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Asymptotic stepwise solutions of the Korteweg–de Vries equation with a singular perturbation and their accuracy

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The paper deals with the Korteweg-de Vries equation with variable coefficients and a small parameter at the highest derivative. The asymptotic step-like solution to the equation is obtained by the non-linear WKB technique. An algorithm of constructing the higher terms of the asymptotic step-like solutions is presented. The theorem on the accuracy of the higher asymptotic approximations is proven. The proposed technique is demonstrated by example of the equation with given variable coefficients. The main term and the first asymptotic approximation of the given example are found, their analysis is done and statement of the approximate solutions accuracy is presented.

Keywords: *Korteweg–de Vries equation, nonlinear WKB technique, asymptotic step-like solution, soliton, singular perturbation, asymptotic analysis.*

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1. Introduction

In [1–3] we began to study new type of asymptotic solutions to the Korteweg–de Vries equation with variable coefficients and a small parameter of the following form

$$\varepsilon u_{xxx} = a(x,t,\varepsilon)u_t + b(x,t,\varepsilon)u\,u_x, \tag{1}$$

where

$$a(x,t,\varepsilon) = \sum_{k=0}^{N} a_k(x,t)\varepsilon^k + \mathrm{O}\big(\varepsilon^{N+1}\big), \quad b(x,t,\varepsilon) = \sum_{k=0}^{N} b_k(x,t)\varepsilon^k + \mathrm{O}\big(\varepsilon^{N+1}\big),$$

$a_k(x,t) \in C^\infty(\mathbf{R}\times[0;T])$, $b_k(x,t) \in C^\infty(\mathbf{R}\times[0;T])$, $k=\overline{0,N}$, $T>0$, ε is a small parameter.

They are called the asymptotic step-like solutions and these ones are constructed by the non-linear WKB technique that is the most suitable method [4] for obtaining such solutions [5–7]. The asymptotic solution contains regular and singular parts. The regular part describes the background of the wave process, while the singular part reflects the specific features of the behavior of the wave at infinity. Special characteristics of the step-like asymptotic solutions are associated with the properties of its singular part. The singular part of the searched asymptotic solution has the main term, which, like the soliton solution, is a quickly decreasing function of the phase variable τ, in contrast to the other terms that do not have this property. It means that higher terms of the singular part of the asymptotic tend to zero as $\tau\to+\infty$ and tend to non-zero as $\tau\to-\infty$.

Now we proceed to constructing asymptotic step-like solutions to the singularly perturbed Korteweg–de Vries equation with variable coefficients. The use of the WKB technique allows one to find asymptotic solutions of equation (1) in the following form

$$u(x,t,\varepsilon) = Y_N(x,t,\tau,\varepsilon) + \mathrm{O}\big(\varepsilon^{N+1/2}\big), \tag{2}$$

where

$$Y_N(x,t,\tau,\varepsilon) = U_N(x,t,\varepsilon) + V_N(x,t,\tau,\varepsilon), \quad \tau = \frac{x-\varphi(t)}{\sqrt{\varepsilon}},$$

$$U_N(x,t,\varepsilon) = \sum_{j=0}^{2N} \varepsilon^{j/2} u_j(x,t), \quad V_N(x,t,\tau,\varepsilon) = \sum_{j=0}^{2N} \varepsilon^{j/2} V_j(x,t,\tau).$$

The regular part $U_N(x,t,\varepsilon)$ and the main term of the singular part $V_N(x,t,\tau,\varepsilon)$ of (2) are have been found in [1]. Below we will obtain the higher terms of asymptotic (2) and establish an estimate of their accuracy.

