A new geometrical method for portfolio optimization

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Risk aversion plays a significant and central role in investors' decisions in the process of developing a portfolio. In this portfolio optimization framework, we determine the portfolio that possesses the minimal risk by using a new geometrical method. For this purpose, we elaborate an algorithm that enables us to compute any Euclidean distance to a standard simplex. With this new approach, we can treat the case of portfolio optimization without short-selling in its entirety, and we also recover in geometrical terms the well-known results on portfolio optimization with allowed short-selling. Then, we apply our results to determine which convex combination of the CAC 40 stocks possesses the lowest risk. Thus, we not only obtain a very low risk compared to the index, but we also get a rate of return that is almost three times better than the one of the index.

Keywords: *portfolio optimization, short-selling, Euclidean distance to a standard simplex, geometrical approach of portfolio optimization, geometrical algorithm.*

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1. Introduction and aims of the article

1.1. Framework

The paper [1] published by Harry Markowitz in 1952 completely changed the methods of portfolio management and gave birth to the so-called "Modern Portfolio Theory", thanks to which its author earned the Nobel Prize in Economics in 1990. Since his works and the paper [2] of Sharpe, this theme centralizes a lot of interest and many developments have been written in this domain. Let us recall some recent and important works to which our article is linked.

In [3], Jón Daníelsson, Bjórn N. Jorgensen, Casper G. de Vries and Xiaoguang Yang study the portfolio allocation under the probabilistic VaR constraint and obtain remarkable topological results: the set of feasible portfolios is not always connected nor convex, and the number of local optima increases in an exponential way with the number of states. They propose a solution to reduce computational complexity due to this exponential increase.

In [4], Claudio Fontana and Martin Schweizer give a simple approach to mean-variance portfolio problems: they change the problems' parametrization from trading strategies to final positions. In this way, they are able to solve many quadratic optimization problems by using orthogonality techniques in Hilbert spaces and providing explicit formulas.

In their important article [5], Hanene Ben Salah, Mohamed Chaouch, Ali Gannoun and Christian De Peretti (see also the thesis [6]) define a new portfolio optimization model in which the risks are measured thanks to conditional variance or semivariance. They use returns prediction obtained by nonparametric univariate methods to make a dynamical portfolio selection and get a better performance.

In [7], Sarah Perrin and Thierry Roncalli show how four algorithms of optimization (the coordinate descent, the alternating direction method of multipliers, the proximal gradient method and the Dykstra's algorithm) can be used to solve problems of portfolio allocation.

In [8], Taras Bodnar, Dmytro Ivasiuk, Nestor Parolya and Wolfgang Schmid make an interesting work about the portfolio choice problem for power and logarithmic utilities: they compute the portfolio weights for these utility functions assuming that the portfolio returns follow an approximate log-normal distribution, as suggested in [9]. It is also noticeable that their optimal portfolios belong to the set of mean-variance feasible portfolios.

1.2. Aims and organization of the paper

There are three aims in this article:

— give a new geometrical algorithm (Algorithm 1) to compute any Euclidean distance to a simplex,

— determine, by making use of this algorithm, a portfolio with minimal variance,

— apply this technique to the CAC 40 stocks, and get a portfolio with a rate of return that is almost three times better than the one of the index.

After having briefly explained the notations in Section 1, we expose the portfolio optimization problem and prove by compactness and convexity arguments that it possesses a unique solution.

Then, in Section 2, we solve the problem in the case where short-selling is allowed: for this, we recall the classical method and give our very simple geometrical method.

Section 3 is the heart of the article: in this section, we solve the portfolio optimization problem in the case where short-selling is not allowed. For this purpose, we give a new geometrical algorithm (Algorithm 1) to compute the distance from a point to a standard simplex, which can be used for every Euclidean distance.

We can eventually apply this algorithm to the example of the CAC 40 stocks and determine the portfolio with the lowest risk. This portfolio also has the property of being almost three times more efficient than the underlying index. This is done in Section 4.

1.3. Notations

We consider n stocks $S_1,\dots,S_n$ and denote by $X_1,\dots,X_n$ the random variables that represent their rate of return (for example, daily, monthly or yearly). For every $i \in [\![1,n]\!]$, we set $m_i = E(X_i)$ (mean of X_i), $\mathbf{m} = (m_1,\dots,m_n)$, and $V_i = V(X_i)$ (variance of X_i). We set $C = (\mathrm{Cov}(X_i,X_j))_{(i,j)\in[\![1,n]\!]^2}$ the covariance matrix of $X_1,\dots,X_n$, and define the random vector $X = (X_1,\dots,X_n)$.

