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Caputo fractional reduced differential transform method for SEIR epidemic model with fractional order

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This paper proposes the Caputo Fractional Reduced Differential Transform Method (CFRDTM) for Susceptible-Exposed-Infected-Recovered (SEIR) epidemic model with fractional order in a host community. CFRDTM is the combination of the Caputo Fractional Derivative (CFD) and the well-known Reduced Differential Transform Method (RDTM). CFRDTM demonstrates feasible progress and efficiency of operation. The properties of the model were analyzed and investigated. The fractional SEIR epidemic model has been solved via CFRDTM successfully. Hence, CFRDTM provides the solutions of the model in the form of a convergent power series with easily computable components without any restrictive assumptions.

Keywords: *Caputo fractional derivative, epidemic model, fractional order, reduced differential transform method.*

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1. Introduction

Fractional calculus has been widely used to describe practical dynamics phenomena arising from computational finance, applied mathematics, physics, economics, medicine biomathematics and engineering; see [1–15], just to mention a few. Fadugba [16] applied homotopy analysis method for the valuation of European call options with time-fractional Black-Scholes equation. The comparison of the reduced differential transform method and Sumudu transform for the solution of fractional Black–Scholes equation for a European call option problem was studied by [17]. RDTM was first proposed by [18] and successfully employed to solve many types of nonlinear PDEs. Similar to the traditional Differential Transformation Method (DTM), RDTM demonstrates feasible progress and efficiency of operation. More on the applications of RDTM, see [19–21], just to mention a few. Unlike other existing approaches, RDTM provides a simple way to ensure the convergence of solution series. Baleanu et al. [22] investigated the existence of solutions for a fractional hybrid integro-differential equation with mixed hybrid integral boundary value condition. The numerical solution of fractional Schistosomiasis disease via q-homotopy analysis transform method was studied by [23]. Gao et al. [24] investigated the infection system of the novel coronavirus (2019-nCoV) with a non local operator in a Caputo sense via a powerful computational technique based on the fractional natural decomposition method. Veeresha et al. [25] investigated and studied the solution of fractional forced KdV equation using fractional natural decomposition method. Arqub et al. [26] investigated the accuracy of the homotopy analysis method for solving the fractional order problem of the spread of a non-fatal disease in a population. In this paper, the solution of epidemic model with fractional order via CFRDTM is proposed. CFRDTM does not require linearization, perturbation or restrictive assumptions and offers solutions with easily computable components as convergent series. Also, it is a powerful tool that overcomes the deficiency that is mainly caused by unsatisfied conditions. The emphasis is given to the Caputo fractional operator which is more suitable for the study of differential equations of fractional order. The rest of the paper is organized as follows: Section Two captures the preliminaries. The Caputo fractional epidemic model is presented in Section 3. In Section Four, the solution of the model via CFRDTM is obtained. In Section Five, concluding remarks are also presented5

2. Preliminaries

This section presents definition of some concepts [27, 28].

Definition 1. *The Riemann–Liouville Fractional Derivative Operator (RLFDO) of $f(t)$ is given by*

$$ {}_0D_t^q f(t) = \frac{1}{\Gamma(n-q)} \int_0^t (t-\tau)^{n-q-1} f(\tau) d\tau,\ t > 0, n-1 < q < n, n \in N. \tag{1}$$

Definition 2. *The Riemann–Liouville Fractional Integral Operator (RLFIO) of $f(t)$ is given by*

$$ J^q f(t) = \frac{1}{\Gamma(q)} \int_0^t (t-\tau)^{q-1} f(\tau) d\tau, \tau > 0, \tag{2}$$

where $q \geqslant 0$ is the order, $f(t) \in C_\rho$ is a function, $\rho \geqslant 1$, C_ρ is a space and $\Gamma(q)$ is the gamma function of q.

Definition 3. *The Caputo Fractional Derivative Operator (CFDO) of $f(t) \in C_{-1}^n$, $n \in N$ is given by*

$$ {}_0^cD_t^q f(t) = \frac{1}{\Gamma(n-q)} \int_0^t (t-\tau)^{n-q-1} f^{(n)}(\tau) d\tau, \tag{3}$$

for $\alpha \in (n-1, n]$, $t > 0$.