2. Higher terms of the singular part on the discontinuity curve

Let come to studying the equation of the singular part of the asymptotics. Firstly, we examine conditions of existence of solutions to equation for the singular terms on the discontinuity curve $x = \varphi(t)$, $t \in [0;T]$, in the space G_1 of the following form [1]

$$\frac{\partial^3 v_j}{\partial \tau^3} + a_0(\varphi,t)\frac{\partial v_j}{\partial \tau}\varphi'(t) - b_0(\varphi,t)\left[u_0(\varphi,t)\frac{\partial v_j}{\partial \tau} + v_j\frac{\partial v_0}{\partial \tau} + v_0\frac{\partial v_j}{\partial \tau}\right] = \mathcal{F}_j(t,\tau), \tag{3}$$

where the functions $\mathcal{F}_j(t,\tau)$, $j = \overline{1,2N}$, are considered to be known.

Let us consider differential operator

$$L = \frac{\partial^3}{\partial \tau^3} + \left(a_0(\varphi,t)\varphi'(t) - b_0(\varphi,t)v_0 - b_0(\varphi,t)u_0(\varphi,t)\right)\frac{\partial}{\partial \tau} - \frac{\partial v_0}{\partial \tau} b_0(\varphi,t). \tag{4}$$

Then for any $j = \overline{1,2N}$ equation (3) can be written as

$$L v_j = \mathcal{F}_j, \quad j = \overline{1,2N}. \tag{5}$$

Representation of operator (4) implies the following property, namely: if the function $v_j(t,\tau) \in G_1$ then $Lv_j(t,\tau) \in G_0$, $j = \overline{1,2N}$. Thus, we can consider a problem of existence a solution of equation (5) in the space G_1 provided that $\mathcal{F}_j(t,\tau) \in G_0$, $j = \overline{1,2N}$.

The following lemma is true.

Lemma 1. *Let us suppose $\mathcal{F}_j(t,\tau) \in G_0$, $j = \overline{1,2N}$. Then a solution to equation (5) exists in the space G_1 if and only if the condition*

$$\int_{-\infty}^{+\infty} \mathcal{F}_j(t,\tau) v_0(t,\tau) d\tau = 0, \quad j = \overline{1,2N} \tag{6}$$

holds.

Proof. Firstly, assume that equation (5) has a solution in G_1. We multiply both sides of (5) by $v_0(t,\tau)$ and integrate the resulting expression in τ. Taking into account the property $v_0 \in G_0$ and the evident equality

$$\int_{-\infty}^{+\infty} v_0(t,\tau) L v_j(t,\tau) d\tau = 0$$

we come to orthogonality condition (6).

Now let us consider the orthogonality condition to be true. We need to prove that under this assumption a solution to equation (5) exists in the space G_1. For this purpose, we demonstrate that solution to equation (5) in the space G_1 can be written as

$$v_j(t,\tau) = \nu_j(t)\eta_j(t,\tau) + \psi_j(t,\tau), \quad j = \overline{1,2N}, \tag{7}$$

where

$$\nu_j(t) = [a_0(\varphi(t),t)\varphi'(t) - b_0(\varphi(t),t)u_0(\varphi(t),t)]^{-1} \lim_{\tau\to-\infty} \Phi_j(t,\tau), \tag{8}$$

$$\Phi_j(t,\tau) = \int_{-\infty}^{\tau} \mathcal{F}_j(t,\xi)d\xi + E_j(t), \tag{9}$$

a constant of integrating $E_j(t)$ satisfies equality

$$\lim_{\tau\to+\infty} \Phi_j(t,\tau) = 0,$$

and $\eta_j(t,\tau) \in G_1$ is such that $\lim\limits_{\tau\to-\infty} \eta_j(t,\tau) = 1$.

We must show that the function $\psi_j(t,\tau) \in G_0$. Indeed, integrating equation (5) from $-\infty$ to τ provides us the following relation

$$L_1 v_j = \Phi_j(t,\tau) \tag{10}$$

with

$$L_1 = \frac{d^2}{d\tau^2} + a_0(\varphi(t),t)\varphi'(t) - b_0(\varphi(t),t)u_0(\varphi(t),t) - b_0(\varphi(t),t)v_0(t,\tau).$$

From (7), (10) we deduce that the function $\psi_j(t,\tau)$, $j = \overline{1,2N}$, satisfies equation

$$L_1\psi_j = \Phi_j - \nu_j L_1 \eta. \tag{11}$$

Because of condition ker $L_1^* = \{v_{0\tau}\}$, accordingly Theorem 3.1 [12] a solution to equation (11) exists in the space G_0 if and only if the following condition

$$\int_{-\infty}^{+\infty} (\Phi_j - \nu_j L_1 \eta)\, v_{0\tau} d\tau = 0, \quad j = \overline{1,2N}, \tag{12}$$

takes place.