Definition 1. *We call portfolio (with allowed short-selling) every linear combination $P_{\mathbf{x}} = \sum_{j=1}^n x_j S_j$, where $\mathbf{x} = (x_1,\dots,x_n) \in \mathbb{R}^n$ and $x_1 + \dots + x_n = 1$.*

If we do not allow short-selling, then every x_i must be nonnegative, and in that case the linear combination is a *convex combination.*

The *rate of return* of the portfolio is the linear combination $R_{\mathbf{x}} = x_1 X_1 + \dots + x_n X_n$.

1.4. Existence and uniqueness of the solution

Let us recall that the mean of $R_{\mathbf{x}}$ is $E(R_{\mathbf{x}}) = {}^t\mathbf{x}\,\mathbf{m}$ and its variance $V(R_{\mathbf{x}}) = {}^t\mathbf{x}\,C\,\mathbf{x}$. Moreover, C is a symmetric positive matrix. This matrix is symmetric definite positive if and only if $X_1,\dots,X_n$ are almost surely affinely independent. In all the following, we assume that C is symmetric *definite* positive, which is true in practice.

Let us denote by H the affine hyperplane of $\mathbb{R}^n$ with equation $x_1 + \dots x_n = 1$, and by K the standard $(n-1)$-simplex, i.e. $K := \{\mathbf{x} \in [0,1]^n \ / \ x_1 + \dots + x_n = 1\}$. This is a Haussdorff compact subset of $\mathbb{R}^n$ that is contained in the hyperplane H.

Minimizing the variance of the portfolio is equivalent to finding the minimum on K of the quadratic form $f \colon \mathbf{x} \mapsto V(R_{\mathbf{x}}) = {}^t\mathbf{x}\,C\,\mathbf{x}$. Let us consider the scalar product $(\mathbf{x},\mathbf{y}) \mapsto \langle \mathbf{x},\mathbf{y}\rangle := {}^t\mathbf{x}\,C\,\mathbf{y}$ and the Euclidean norm $\mathbf{x} \mapsto \|\mathbf{x}\| := \sqrt{{}^t\mathbf{x}\,C\,\mathbf{x}}$. The aim is to determine the point of K that realizes the minimal distance from the origin point to K in the sense of $\|\cdot\|$. As K is a Haussdorff compact subset, and as f is continuous, we know that this minimum does exist. Moreover, the map f is strictly convex, so that it possesses at most one minimum, and every local minimum of f is global (see for example [10]). Therefore, f possesses exactly one minimum on K, and this minimum is global.

2. Minimization of f on the hyperplane H: allowed short-selling

In this section, we give two methods to compute the portfolio that possesses the lowest risk: the classical one, and our geometrical approach. Let us denote by $\mathcal{E} = (\mathbf{e}_1, \ldots, \mathbf{e}_n)$ the canonical basis of $\mathbb{R}^n$. Then, a basis of the vector hyperplane that directs H is $\mathcal{B} = (\mathbf{e}_1 - \mathbf{e}_2, \mathbf{e}_1 - \mathbf{e}_3, \ldots, \mathbf{e}_1 - \mathbf{e}_n)$, and for every $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{x}$ belongs to H if and only if $x_1 + \cdots + x_n = 1$. Let us set $\mathbf{u} = \mathbf{e}_1 + \cdots + \mathbf{e}_n$ and $h \colon \mathbf{x} \mapsto {}^t\mathbf{x}\,\mathbf{u} - 1$.

2.1. Minimization of f on H by the classical method

Here we briefly recall the classical method to compute the portfolio that possesses the lowest risk. Several sources, such as [11, 12] and [13], provide a clear presentation of these well-known tools. The classical method using Lagrange's multipliers theorem applied to h provides the following proposition.

Proposition 3. $\mathbf{x}_0 = \frac{C^{-1}\mathbf{u}}{{}^t\mathbf{u}C^{-1}\mathbf{u}}$ is the unique solution that minimizes f on H.

For example, for $n = 2$ the unique solution is given by $\mathbf{x}_0 = \frac{V(X_2) - \mathrm{Cov}(X_1, X_2)}{v}\mathbf{e}_1 + \frac{V(X_1) - \mathrm{Cov}(X_1, X_2)}{v}\mathbf{e}_2$, where $v = V(X_2) - 2\,\mathrm{Cov}(X_1, X_2) + V(X_1)$.