Definition 4. *The Caputo Time-Fractional Derivative Operator (CTFDO) of order $\alpha > 0$ is defined as*

$$ {}_o^cD_t^q u = \begin{cases} \dfrac{1}{\Gamma(n-q)} \displaystyle\int_0^t (t-\tau)^{n-q-1} u^{(n)}(x,\tau) d\tau, & q \in (0,1], \\ \dfrac{\partial^n u(x,t)}{\partial t^n}, & q = n, \end{cases} \tag{4}$$

where n is the smallest integer that exceeds α, $u = u(x,t)$ and $u^{(n)}(x,\tau) = \frac{\partial^n u(x,\tau)}{\partial \tau^n}$. The superiority of CFDO over RLFO is that CFD of a constant is zero. Also, in the CFD, initial conditions have clear physical interpretation [29].

Definition 5. *The series representation of the form*

$$ E_q(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(nq+1)} = \sum_{n=0}^{\infty} \frac{z^n}{(nq)!}, \quad z \in C, \tag{5}$$

is called the Mittag–Leffler Function (MLF).

Definition 6. *The Caputo Fractional Reduced Differential Transform (CFRDT) of the function $\psi(x,t)$ is defined as*

$$ \Psi_k(x) = \frac{\left[{}_0^cD_t^{kq} \psi(x,t)\right]_{t=t_0}}{\Gamma(1+kq)}, \quad q \in (0,1], \quad k = 0, 1, \ldots, n, \tag{6}$$

where ${}_0^cD_t^{kq} a(x,t) = \frac{\partial^{kq} \psi(x,t)}{\partial t^{kq}}$.

Definition 7. *The inverse CFRDT of $\Psi_k(x)$ is defined as*

$$\psi(x,t)=\sum_{k=0}^{\infty}\Psi_k(x)(t-t_0)^{kq}, \quad 0<q\leqslant 1. \tag{7}$$

By means of (6) and (7), the fundamental properties of CFRDTM were given in Table 1.

Table 1. The fundamental properties of CFRDTM.

Functional form	Transformed form
$\psi(x,t)=\varphi(x,t)\pm\xi(x,t)$	$\Psi_k(x)=\Phi_k(x)\pm\Xi_k(x)$
$\psi(x,t)=c\varphi(x,t)$	$\Psi_k(x)=c\Phi_k(x)$, c is a constant.
$\psi(x,t)=\varphi(x,t)\xi(x,t)$	$\Psi_k(x)=\sum_{i=0}^{k}\Phi_i(x)\Xi_{k-i}(x)$
$\psi(x,t)=\frac{x^{mq}t^{nq}}{\Gamma(1+mq)\Gamma(1+nq)}$	$\Psi_k(x)=\frac{x^{mq}}{\Gamma(1+mq)}\frac{\delta_q(k-n)}{\Gamma(1+\alpha)}$, $m,n\in N$
$\psi(x,t)=\frac{\partial^{nq}\varphi(x,t)}{\partial t^{nq}}$	$\Psi_k(x)=\frac{\Gamma(1+(k+n)q)}{\Gamma(1+kq)}\Phi_{k+n}(x), n\in N$
$\psi(x,t)=\frac{\partial^{mq}\varphi(x,t)}{\partial x^{mq}}$	$\Psi_k(x)=\frac{\partial^{mq}\Phi_k(x)}{\partial x^{mq}}$, $m\in N$
$\psi(x,t)=x_i^n t^r$	$\Psi_k(x)=x_i^n\delta_{(k-r)}, i=1,...,m$
$\psi(x,t)=e^{\lambda t}$	$\Psi_k(x)=\frac{\lambda^k}{k!}$

3. Caputo fractional Susceptible-Exposed-Infected-Recovered (SEIR) epidemic model

This section captures classical and fractional SEIR epidemic models as follows.

3.1. Classical SEIR epidemic model

Let the total population of a host community be denoted by N. The total population is subdivided into four classes namely: (a) Susceptible population, S; (b) Exposed population, E; (c) Infected population, I; (d) Recovered population, R.

The above four classes describe the model equation of Measles in a host community given by [30]

$$\begin{aligned}\frac{dS}{dt}&=B-\beta SI-\mu S,\\ \frac{dE}{dt}&=\beta SI-(\sigma+\mu+\alpha)E,\\ \frac{dI}{dt}&=\alpha E-(\gamma+\mu)I,\\ \frac{dR}{dt}&=\gamma I+\sigma E-\mu R,\end{aligned} \tag{8}$$

where β is the infected individual rate, B is the birth rate, μ is the natural death rate, σ is the Measles therapy rate, α is the infected class rate and γ is the recovery rate.