Relation (12) is equivalent to the following equality

$$\int_{-\infty}^{+\infty} \mathcal{F}_j(t,\tau) v_0(t,\tau) d\tau = 0, \quad j = \overline{1,2N}.$$

So, a solution to (5) exists in G_0 if and only if orthogonality condition (6) is hold. Moreover, the solution is written as (7), where the function $\psi_j(t,\tau) \in G_0$.

The lemma is proven. ■

Remark 1. The formula gives us a representation of the terms of the singular part of the asymptotics on the discontinuity curve. This formula plays an essential role in the continuation of these terms from the curve.

The following statement presents the important particular case of Lemma 1.

Lemma 2. *Let $\mathcal{F}_j(t,\tau) \in G_0$, $j = \overline{1,2N}$, and condition (6) take place. The solution $v_j(t,\tau)$ to equation (3) belongs to G_0, $j = \overline{1,2N}$, iff*

$$\lim_{\tau\to-\infty} \Phi_j(t,\tau) = 0, \quad j = \overline{1,N}. \tag{13}$$

Proof. The proof of the statement follows from formula (7). Indeed, under condition (13) we have $\nu_j(t) = 0$. It means that $v_j(t,\tau) = \psi_j(t,\tau) \in G_0$ for any $j = \overline{1,2N}$.

The lemma is proven. ■

Remark 2. Inclusion $v_j(t,\tau)\in G_0$ implies the following relation

$$\lim_{\tau\to\pm\infty} v_j(t,\tau)=0, \quad j=\overline{1,2N}.$$

This allows us to set $V_j(t,\tau)=v_j(t,\tau)$ and thereby to obtain a singular part of the asymptotics of a special form called an asymptotic soliton-like solution of soliton type [8].

Remark 3. The general solution to equation (3) can be written in closed form using the following formula

$$v_j(t,\tau)=\Bigg[\bigg(\int_{\tau_0}^{\tau}\Phi_j(t,\xi)v_{0\xi}(t,\xi)\,d\xi+c_1\bigg)\int_{\tau_0}^{\tau}v_{0\xi}^{-2}(t,\xi)\,d\xi - \bigg(\int_{\tau_0}^{\tau}\Phi_j(t,\eta)v_{0\eta}(t,\eta)\int_{\tau_0}^{\eta}v_{0\xi}^{-2}(t,\xi)\,d\xi\,d\eta+c_2\bigg)\Bigg]v_{0\tau}(t,\tau). \quad (14)$$

It deduced by the method of variation of constants.

Remark 4. From the orthogonality condition (6) as $j=1$ we obtain the second order ordinary differential equation for the phase function $\varphi=\varphi(t)$ in the form:

$$15a_0(\varphi,t)\,b_0(\varphi,t)\frac{d}{dt}A(\varphi,\varphi',t)+\big[(10a_{0x}(\varphi,t)\,b_0(\varphi,t)-36a_0(\varphi,t)\,b_{0x}(\varphi,t))\,\varphi' +10b_0^2(\varphi,t)u_{0x}(\varphi,t)+3(b_0^2(\varphi,t))_x u_0(\varphi,t)-20a_0(\varphi,t)b_{0t}(\varphi,t)\big]\,A(\varphi,\varphi',t)=0, \quad (15)$$

where condition

$$A(\varphi,\varphi',t)=-a_0(\varphi,t)\varphi'(t)+b_0(\varphi,t)u_0(\varphi,t)>0 \quad (16)$$

is supposed to be satisfied.