2.2. Minimization of f on H by the geometrical approach

We can recover the classical results on the portfolio with minimal variance and with allowed short-selling by making use of an Euclidean interpretation. This portfolio is $P_{\mathbf{x}_0}$, where $\mathbf{x}_0$ is the orthogonal projection onto H of the origin point. In order to compute $\mathbf{x}_0$, let us define the (n, n)-matrix

$$A = \begin{bmatrix} c_{1,1} - c_{1,2} & c_{1,2} - c_{2,2} & \cdots & c_{1,n} - c_{n,2} \\ c_{1,1} - c_{1,3} & c_{1,2} - c_{2,3} & \cdots & c_{1,n} - c_{n,3} \\ \vdots & \vdots & & \vdots \\ c_{1,1} - c_{1,n} & c_{1,2} - c_{2,n} & \cdots & c_{1,n} - c_{n,n} \\ 1 & 1 & \cdots & 1 \end{bmatrix}.$$

Theorem 1. *The unique solution $\mathbf{x}_0$ that minimizes f on H is the vector whose coordinates in $\mathcal{E}$ are given by the last column of the inverse of matrix A.*

Proof. For every $\mathbf{x}$ in H, $\mathbf{x}$ is the orthogonal projection onto H of the origin point if and only if $\mathbf{x}$ is orthogonal to H, that is to say, for every $i \in [\![2, n]\!]$, $\langle \mathbf{x}, \mathbf{e}_1 - \mathbf{e}_i \rangle = 0$.

Since $\langle \mathbf{x}, \mathbf{e}_1 - \mathbf{e}_i \rangle = \sum_{j=1}^n x_j \langle \mathbf{e}_j, \mathbf{e}_1 - \mathbf{e}_i \rangle = \sum_{j=1}^n x_j (c_{1,j} - c_{i,j})$, we deduce that $\mathbf{x}$ is the solution if and only if $A\mathbf{x} = \left[\begin{smallmatrix} \mathbf{0}_{n-1,1} \\ 1 \end{smallmatrix}\right]$, i.e. $\mathbf{x} = A^{-1}\left[\begin{smallmatrix} \mathbf{0}_{n-1,1} \\ 1 \end{smallmatrix}\right]$, which means that the coordinates of $\mathbf{x}$ in $\mathcal{E}$ are given by the last column of A^{-1}. ∎

3. Minimization of f on the simplex K: not allowed short-selling

In this section, we solve the problem of portfolio optimization without short-selling, by giving an explicit and calculable solution that doesn't seem to appear in the literature.

3.1. Projections onto affine hyperplanes of $\mathbb{R}^m$

Let us now consider J a subset of $[\![1, n]\!]$ and the vector subspace $E = \oplus_{j \in J} \mathbb{R} e_j$ of $\mathbb{R}^n$, identified with $\mathbb{R}^m$, where $m = |J|$. Let H' be the affine hyperplane of E defined by the equation $\sum_{j \in J} x_j = 0$. Let us fix $i_0 \in J$ and define J_1 by $J_1 = J \setminus \{i_0\}$, and let us denote by $\overline{J}$ the complementary of J in $[\![1, n]\!]$. Then, a basis of the vector hyperplane of E parallel to H' is $\mathcal{B}' = (\mathbf{e}_{i_0} - \mathbf{e}_i \;/\; i \in J_1)$.

Let $\mathbf{a} \in \mathbb{R}^n$, and let us set $B' = (c_{i,j} - c_{i_0,j})_{(i,j)\in J_1\times J}$ and $B = \begin{bmatrix} B' \\ \mathbf{1}_{1,m} \end{bmatrix}$, then

$$\mathbf{b}' = \left(\sum_{j=1}^{n} a_j (c_{i,j} - c_{i_0,j}) \right)_{i\in J_1} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} \mathbf{b}' \\ 1 \end{bmatrix}.$$

Theorem 2. *The orthogonal projection of* $\mathbf{a}$ *onto* H' *is the vector* $\mathbf{x}$ *whose nonzero coordinates in* $\mathcal{E}$ *are given by* $B^{-1}\mathbf{b}$, *i.e.* $(x_i)_{i\in J} = B^{-1}\mathbf{b}$ *and* $(x_i)_{i\in\overline{J}} = \mathbf{0}_{n-m,1}$.

Proof. For every $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{x}$ is the orthogonal projection of $\mathbf{a}$ onto H' if and only if the three following conditions hold

(i) for every $j \in \overline{J}$, $x_j = 0$,
(ii) $x_1 + \ldots + x_n = 1$,
(iii) for every $i \in J_1$, $\mathbf{x} - \mathbf{a}$ is orthogonal to $\mathbf{e}_{i_0} - \mathbf{e}_i$.