3.2. SEIR epidemic model with fractional order in a Caputo sense

It is note worthy to say that the fractional extension of (8) was first studied by [26, 31]. Here, we consider the Caputo fractional epidemic model of the form

$$\begin{aligned}{}_0^cD_t^qS&=B-\beta SI-\mu S,\\ {}_0^cD_t^qE&=\beta SI-(\sigma+\mu+\alpha)E,\\ {}_0^cD_t^qI&=\alpha E-(\gamma+\mu)I,\\ {}_0^cD_t^qR&=\gamma I+\sigma E-\mu R.\end{aligned} \tag{9}$$

Subject to the initial conditions $S_0=a_1$, $E_0=a_2$, $I_0=a_3$, $R_0=a_4$, where q is the fractional order and the initial population $N_0=S_0+E_0+I_0+R_0$.

3.2.1. Existence and uniqueness of the solution of Caputo fractional epidemic model

Here, we present the proof of existence and uniqueness of the solution of (9). Similar approach in [32, 33], where the Banach fixed point theorem and Picard's operators have been employed in this scenario.

Consider the functions H, K, L, M defined as $(H, K, L, M)\colon [0, T] \times \mathbf{R} \to \mathbf{R}$. Let

$$\begin{aligned} H(S) &= B - \beta SI - \mu S, \\ K(E) &= \beta SI - (\sigma + \mu + \alpha)E, \\ L(E) &= \alpha E - (\gamma + \mu)I, \\ M(R) &= \gamma I + \sigma E - \mu R. \end{aligned} \tag{10}$$

Taking the norms of (10), yields

$$\begin{aligned} ||H(S_1) - H(S_2)|| &\leqslant (a\beta + \mu)||S_1 - S_2||, \\ ||K(E_1) - K(E_2)|| &\leqslant (\sigma + \mu + \alpha)||E_1 - E_2||, \\ ||L(I_1) - L(I_2)|| &\leqslant (\gamma + \mu)||I_1 - I_2||, \\ ||M(R_1) - M(R_2)|| &\leqslant \mu||R_1 - R_2||. \end{aligned} \tag{11}$$

Equation (11) shows that H, K, L, M are Lipschitz continuous. Suppose that S, E, I, R are normed spaces and $p > 0$, define the subset of the Banach space of all continuous functions on $t \in [0, T]$ as follows

$$\begin{aligned} B_S &= \{S \in C[0, T]\colon ||S_1 - S_2|| \leqslant p\}, \\ B_E &= \{E \in C[0, T]\colon ||E_1 - E_2|| \leqslant p\}, \\ B_I &= \{I \in C[0, T]\colon ||I_1 - I_2|| \leqslant p\}, \\ B_R &= \{R \in C[0, T]\colon ||R_1 - R_2|| \leqslant p\}, \end{aligned} \tag{12}$$

where C is defined as the set of all continuous functions. Next, define the Picard's operators on B_S, B_E, B_I, B_R by help of the Volterra integral equations of second kind,

$$\begin{aligned} U_1(S) &= S_0 +_0^c V_t^q H(S) \\ U_2(E) &= E_0 +_0^c V_t^q K(E), \\ U_3(I) &= I_0 +_0^c V_t^q L(I), \\ U_4(R) &= R_0 +_0^c V_t^q M(R). \end{aligned} \tag{13}$$

In the sequel, we show that the LHS of (12) is a contraction. Let $S_1, S_2 \in B_S$, $E_1, E_2 \in B_E$, $I_1, I_2 \in B_I$, $R_1, R_2 \in B_R$. By means of (11) and the Chebyshev norm, one obtains

$$\begin{aligned} ||U_1(S_1) - U_1(S_2)|| &\leqslant \frac{(a\beta + \mu)||S_1 - S_2||T^q}{\Gamma(q+1)}, \\ ||U_2(E_1) - U_2(E_2)|| &\leqslant \frac{(\sigma + \mu + \alpha)||E_1 - E_2||T^q}{\Gamma(q+1)}, \\ ||U_3(I_1) - U_3(I_2)|| &\leqslant \frac{(\gamma + \mu)||I_1 - I_2||T^q}{\Gamma(q+1)}, \\ ||U_4(R_1) - U_4(R_2)|| &\leqslant \frac{\mu||R_1 - R_2||T^q}{\Gamma(q+1)}. \end{aligned} \tag{14}$$