In a neighborhood of initial point equation (15) has a solution existing on finite or infinite interval in general. We assume that the solution is defined on some interval $[0;T]$, where $T>0$ is a real. It should be also noted that for certain coefficients $a_0(x,t), b_0(x,t)$ equation (15) can be written in simpler form.

Prolonging the singular part from the discontinuity curve

Now we construct the terms $V_j(x,t,\tau)$, $j=\overline{0,2N}$, by prolongation of the function $v_j(t,\tau)$, $j=\overline{0,2N}$, from the curve $x=\varphi(t)$, $t\in[0;T]$.

Because $v_0(t,\tau)\in G_0$ we put

$$V_0(x,t,\tau)=v_0(t,\tau)=-3\frac{A(\varphi,\varphi',t)}{b_0(\varphi,t)}\cosh^{-2}\left(\frac{\sqrt{A(\varphi,\varphi',t)}}{2}(\tau+c_0)\right). \quad (17)$$

Similarly the coefficients $V_j(x,t,\tau)=v_j(t,\tau)$ if $v_j(t,\tau)\in G_0$ [3].

In another case let us consider the Cauchy problem

$$\Lambda u_j^-(x,t)=f_j^-(x,t), \quad (18)$$

$$u_j^-(x,t)\big|_\Gamma=\nu_j(t), \quad (19)$$

where differential operator Λ is written as

$$\Lambda=a_0(x,t)\frac{\partial}{\partial t}+b_0(x,t)u_0(x,t)\frac{\partial}{\partial x}+b_0(x,t)u_{0x}(x,t)$$

and initial function $\nu_j(t)$ is given by formula (8).

Equation (18) is found by substituting (2) in (1) and calculating limits as $\tau \to -\infty$. For example we have

$$f_1^-(x,t) = 0, \quad f_2^-(x,t) = -a_1(x,t)\frac{\partial u_1^-}{\partial t} - b_0(x,t)\frac{\partial}{\partial x}\left(u_1\, u_1^-\right) - b_0(x,t)u_1^-\frac{\partial u_1^-}{\partial x} - b_1(x,t)\frac{\partial}{\partial x}\left(u_0\, u_1^-\right).$$

For all $t \in [0;T]$ the curve Γ is transversal to characteristics of the operator Λ. So, the Cauchy problem (18), (19) has a solution $u_j^-(x,t) \in C^{(\infty)}(\Omega_\mu(\Gamma))$, where $\Omega_\mu(\Gamma) = \{(x,t) \in \mathbf{R} \times [0;T] : |x - \varphi(t)| < \mu\}$ and μ is a certain positive value.

According to representation (7), prolongation of the $v_j(t,\tau)$, $j = \overline{1,2N}$, from the curve Γ is given by the following formula [9]

$$V_j(x,t,\tau) = u_j^-(x,t)\eta(t,\tau) + \psi_j(t,\tau). \tag{20}$$

Thus, the problem of constructing asymptotic step like solution is completed.

3. Justification of the algorithm of constructing the solution

Now we come to estimation of higher asymptotic approximations. Let denote sets

$$D^- = \{(x,t) \in \mathbf{R} \times [0;T] : x - \varphi(t) \leqslant 0\},$$
$$D^+ = \{(x,t) \in \mathbf{R} \times [0;T] : x - \varphi(t) \geqslant 0\}.$$

The following statement is correct.

Theorem 1. *Let be the following conditions take place:*
1) the functions $a_k(x,t)$, $b_k(x,t) \in C^\infty(\mathbf{R} \times [0;T])$, $k = \overline{0,N}$;
2) the functions $\mathcal{F}_j(t,\tau) \in G_0$, $j = \overline{1,2N}$, and satisfy orthogonality conditions (6);
3) function $\varphi(t)$, $t \in [0;T]$, is a solution of differential equation (15) and satisfies inequality (16);
4) the Cauchy problem (18), (19) has a solution $u_j^-(x,t)$, $j = \overline{1,2N}$, in the set D^-.