Since $\langle \mathbf{x} - \mathbf{a}, \mathbf{e}_{i_0} - \mathbf{e}_j \rangle = \sum_{j=1}^{n} (x_j - a_j)\langle \mathbf{e}_j, \mathbf{e}_{i_0} - \mathbf{e}_i \rangle$, we have $\langle \mathbf{x} - \mathbf{a}, \mathbf{e}_{i_0} - \mathbf{e}_j \rangle = 0$ if and only if $\sum_{j\in J} x_j \langle \mathbf{e}_i - \mathbf{e}_{i_0}, \mathbf{e}_j \rangle = \sum_{j=1}^{n} a_j \langle \mathbf{e}_i - \mathbf{e}_{i_0}, \mathbf{e}_j \rangle$, i.e. $\sum_{j\in J} x_j (c_{i,j} - c_{i_0,j}) = \sum_{j=1}^{n} a_j (c_{i,j} - c_{i_0,j})$. ∎

3.2. The algorithm for computing any Euclidean distance to K from a point of $\mathbb{R}^n$

We now propose a recursive algorithm[1] to compute the point $\mathbf{x}_0$ realizing the distance to K from a point $\mathbf{a} \in \mathbb{R}^n$. In his article [14], L. Condat gave a fast algorithm to project a vector onto a simplex. However, his algorithm was made only for the usual Euclidean distance. Our algorithm can be used for every Euclidean distance, and this is necessary for our purpose. The reader can also have a look at the paper [15] about the projection onto a simplex.

Algorithm 1 Compute $\mathbf{x}$ the orthogonal projection of $\mathbf{a}$ onto K.

Require: ($\mathbf{a}$, K)
if $\mathbf{x}$ belongs to K **then**
  return $\mathbf{x}$
else
  if K is a 1-simplex (i.e. K possesses exactly 2 vertices) **then**
    return the vertex that is the closest to $\mathbf{x}$
  else
    Determine the hyperface K' of K that is the closest to $\mathbf{x}$
    Compute $\mathbf{y}$ the orthogonal projection of $\mathbf{x}$ onto H' (the affine subspace defined by K')
    Apply recursively the algorithm to ($\mathbf{y}$, K')

Proposition 4. Algorithm 1 ends.

Proof. This is straightforward since at each step of the algorithm the dimension of the simplex decreases of one unit. ∎

Lemma 1. *If* $\mathbf{x}$ *belongs to* $H \setminus K$, *then the distance from* $\mathbf{x}$ *to* K *is realized in a point of the frontier of* K.

Proof. Let us proceed by contradiction by assuming that the distance from $\mathbf{x}$ to K is realized in a point $\mathbf{z}$ in the interior of K. Let us denote by $\mathbf{y}$ the intersection of the line $(\mathbf{x}, \mathbf{z})$ with an hyperface of K crossed by this line. Then, by Minkowski, we get $\|\mathbf{x} - \mathbf{z}\| = \|\mathbf{x} - \mathbf{y}\| + \|\mathbf{y} - \mathbf{z}\| > \|\mathbf{x} - \mathbf{y}\|$, which is absurd since $\mathbf{z}$ realizes the minimal distance from $\mathbf{x}$ to K. ∎

As a consequence, if $\mathbf{x}$ belongs to $H \setminus K$, then the distance from $\mathbf{x}$ to K is the distance from $\mathbf{x}$ to the hyperface of K that is the closest to $\mathbf{x}$.

[1] A Python version of this algorithm is proposed in the appendix at the end of the paper.

Theorem 3. *Algorithm 1 is correct.*

Proof. Let us prove by induction on the dimension of K that the algorithm provides us $\mathbf{x}_0 \in K$ such that $d(\mathbf{a}, K) = \|\overrightarrow{\mathbf{a}\,\mathbf{x}_0}\|$.

• If K has dimension 1, the result is clear.

• Now assume that the algorithm is correct for every $(n-1)$-simplex. Let us consider K a n-simplex (with $n \geqslant 2$), and prove that the algorithm is correct for K. Let $\mathbf{x}$ be the orthogonal projection of $\mathbf{a}$ onto H.

— If $\mathbf{x}$ belongs to K, then $\mathbf{x}$ is the solution, and the algorithm is correct.

— If $\mathbf{x}$ does not belong to K, as $n \geqslant 2$, we consider the simplex K' defined above, the affine subspace H' and $\mathbf{y}$ the orthogonal projection of $\mathbf{x}$ onto H'. By induction hypothesis applied to $\mathbf{y}$ and the $(n-1)$-simplex K', the algorithm provides us $\mathbf{x}_0 \in K'$ such that $d(\mathbf{y}, K') = \|\overrightarrow{\mathbf{y}\,\mathbf{x}_0}\|$. In particular, $\mathbf{x}_0$ belongs to K.