Thus, (14) shows that $U_1(S)$, $U_2(E)$, $U_3(I)$, $U_4(R)$ are bounded and continuous. Also from (14), it follows that

$$\frac{(a\beta + \mu)T^q}{\Gamma(q+1)} < 1, \frac{(\sigma + \mu + \alpha)T^q}{\Gamma(q+1)} < 1, \frac{(\gamma + \mu)T^q}{\Gamma(q+1)} < 1, \frac{\mu T^q}{\Gamma(q+1)} < 1. \tag{15}$$

Therefore,

$$\begin{aligned} ||U_1(S_1) - U_1(S_2)|| &< ||S_1 - S_2||, \\ ||U_2(E_1) - U_2(E_2)|| &< ||E_1 - E_2||, \\ ||U_3(I_1) - U_3(I_2)|| &< ||I_1 - I_2||, \\ ||U_4(R_1) - U_4(R_2)|| &< ||R_1 - R_2||. \end{aligned} \tag{16}$$

Using (16) and the Banach contraction principle, operators $U_1(S)$, $U_2(E)$, $U_3(I)$, $U_4(R)$ are contractions. Hence, it is concluded that the Caputo fractional epidemic model (9) exists and has a unique solution in B_S, B_E, B_I, B_R, respectively.

3.3. Properties of the model

The properties of (9) were analyzed and investigated as follows.

3.3.1. Disease free equilibrium (DFE)

DFE is defined as the point where there is total absence of the disease (Measles). It is denoted by Z_0.

Theorem 1. *The DFE of the system* (9) *exists at the point*

$$Z_0 = (S^*, E^*, I^*, R^*) = \left(\frac{B}{\mu}, 0, 0, 0\right). \tag{17}$$

Proof. At the equilibrium points, the fractional derivative of each class becomes

$${}_0^cD_t^q S = {}_0^c D_t^q E = {}_0^c D_t^q I = {}_0^c D_t^q R = 0. \tag{18}$$

Suppose at the equilibrium state, $S = S^*$, $E = E^*$, $I = I^*$, $R = R^*$, then (18) becomes

$${}_0^cD_t^q S^* = {}_0^cD_t^q E^* = {}_0^cD_t^q I^* = {}_0^cD_t^q R^* = 0. \tag{19}$$

Using (9), (19) becomes

$$\begin{aligned} {}_0^cD_t^q S^* &= B - \beta S^* I^* - \mu S^* = 0, \\ {}_0^cD_t^q E^* &= \beta S^* I^* - (\sigma + \mu + \alpha) E^* = 0, \\ {}_0^cD_t^q I^* &= \alpha E^* - (\gamma + \mu) I^* = 0, \\ {}_0^cD_t^q R^* &= \gamma I^* + \sigma E^* - \mu R^* = 0. \end{aligned} \tag{20}$$

This implies that

$$\begin{aligned} & B - \beta S^* I^* - \mu S^* = 0, \\ & \beta S^* I^* - (\sigma + \mu + \alpha) E^* = 0, \\ & \alpha E^* - (\gamma + \mu) I^* = 0, \\ & \gamma I^* + \sigma E^* - \mu R^* = 0. \end{aligned} \tag{21}$$

At DFE state, $E^* = 0$, $I^* = 0$, the other two classes are obtained as $R^* = 0$, $S^* = \frac{B}{\mu}$. Hence

$$Z_0 = (S^*, E^*, I^*, R^*) = \left(\frac{B}{\mu}, 0, 0, 0\right).$$

This completes the proof. ■

3.3.2. Basic reproduction number

Basic reproduction number, R_{rb} is defined as a threshold parameter to determine the number of equilibrium. By means of the next generation matrix, R_{rb} is expressed as

$$R_{rb} = \rho(FV^{-1}), \tag{22}$$

where

$$F=\begin{bmatrix} 0 & \frac{\beta B}{\mu} \\ 0 & 0 \end{bmatrix}, \tag{23}$$

$$V=\begin{bmatrix} \mu+\alpha+\sigma & 0 \\ -\mu & \mu+\gamma \end{bmatrix}. \tag{24}$$