Then the asymptotic step-like solution to equation (1) can be written as follows

$$u_N(x,t,\varepsilon) = \begin{cases} Y_N^-(x,t,\varepsilon), & (x,t) \in D^- \backslash \Omega_\mu(\Gamma), \\ Y_N(x,t,\varepsilon), & (x,t) \in \Omega_\mu(\Gamma), \\ Y_N^+(x,t,\varepsilon), & (x,t) \in D^+ \backslash \Omega_\mu(\Gamma), \end{cases} \tag{21}$$

where

$$Y_N^-(x,t,\varepsilon) = u_0(x,t) + \sum_{j=1}^{2N} \varepsilon^{j/2}\left[u_j(x,t) + u_j^-(x,t)\right], \tag{22}$$

$$Y_N(x,t,\varepsilon) = \sum_{j=0}^{2N} \varepsilon^{j/2}\left[u_j(x,t) + V_j(x,t,\tau)\right], \quad \tau = \frac{x - \varphi(t)}{\sqrt{\varepsilon}}, \tag{23}$$

$$Y_N^+(x,t,\varepsilon) = \sum_{j=0}^{2N} \varepsilon^{j/2} u_j(x,t). \tag{24}$$

Function (21) satisfies the Korteweg–de Vries equation (1) with accuracy $\mathrm{O}(\varepsilon^N)$ on the set $\mathbf{R} \times [0;T]$. Moreover, it satisfies equation (1) with accuracy $\mathrm{O}(\varepsilon^{N+1/2})$ as $\tau \to \pm\infty$ on the set $\mathbf{R} \times [0;T]$.

Proof. To prove the theorem let us consider the domain $\Omega_\mu(\Gamma)$ firstly. After substituting expression (23) into (1) we get the following equality

$$\varepsilon\left(\sum_{j=0}^{2N}\varepsilon^{j/2}\frac{\partial^3 u_j}{\partial t^3}+\sum_{j=0}^{2N}\varepsilon^{j/2}\frac{\partial^3 V_j}{\partial x^3}+\frac{1}{\sqrt{\varepsilon}}\sum_{j=0}^{2N}\varepsilon^{j/2}\frac{\partial^3 V_j}{\partial x^2\partial\tau}+\frac{1}{\varepsilon}\sum_{j=0}^{2N}\varepsilon^{j/2}\frac{\partial^3 V_j}{\partial x\partial\tau^2}+\frac{1}{\varepsilon\sqrt{\varepsilon}}\sum_{j=0}^{2N}\varepsilon^{j/2}\frac{\partial^3 V_j}{\partial\tau^3}\right)$$
$$=\sum_{k=0}^{N}\varepsilon^k a_k(x,t)\left(\sum_{j=0}^{2N}\varepsilon^{j/2}\frac{\partial u_j}{\partial t}+\sum_{j=0}^{2N}\varepsilon^{j/2}\frac{\partial V_j}{\partial t}-\varphi'(t)\frac{1}{\sqrt{\varepsilon}}\sum_{j=0}^{2N}\varepsilon^{j/2}\frac{\partial V_j}{\partial\tau}\right)$$
$$+\sum_{k=0}^{N}\varepsilon^k b_k(x,t)\left(\sum_{j=0}^{2N}\varepsilon^{j/2}u_j(x,t)+\sum_{j=0}^{2N}\varepsilon^{j/2}V_j(x,t,\tau)\right)$$
$$\times\left(\sum_{j=0}^{2N}\varepsilon^{j/2}\frac{\partial u_j}{\partial x}+\sum_{j=0}^{2N}\varepsilon^{j/2}\frac{\partial V_j}{\partial x}+\frac{1}{\sqrt{\varepsilon}}\sum_{j=0}^{2N}\varepsilon^{j/2}\frac{\partial V_j}{\partial\tau}\right)+\mathrm{O}\big(\varepsilon^{N+1/2}\big). \quad (25)$$