Let us now prove that $d(\mathbf{a}, K) = \|\overrightarrow{\mathbf{a}\,\mathbf{x}_0}\|$. According to the Pythagorean theorem, as $\overrightarrow{\mathbf{a}\mathbf{x}}$ is orthogonal to H, we have

$$d(\mathbf{a}, K)^2 = \|\overrightarrow{\mathbf{a}\,\mathbf{x}}\|^2 + d(\mathbf{x}, K)^2 = \|\overrightarrow{\mathbf{a}\,\mathbf{x}}\|^2 + d(\mathbf{x}, K')^2,$$

thanks to Lemma 1.

Moreover, as $\overrightarrow{\mathbf{x}\,\mathbf{y}}$ is orthogonal to H', we have

$$d(\mathbf{x}, K')^2 = \|\overrightarrow{\mathbf{x}\,\mathbf{y}}\|^2 + d(\mathbf{y}, K')^2 = \|\overrightarrow{\mathbf{x}\,\mathbf{y}}\|^2 + \|\overrightarrow{\mathbf{x}_0\mathbf{y}}\|^2 = \|\overrightarrow{\mathbf{x}\,\mathbf{x}_0}\|^2$$

since $\overrightarrow{\mathbf{x}\,\mathbf{y}}$ is orthogonal to $\overrightarrow{\mathbf{x}_0\,\mathbf{y}}$.

Finally, $d(\mathbf{a}, K)^2 = \|\overrightarrow{\mathbf{a}\,\mathbf{x}}\|^2 + \|\overrightarrow{\mathbf{x}\,\mathbf{x}_0}\|^2 = \|\overrightarrow{\mathbf{a}\,\mathbf{x}_0}\|^2$ as $\overrightarrow{\mathbf{a}\,\mathbf{x}}$ is orthogonal to $\overrightarrow{\mathbf{x}\,\mathbf{x}_0}$, hence $d(\mathbf{a}, K) = \|\overrightarrow{\mathbf{a}\,\mathbf{x}_0}\|$ and the algorithm is correct for K. ■

Remark 1. Let $\mathbf{x}$ be in $H \setminus K$. Then the hyperface of K that is the closest to $\mathbf{x}$ is not necessarily the hyperface of K obtained by suppressing the (or one) vertex of K that is the furthest from $\mathbf{x}$.

According to Algorithm 1, finding the solution that minimizes f on K is now a straightforward application of the following proposition.

Theorem 4. *The unique portfolio that possesses the lowest risk is $P_{\mathbf{x}_0}$, where $d(0, K) = \|\mathbf{x}_0\|$.*

4. Application to portfolio optimization

Here we determine the portfolio with the lowest risk: we find the convex combination of CAC 40 stocks (we use the abbreviations given in Table A at the end of the article) for which the variance is minimal[2]. We use the mean and the standard deviation of monthly[3] variation.

4.1. Portfolio optimization from 2007-04-23 to 2020-07-21

Here we consider the period from 2007-04-23 to 2020-07-21, that is to say we start from the highest point of CAC 40 index. By using Algorithm 1, we determine the portfolio with allowed short-selling that possesses the lowest risk: this is the linear combination given by Table 1, which also provides the mean and the standard deviation of stocks' rates of return. The mean of this portfolio's monthly variation is 0.44% and its standard-deviation 4.35%. We observe here that the linear combination obtained is already a convex combination, which means that this portfolio is also the portfolio without short-selling that possesses the lowest risk. In geometrical terms, this means that the orthogonal projection of the origin point onto the hyperplane H already belongs to the simplex K.

[2]For this computation, we do not consider EL, GLE and WLN, for which we don't have enough data.

[3]The French stock market's month (that ends the third Friday in the month) is used.

Table 1. Portfolio with allowed short-selling that possesses the lowest risk (in %).

Stock	AI	BN	CA	DSY	ENGI	HO	ORA	RMS	SAN	VIV
$\mathbf{x}_0$	7.28	22.69	3.90	12.63	2.41	3.81	21.05	8.96	16.57	0.70
Mean	0.73	0.18	−0.559	1.53	−0.43	0.53	−0.18	1.6	0.4	0.08
Std. Dev.	5.73	5.55	8.02	6.58	7.06	6.57	6.74	8.63	6.25	6.75

Table 2 and Figure 1 give the yearly rate of return of this portfolio. We notice that the portfolio is more profitable and more regular than the index: indeed, its mean is 7.41% and its standard deviation 11.93%, whereas for the index, the mean is 1.12% and the standard deviation 19.61%. Moreover, the rate of return of the portfolio is almost never negative. Finally, if we fix at 100 the value of CAC 40 index and portfolio on 2007-04-23, then on 2020-07-21, the value of CAC 40 index is 86.26 whereas the value of the portfolio is 248.16.