The inverse of V is obtained as

$$V^{-1}=\begin{bmatrix} \frac{1}{(\mu+\alpha+\sigma)} & 0 \\ \frac{\mu}{(\mu+\gamma)(\mu+\alpha+\sigma)} & \frac{1}{(\mu+\gamma)} \end{bmatrix}. \tag{25}$$

So, the next generation matrix is computed as

$$\begin{aligned} FV^{-1} &= \begin{bmatrix} 0 & \frac{\beta B}{\mu} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{(\mu+\alpha+\sigma)} & 0 \\ \frac{\mu}{(\mu+\gamma)(\mu+\alpha+\sigma)} & \frac{1}{(\mu+\gamma)} \end{bmatrix} \\ &= \begin{bmatrix} \frac{B\beta\alpha}{\mu(\mu+\gamma)(\mu+\alpha+\sigma)} & \frac{B\beta}{\mu(\mu+\gamma)} \\ 0 & 0 \end{bmatrix}. \end{aligned} \tag{26}$$

Therefore, the characteristic equation is obtained as

$$\begin{vmatrix} \frac{B\beta\alpha}{\mu(\mu+\gamma)(\mu+\alpha+\sigma)}-\lambda & \frac{B\beta}{\mu(\mu+\gamma)} \\ 0 & -\lambda \end{vmatrix}=0. \tag{27}$$

Thus, the characteristic values are obtained as

$$\lambda_1=0, \quad \lambda_2=\frac{B\beta\alpha}{\mu(\mu+\gamma)(\mu+\alpha+\sigma)}. \tag{28}$$

The spectral radius of FV^{-1} is calculated as the largest characteristic value;

$$\rho(FV^{-1})=\lambda_2=\frac{B\beta\alpha}{\mu(\mu+\gamma)(\mu+\alpha+\sigma)}. \tag{29}$$

Hence, the reproduction number is

$$R_{rb}=\frac{B\beta\alpha}{\mu(\mu+\gamma)(\mu+\alpha+\sigma)}. \tag{30}$$

3.3.3. Stability analysis of DFE

To examine the local stability of DFE, we evaluate the Jacobian matrix of (17) as follows.

$$J(Z_0)=\begin{bmatrix} -\mu & 0 & -\frac{\beta B}{\mu} & 0 \\ 0 & -(\mu+\alpha+\sigma) & \frac{\beta B}{\mu} & 0 \\ 0 & \mu & -(\mu+\gamma) & 0 \\ 0 & \sigma & \gamma & -\mu \end{bmatrix}. \tag{31}$$

The characteristic equation of (31) is given by

$$\begin{vmatrix} -\mu-\lambda & 0 & -\frac{\beta B}{\mu} & 0 \\ 0 & -(\mu+\alpha+\sigma)-\lambda & \frac{\beta B}{\mu} & 0 \\ 0 & \mu & -(\mu+\gamma)-\lambda & 0 \\ 0 & \sigma & \gamma & -\mu-\lambda \end{vmatrix}=0. \tag{32}$$

The characteristic values of $J(Z_0)$ are obtained as

$$\lambda_1 = -\mu, \quad \lambda_2 = -\mu, \quad \lambda_3 = -(\mu+\alpha+\sigma), \quad \lambda_4 = -\left(\frac{\beta B}{\mu} + \mu + \gamma\right). \tag{33}$$

Therefore,

$$\mathrm{Re}(\lambda_1),\ \mathrm{Re}(\lambda_2),\ \mathrm{Re}(\lambda_3),\ \mathrm{Re}(\lambda_4) < 0.$$

Since all the model parameters are positive. Hence, Z_0 is locally asymptotically stable.

Remark 1. The DFE, Z_0 is locally asymptotically stable if $R_{rb} < 1$.

