, Newnes-Butterworth (1976).

[6] Lyashko I. I., Velikoivanenko I. M., Lavrik V. I., Misteckij G. E. Metod mazhorantnyh oblastej v teorii filtracii. Kiev, Naukova dumka (1974), (in Russian).

[7] Lyashko N. I., Velikoivanenko N. M. Chislenno-analiticheskoe reshenie kraevyh zadach teorii filtracii. Kiev, Naukova dumka (1973), (in Russian).

[8] Kravchenko V. F., Rvachev V. L. Algebra logiki, atomarnye funkcii i vejvlety v fizicheskih prilozheniyah. Moskva, Fizmatlit (2006), (in Russian).

[9] Rvachev V. L. Teorija R-funkcij i nekotorye ego prilozhenija. Kiev, Naukova dumka (1982), (in Russian).

[10] Blishun A. P., Sidorov M. V. Metod chislennogo analiza stacionarnogo filtracionnogo techeniya pod gidrotehnicheskim sooruzheniem v kusochno-odnorodnomu grunte. Visnik Zaporizkogo nacionalnogo universitetu. Seriya: fiziko-matematichni nauki. **2**, 5–12 (2012), (in Russian).

[11] Blishun A. P., Sidorov M. V., Jalovega I. G. Matematicheskoe modelirovanie i chislennyj analiz filtracionnyh techenij pod gidrotehnicheskimi sooruzheniyami s pomoshyu. Radioelektronika i informatika. **2**, 40–46 (2010), (in Russian).

[12] Blishun A. P., Sidorov M. V., Jalovega I. G. Primenenie metoda R-funkcij k chislennomu analizu filtracionnyh techenij pod gidrotehnicheskimi sooruzheniyami. Visnik Zaporizkogo nacionalnogo universitetu. Seriya: fiziko-matematichni nauki. **1**, 50–56 (2012), (in Russian).

[13] Sidorov M. V., Storozhenko A. V. Matematicheskoe kompyuternoe modelirovanie nekotoryh filtracionnyh techenij. Radioelektronika i informatika. **4**, 58–61 (2004), (in Russian).

[14] Podhornyi O. R. Matematichni modeli filtracijnih techij ta zastosuvannya metodu R-funkcij dlya yih chiselnogo analizu. Radioelektronika ta informatika. **1**, 40–47 (2018), (in Ukrainian).

[15] Podhornyi O. R. Chiselnij analiz metodom R-funkcij filtracijnih techij u neodnoridnomu grunti. Matematichne ta kompyuterne modelyuvannya. Seriya: Fiziko-matematichni nauki. **18**, 147–162 (2018), (in Ukrainian).

[16] Mihlin S. G. Variacionnye metody v matematicheskoj fizike. Moskva, Nauka (1970), (in Russian).

[17] Rektoris K. Variacionnye metody v matematicheskoj fizike i tehnike. Moskva, Mir (1985), (in Russian).

Математичне моделювання фільтраційних течій у кусково-однорідному середовищі методом R-функцій

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Розглядається стаціонарна фільтраційна течія у кусково-однорідному ґрунті у припущенні, що виконується закон Дарсі. Математичною моделлю цієї задачі є еліптичне рівняння для функції течії, доповнене крайовими умовами другого роду на ділянках межі водойми і крайовими умовами першого роду на ділянках межі, що є непроникними для рідини. Також до постановки задачі входять умови спряження на лінії розділу двох ґрунтів. При цьому у постановку задачі входить невідоме значення повних витрат рідини, для визначення якого формулюється додаткове інтегральне співвідношення. Для чисельного аналізу розглядуваної крайової задачі пропонується використати структурно-варіаційний метод (метод R-функцій), що дозволить найбільш повно урахувати у обчислювальному алгоритмі усю геометричну та аналітичну інформацію, яка входить у постановку задачі. Від вихідної задачі здійснено перехід до крайової задачі з відомими крайовими умовами. Відповідно до методу R-функцій для побудованої структури розв'язку, яка точно враховує всі крайові умови отриманої задачі, обґрунтовано використання варіаційного метода Рітца для апроксимації невизначеної компоненти. Після цього з додаткового інтегрального співвідношення знаходиться наближене значення невідомих витрат рідини і наближений розв'язок вихідної задачі. Обчислювальний експеримент було проведено у області, яка має вигляд нижньої половини кільця для різних значень коефіцієнта фільтрації, якщо координатні функції побудовані на основі поліномів Лежандра. Отримано, що зі збільшенням кількості координатних функцій значення повних витрат має тенденцію до збіжності.

Ключові слова: *закон Дарсі, течії рідини у пористому середовищі, кусково-однорідне середовище, метод R-функцій, метод Рітца.*