Table 2. Yearly rate of return of the portfolio (in %).

Year	2007	2008	2009	2010	2011	2012	2013	2014	2015	2016	2017	2018	2019	2020
Porfolio's r. r.	1.85	−23.04	5.09	12.84	−0.80	14.67	7.81	12.89	16.16	3.95	14.38	1.47	33.12	3.37
CAC 40's r. r.	−5.12	−43.87	22.87	0.34	−23.45	23.54	14.72	0.93	7.30	5.64	12.40	−14.65	30.33	−15.34

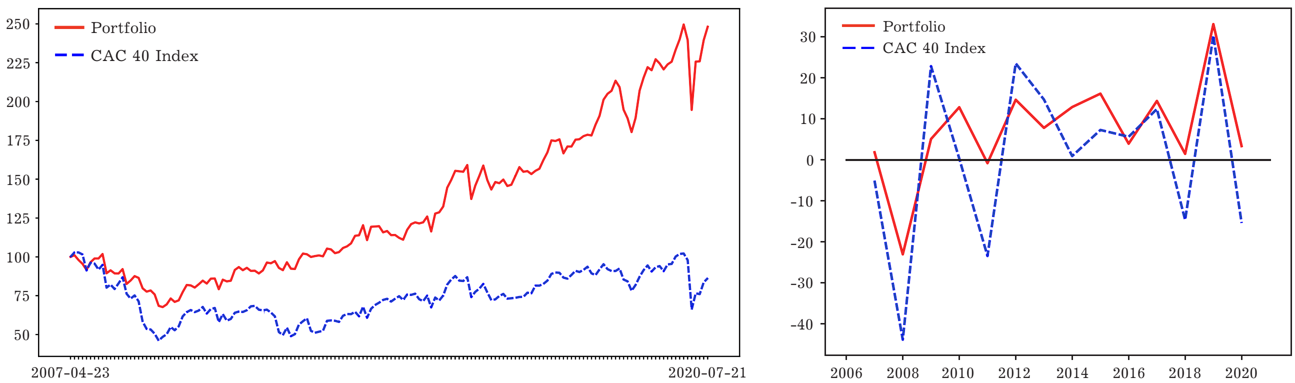

Fig. 1. Portfolio optimization from 2007-04-23 to 2020-07-21 and yearly rate of return.

4.2. Portfolio optimization from 2009-01-19 to 2020-07-21

Here we consider the period from 2009-01-19 to 2020-07-21, that is to say we start from the lowest point of CAC 40 index. The portfolio with allowed short-selling that possesses the lowest risk is given by the following linear combination in Table 3. The mean of its monthly variation is 0.76% and its standard-deviation 4.17%. As in previous section, Table 3 also provides the mean and the standard deviation of stocks' rates of return that appear in the results.

Table 3. Portfolio with allowed short-selling that possesses the lowest risk (in %).

Stock	AI	BN	CA	DSY	ENGI	HO	OR	ORA	RI	RMS	SAN	VIV
$\mathbf{x}_0$	10.70	28.25	5.42	20.91	−2.34	1.88	−20.66	19.75	−3.07	17.26	18.36	3.54
Mean	1.03	0.44	−0.13	1.93	−0.470	0.79	1.36	−0.18	1.0	1.86	0.64	0.37
Std. Dev.	5.36	5.34	7.97	6.00	7.09	6.48	5.42	6.77	5.93	7.44	6.03	6.77

Now, according to Algorithm 1, the portfolio without short-selling that possesses the lowest risk is given by the convex combination in Table 4. The mean of its monthly variation is 0.87% and its standard-deviation 4.22%. Let us note that some stocks that were in the linear combination of the portfolio with allowed short-selling now disappear: the geometric explanation of this fact immediately comes from Lemma 1.

Table 4. Portfolio without allowed short-selling that possesses the lowest risk (in %).

Stock	AI	BN	CA	DSY	HO	ORA	RMS	SAN	VIV
$\mathbf{x}_0$	6.17	19.22	3.43	17.37	2.96	17.15	15.60	16.73	1.37

The yearly rate of return of this portfolio is given by Table 5. Here again, as shown by this table and Figure 2, the portfolio is more profitable and more regular than the index: its mean is 13.27% and its standard deviation 11.99%, whereas for the index, the mean is 5.94% and the standard deviation 16.78%. Moreover, the rate of return of the portfolio is negative for only two years, and the absolute value of these negative rates of return very small. Finally, if we fix at 100 the value of CAC 40 index and portfolio on 2009-01-19, then on 2020-07-21, the value of CAC 40 index is 170.73 whereas the value of the portfolio is 417.97.