Theorem 2. *The closed region* $\Omega = \left\{(S,E,I,R) \in \mathbf{R}^4_+ \colon N \to \frac{B}{\mu}\right\}$ *is positively invariant and attracting.*

Proof. The total population is defined as

$$N = S + E + I + R. \tag{34}$$

Thus,

$$\begin{aligned} {}^c_0D^q_t N &= {}^c_0D^q_t S + {}^c_0D^q_t E + {}^c_0D^q_t I + {}^c_0D^q_t R \\ &= B - \mu N. \end{aligned} \tag{35}$$

Applying the Laplace transform to both sides of (35), we obtain

$$\hat{N} = \frac{B}{s(s^q+\mu)} + N_0 \frac{(s^q-1)}{(s^q+\mu)}. \tag{36}$$

Solving (36) further, one obtains

$$N = \frac{B}{\mu}\left(\frac{1}{s}\right) - \left(\frac{B}{\mu} - N_0\right) \sum_{k=0}^{\infty} \frac{(-\mu)^k}{s^{qk+1}}. \tag{37}$$

By means of the Laplace inversion formula, (37) becomes

$$N = \frac{B}{\mu} + \left(N_0 - \frac{B}{\mu}\right) \sum_{k=0}^{\infty} \frac{(-\mu t^q)^k}{\Gamma(qk+1)}. \tag{38}$$

It follows that as $t \to \infty$

$$N = N(t) \to \frac{B}{\mu}. \tag{39}$$

Hence, the region Ω is positively invariant and attracts all solutions in $\mathbf{R}^4_+$. ∎

Theorem 3. *Every solution of system* (9) *with initial conditions remain positive for all* $t \geqslant 0$.

Proof. Using the approach of [34], assume by contradiction that the second equation in (9) is not true. Let $t^* = \min\{t \colon S(t), E(t), I(t), R(t) = 0\}$. Now if $E(t^*) = 0$, it then implies that $S(t), I(t), R(t) \geqslant 0$ for all $0 \leqslant t^* \leqslant t$. Assume that the following expressions exist;

$$\begin{aligned} A_1 &= \min_{0 \leqslant t^* \leqslant t} \left[\frac{1}{S}(B - \beta S I - \mu S)\right], \\ A_2 &= \min_{0 \leqslant t^* \leqslant t} \left[\frac{1}{E}(\beta S I - (\sigma+\mu+\alpha)E)\right], \\ A_3 &= \min_{0 \leqslant t^* \leqslant t} \left[\frac{1}{I}(\alpha E - (\gamma+\mu)I)\right], \\ A_4 &= \min_{0 \leqslant t^* \leqslant t} \left[\frac{1}{R}(\gamma I + \sigma E - \mu R)\right]. \end{aligned} \tag{40}$$

From (9), it follows that

$$\begin{aligned} {}_0^cD_t^q S - A_1 S \geqslant 0, \\ {}_0^cD_t^q E - A_2 E \geqslant 0, \\ {}_0^cD_t^q I - A_3 I \geqslant 0, \\ {}_0^cD_t^q R - A_4 R \geqslant 0. \end{aligned} \tag{41}$$

By means of the Laplace transform and its inversion formula, (41) yields, respectively;

$$\begin{aligned} S = S(t) \geqslant S_0 \sum_{k=0}^{\infty} \frac{(A_1 t^q)^k}{\Gamma(qk+1)}, \\ E = E(t) \geqslant E_0 \sum_{k=0}^{\infty} \frac{(A_2 t^q)^k}{\Gamma(qk+1)}, \\ I = I(t) \geqslant I_0 \sum_{k=0}^{\infty} \frac{(A_3 t^q)^k}{\Gamma(qk+1)}, \\ R = R(t) \geqslant R_0 \sum_{k=0}^{\infty} \frac{(A_4 t^q)^k}{\Gamma(qk+1)}. \end{aligned} \tag{42}$$

From (42), we conclude that every solution of (9) is positive for all $t \geqslant 0$. ■

Theorem 4. *The fractional epidemic model* (9) *in a Caputo sense admits a unique endemic equilibrium state* $Y^* = (S^*, E^*, I^*, R^*) \neq (0,0,0,0)$ *if the basic reproduction number* $R_{rb} > 1$.