Table 5. Yearly rate of return of the portfolio (in %).

Year	2009	2010	2011	2012	2013	2014	2015	2016	2017	2018	2019	2020
Portfolio's r. r.	17.95	15.82	−1.49	27.80	5.76	7.87	23.81	−2.25	18.18	2.86	38.27	4.72
CAC 40's r. r.	29.51	0.34	−23.45	23.54	14.72	0.93	7.30	5.64	12.40	−14.65	30.33	−15.34

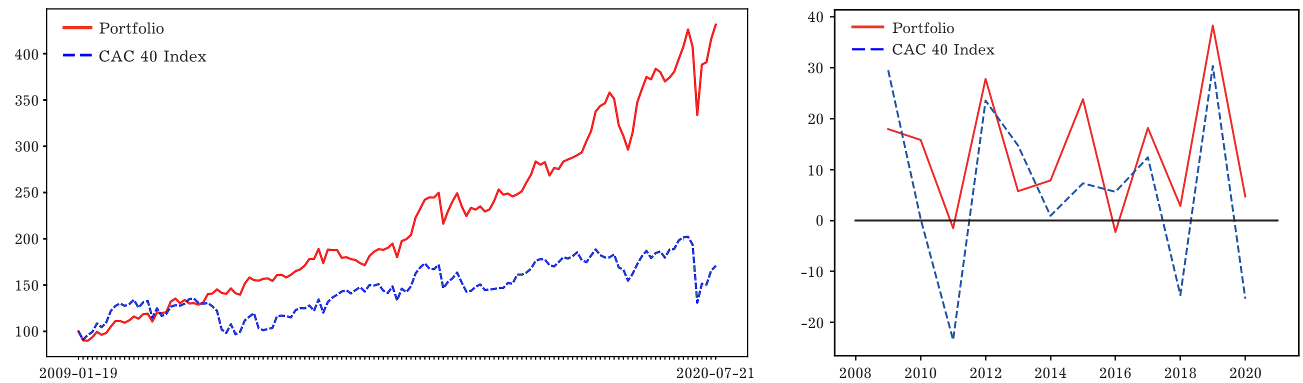

Fig. 2. Portfolio optimization from 2009-01-19 to 2020-07-21 and yearly rate of return.

5. Conclusions and perspectives

Thanks to our geometrical approach, we transformed the portfolio optimization problem into a problem of computation of a distance to a simplex, where the distance is a Euclidean distance defined from the covariance matrix of the stocks used. To solve this problem, we wrote an algorithm that can be used for every Euclidean distance, and that provided us a method to determine the portfolio without short-selling that possesses the minimal risk.

The good rate of return that we got when we applied our method to the example of the CAC 40 stocks can prompt us to use it in a more dynamic way for prediction algorithms. Moreover, it would be very usefull to optimize the time complexity of the algorithm by reducing the number of cases to be processed during the computation of the distance. It could also be fruitful to study how this geometrical method can be used not only in finance, but in various areas where such optimization problems occur.

Appendix — Python programs

Here we give a possible way to program Algorithm 1 in Python as well as the subroutine used to compute an orthogonal projection.

The function `orth_proj(c,a,J)` computes an orthogonal projection, where
— `c` is the covariance matrix,
— `a` is the point of which we want to compute the orthogonal projection,
— `J` is the list of indices of p vectors of $\mathcal{E}$ that define the affine subspace onto which we want to project `a`.

```
def orth_proj(c,a,J):
    p=len(J); n=len(c); i0=J[0]; L=list(set(range(n))-set(J)) #Complementary of J
    Mpart1=np.array([[c[i,j]-c[i0,j] for j in J] for i in J[1:]])
    Mpart2=np.ones((1,p))
    M=np.concatenate((Mpart1,Mpart2),axis=0) #Matrix of the system
    b=np.array([sum(a[j]*(c[i,j]-c[i0,j]) for j in range(n))for i in J[1:]]+[1])
    sol=np.linalg.solve(M,b); x=np.zeros(n); x[J]=sol; return x #Solving
```