Proof. Suppose ${}_0^cD_t^q S = {}_0^cD_t^q E = {}_0^cD_t^q I = {}_0^cD_t^q R = 0$ such that $Y^* = (S^*, E^*, I^*, R^*)$ is the non-trivial solution of the model. Using the last two equations in (9), one gets

$$I^* = \frac{\alpha E^*}{\gamma + \mu}, \tag{43}$$

$$R^* = \frac{\gamma I^* + \sigma E^*}{\mu} = \frac{E^*(\alpha\gamma I^* + \sigma(\gamma+\mu))}{\mu(\gamma+\mu)}. \tag{44}$$

Adding the first two equations in (9) and equating to zero, yields

$${}_0^cD_t^q S + {}_0^cD_t^q E = B - \beta S^* I^* - \mu S^* + \beta S^* I^* - (\sigma+\mu+\alpha)E^* = 0. \tag{45}$$

It is clearly seen from (45) that

$$S^* = \frac{B}{\mu} - \frac{(\sigma+\mu+\alpha)E^*}{\mu}. \tag{46}$$

For the existence of E^*, we assume that h is a continuous function defined as $h\colon \mathbf{R}^+ \to \mathbf{R}$, then

$$h(E) = \frac{\beta(B - (\sigma+\mu+\alpha)E)\alpha}{\mu(\mu+\sigma)} - (\sigma+\mu+\alpha). \tag{47}$$

At the point $E = \frac{B}{(\sigma+\mu+\alpha)}$, one obtains

$$h\left(\frac{B}{(\sigma+\mu+\alpha)}\right) = -(\sigma+\mu+\alpha) < 0. \tag{48}$$

Also at $E = 0$, one gets

$$h(0) = \frac{\beta B\alpha - \mu(\sigma+\mu+\alpha)(\gamma+\mu)}{\mu(\mu+\gamma)}. \tag{49}$$

From (30), we write that

$$B\beta\alpha = R_{rb}[\mu(\sigma+\mu+\alpha)(\gamma+\mu)].$$

Therefore, (49) becomes

$$h(0) = \frac{(R_{rb}-1)\mu(\sigma+\mu+\alpha)(\gamma+\mu)}{\mu(\mu+\gamma)} = (R_{rb}-1)(\sigma+\mu+\alpha). \tag{50}$$

Equation (50) can also be written as

$$h(0) = \frac{\left(1-\frac{1}{R_{rb}}\right)B\beta\alpha}{\mu(\mu+\gamma)}. \tag{51}$$

Equations (49) and (50) are always positive if $R_{rb} > 1$. Thus from the intermediate value theorem, there exists a unique number E^* such that $0 \leqslant E^* \leqslant \frac{B}{(\sigma+\mu+\alpha)}$ and $h(E^*) = 0$. It is observed from the above conditions that (43), (44) and (46) exist and unique. Hence, $Y^* = (S^*, E^*, I^*, R^*)$ is the unique endemic equilibrium state of system (9). ■

Theorem 5. *The endemic equilibrium Y^* is globally asymptotically stable if $R_{rb} > 1$.*

Proof. Now, to investigate the global stability of Y^*. Consider the following Lyapunov function

$$L = \left[S - S^* - S^*\ln\left(\frac{S}{S^*}\right)\right] + \left[E - E^* - E^*\ln\left(\frac{E}{E^*}\right)\right] + \frac{(\mu+\alpha+\sigma)}{\mu}\left[I - I^* - I^*\ln\left(\frac{I}{I^*}\right)\right]. \tag{52}$$

The last equation in (9) has been excluded, since it has no effect on the other three equations. Taking the CFD of (52), one obtains

$${}_0^cD_t^qL = \left[1-\left(\frac{S}{S^*}\right)\right]{}_0^cD_t^qS + \left[1-\left(\frac{E}{E^*}\right)\right]{}_0^cD_t^qE + \frac{(\mu+\alpha+\sigma)}{\mu}\left[1-\left(\frac{I}{I^*}\right)\right]{}_0^cD_t^qI. \tag{53}$$

Substituting the values of ${}_0^cD_t^qS$, ${}_0^cD_t^qE$, ${}_0^cD_t^qI$ into (53) and using the relation $B = \beta S^*I^* + \mu S^*$, one gets

$$\begin{aligned}{}_0^cD_t^qL = &\left[1-\left(\frac{S}{S^*}\right)\right](\beta S^*I^* + \mu S^* - \beta SI - \mu S) + \left[1-\left(\frac{E}{E^*}\right)\right](\beta SI - (\sigma+\mu+\alpha)E)\\ &+ \frac{(\mu+\alpha+\sigma)}{\mu}\left[1-\left(\frac{I}{I^*}\right)\right](\alpha E - (\gamma+\mu)I).\end{aligned} \tag{54}$$

Notice that $\mu E^* - (\mu+\sigma)I^* = 0$, $\beta S^*I - \frac{(\mu+\alpha+\sigma)}{\mu}(\mu+\gamma) = 0$, $\beta S^*I - (\mu+\alpha+\sigma)E^*\frac{I}{I^*} = 0$, $(\beta S^*I^* - (\mu+\alpha+\sigma)E^*)\frac{I}{I^*}$. So, (54) yields

$${}_0^cD_t^qL = \frac{-\mu(S-S^*)^2}{S} + (\mu+\alpha+\sigma)E^*\left(3 - \frac{EI^*}{E^*I} - \frac{S^*}{S} - \frac{SE^*I}{S^*EI^*}\right) \leqslant 0. \tag{55}$$

Since the arithmetic mean is greater or equal to the geometric mean of the three quantities $\frac{S^*}{S}$, $\frac{EI^*}{E^*I}$, $\frac{SE^*I}{S^*EI^*}$, then

$$\left(\frac{EI^*}{E^*I} + \frac{S^*}{S} + -\frac{SE^*I}{S^*EI^*} - 3\right) \leqslant 0. \tag{56}$$

This implies that ${}_0^cD_t^qL = 0$ holds when $S = S^*$, $E = E^*$ and $I = I^*$. Thus the maximal compact variant set in $\{(S,E,I) \in \Omega \colon {}_0^cD_t^qL = 0\}$ is the singleton set $\{Y^*\}$. Hence, by means of the LaSalle's invaraince principle [35], the result follows. ■

4. CFRDTM for the solution of fractional order SEIR model

In this section, we employ CFRDTM to obtain the series solution for the fractional order SEIR model in (9). Applying CFRDTM to (9), yields, respectively

$$\begin{aligned} S_{k+1} &= \frac{\Gamma(qk+1)}{\Gamma(q(k+1)+1)}\left(B\delta(k) - \beta\sum_{j=0}^{k} S_j I_{k-j} - \mu S_k\right), \\ E_{k+1} &= \frac{\Gamma(qk+1)}{\Gamma(q(k+1)+1)}\left(\beta\sum_{j=0}^{k} S_j I_{k-j} - (\sigma+\mu+\alpha)E_k\right), \\ I_{k+1} &= \frac{\Gamma(qk+1)}{\Gamma(q(k+1)+1)}\left(\alpha E_k - (\gamma+\mu)I_k\right), \\ R_{k+1} &= \frac{\Gamma(qk+1)}{\Gamma(q(k+1)+1)}\left(\gamma I_k + \sigma E_k - \mu R_k\right), \end{aligned} \tag{57}$$

with the following conditions

$$S(0) = S_0, E(0) = E_0, I(0) = I_0, R(0) = R_0. \tag{58}$$

5. Concluding remarks

In this paper, CFRDTM is employed for the solution of epidemic model with fractional order in a host community. The properties of the model were discussed, analyzed and investigated. CFRDTM provides the solution of the model in the form of a convergent series without any restrictive assumptions. Moreover, it is noteworthy to conclude that CFRDTM is found to be effective and suitable in obtaining solutions for SEIR epidemic model in a Caputo sense. Hence, CFRDTM provides an approximate-analytical solution in terms of an infinite power series. In the future research, CFRDTM can be extended for the solution of fractional epidemic model of a system of five equations (SEIRP).

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Метод дробового скороченого диференціального перетворення Капуто для моделі епідемії SEIR з дробовим порядком

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У статті запропоновано метод дробового скороченого диференціального перетворення Капуто для моделі епідемії "уразливі–схильні–інфіковані–видужалі" з дробовим порядком у спільноті–хазяїні. Цей метод — це поєднання дробової похідної Капуто та відомого методу скороченого диференціального перетворення. Він демонструє можливий прогрес та ефективність роботи. Властивості моделі були проаналізовані та досліджені. Дробова модель епідемії успішно розв'язана за допомогою цього методу. Отже, цей метод подає розв'язок моделі у вигляді збіжного степеневого ряду з легко обчислюваними компонентами без будь-яких обмежуючих припущень.

Ключові слова: *дробова похідна Капуто, епідемічна модель, дробовий порядок, метод скороченого диференціального перетворення.*