The function `mini_dist_fct(c,a)` finds the point that realizes the minimal distance from **a** to the standard $(n-1)$-simplex and also returns the square of this distance: this is a possible version of Algorithm 1.

```
def mini_dist_fct(c,a):
    #Scalar product
    def phi(x,y):
        return np.dot(x,np.dot(c,y))
    n=len(c); dico={}
    e=[np.array(j*[0]+[1]+(n-j-1)*[0]) for j in range(n)] #Canonical basis
    #Recursive function mini_dist(c,a,J)
    def mini_dist(c,a,J):
        x=orth_proj(c,a,J) #Orthogonal projection of a
        #Case 1: the orthogonal projection belongs to the simplex
        if all(t>=0 for t in x):
            return [x,phi(x-a,x-a)]
        #Case 2: the orth. proj. doesn’t belong to the simplex (dim 1)
        elif len(J)==2:
            d0=phi(x-e[J[0]],x-e[J[0]]); d1=phi(x-e[J[1]],x-e[J[1]])
            if d0<=d1:
                return [e[J[0]],phi(e[J[0]]-a,e[J[0]]-a)]
            else:
                return [e[J[1]],phi(e[J[1]]-a,e[J[1]]-a)]
        #Case 3: the orth. proj. doesn’t belong to the simplex (dim >1)
        else:
            #Looking for the hyperface that is the closest to x
            s=J[0]
            if str(set(J)-{s})+str(x) in dico:
                delta=dico[str(set(J)-{s})+str(x)]
            else:
                delta=mini_dist(c,x,list(set(J)-{s}))
                dico[str(set(J)-{s})+str(x)]=delta
            d=delta[1]
            for j in J[1:]:
                if str(set(J)-{j})+str(x) in dico:
                    delta0=dico[str(set(J)-{j})+str(x)]
                else:
                    delta0=mini_dist(c,x,list(set(J)-{j}))
```

```
                dico[str(set(J)-{j})+str(x)]=delta0
            d0=delta0[1]
            if d0<d:
                s=j; d=d0
        #Projection onto the simplex defined by the closest hyperface
        J=list(set(J)-{s})
        if str(set(J))+str(x) in dico:
            delta=dico[str(set(J))+str(x)]
        else:
            delta=mini_dist(c,x,J); dico[str(set(J))+str(x)]=delta
        x=delta[0]; return [x,phi(x-a,x-a)]
return mini_dist(c,a,list(range(n))) #Point realizing the minimal distance
```


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Table A. CAC 40 stocks.

Symbol	Company	Symbol	Company
AC	Accor SA	ACA	Credit Agricole S.A.
AI	L'Air Liquide S.A.	AIR	Airbus SE
ATO	Atos SE	BN	Danone S.A.
BNP	BNP Paribas SA	CA	Carrefour SA
CAP	Capgemini SE	CS	AXA SA
DG	VINCI SA	DSY	Dassault Systemes SE
EL	EssilorLuxottica Societe anonyme	EN	Bouygues SA
ENGI	ENGIE SA	FP	TOTAL S.A.
GLE	Societe Generale Societe anonyme	HO	Thales S.A.
KER	Kering SA	LR	Legrand SA
MC	LVMH Moet Hennessy - Louis Vuitton	ML	Cie G$^{\text{le}}$ des Et. Michelin
MT	ArcelorMittal	OR	L'Oreal S.A.
ORA	Orange S.A.	PUB	Publicis Groupe S.A.
RI	Pernod Ricard SA	RMS	Hermes International
RNO	Renault SA	SAF	Safran SA
SAN	Sanofi	SGO	Compagnie de Saint-Gobain S.A.
STM	STMicroelectronics N.V.	SU	Schneider Electric
SW	Sodexo S.A.	UG	Peugeot S.A.
URW	Unibail-Rodamco-Westfield	VIE	Veolia Environnement S.A.
VIV	Vivendi	WLN	Worldline

Новий геометричний метод оптимізації портфеля

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Запобігання ризиків відіграє важливу та центральну роль у прийнятті рішень інвесторами в процесі формування портфеля. У межах оптимізації портфеля визначено портфель, який має мінімальний ризик, використовуючи новий геометричний метод. Для цього розроблено алгоритм, який дозволяє нам обчислити будь-яку евклідову відстань до стандартного симплексу. Завдяки цьому новому підходу можна розглянути випадок оптимізації портфеля без коротких продажів у цілому, а також відновити в геометричному вигляді добре відомі результати оптимізації портфеля з дозволеними короткими продажами. Потім застосовано отримані результати для того, щоб визначити, яка опукла комбінація акцій CAC 40 має найнижчий ризик: не тільки отримуємо дуже низький ризик порівняно з індексом, але також отримуємо коефіцієнт прибутковості, який майже втричі кращий, ніж в індекса.

Ключові слова: *оптимізація портфеля без коротких продажів, евклідова відстань до стандартного симплексу, геометричний підхід до оптимізації портфеля, геометричний алгоритм.*