Taking into account equations for the terms of the regular and the singular parts [1], we pass to the asymptotic estimation of the residual function $g_N(x,t,\varepsilon)$ in μ-neighborhood of the curve Γ. The function $g_N(x,t,\varepsilon)$ can be written as follows

$$g_N(x,t,\varepsilon)=\left[a_0(x,t)-\sum_{k=0}^{2N}\frac{(\sqrt{\varepsilon}\tau)^k}{k!}\frac{\partial^k a_0(\varphi,t)}{\partial x^k}+\varepsilon\left(a_1(x,t)-\sum_{k=0}^{2N-1}\frac{(\sqrt{\varepsilon}\tau)^k}{k!}\frac{\partial^k a_1(\varphi,t)}{\partial x^k}\right)\right.$$
$$\left.+\ldots+\varepsilon^N\left(a_N(x,t)-a_N(\varphi,t)\right)\right]\times\left(\sum_{j=0}^{2N}\varepsilon^{j/2}\frac{\partial V_j}{\partial t}-\frac{1}{\sqrt{\varepsilon}}\varphi'(t)\sum_{j=0}^{2N}\varepsilon^{j/2}\frac{\partial V_j}{\partial\tau}\right)$$
$$+\left[b_0(x,t)-\sum_{k=0}^{2N}\frac{(\sqrt{\varepsilon}\tau)^k}{k!}\frac{\partial^k b_0(\varphi,t)}{\partial x^k}+\varepsilon\left(b_1(x,t)-\sum_{k=0}^{2N-1}\frac{(\sqrt{\varepsilon}\tau)^k}{k!}\frac{\partial^k b_1(\varphi,t)}{\partial x^k}\right)\right.$$
$$\left.+\ldots+\varepsilon^N\left(b_N(x,t)-b_N(\varphi,t)\right)\right]\times\left[u_0(x,t)-\sum_{k=0}^{2N}\frac{(\sqrt{\varepsilon}\tau)^k}{k!}\frac{\partial^k u_0(\varphi,t)}{\partial x^k}\right.$$
$$\left.+\varepsilon\left(u_1(x,t)-\sum_{k=0}^{2N-1}\frac{(\sqrt{\varepsilon}\tau)^k}{k!}\frac{\partial^k u_1(\varphi,t)}{\partial x^k}\right)+\ldots+\varepsilon^N\left(u_N(x,t)-u_N(\varphi,t)\right)\right]$$
$$\times\left(\sum_{j=0}^{2N}\varepsilon^{j/2}\frac{\partial V_j}{\partial x}+\frac{1}{\sqrt{\varepsilon}}\sum_{j=0}^{2N}\varepsilon^{j/2}\frac{\partial V_j}{\partial\tau}\right)+\left[b_0(x,t)-\sum_{k=0}^{2N}\frac{(\sqrt{\varepsilon}\tau)^k}{k!}\frac{\partial^k b_0(\varphi,t)}{\partial x^k}\right.$$
$$\left.+\varepsilon\left(b_1(x,t)-\sum_{k=0}^{2N-1}\frac{(\sqrt{\varepsilon}\tau)^k}{k!}\frac{\partial^k b_1(\varphi,t)}{\partial x^k}\right)+\ldots+\varepsilon^N\left(b_N(x,t)-b_N(\varphi,t)\right)\right]$$
$$\times\left[\frac{\partial u_0(x,t)}{\partial x}-\sum_{k=0}^{2N}\frac{(\varepsilon\tau)^k}{k!}\frac{\partial^{k+1}u_0(\varphi,t)}{\partial x^{k+1}}+\varepsilon\left(\frac{\partial u_1(x,t)}{\partial x}-\sum_{k=0}^{2N-1}\frac{(\varepsilon\tau)^k}{k!}\frac{\partial^{k+1}u_1(\varphi,t)}{\partial x^{k+1}}\right)\right.$$
$$\left.+\ldots+\varepsilon^N\left(\frac{\partial u_N(x,t)}{\partial x}-\frac{\partial u_N(\varphi,t)}{\partial x}\right)\right]\sum_{j=0}^{2N}\varepsilon^{j/2}V_j+\left[b_0(x,t)-\sum_{k=0}^{2N}\frac{(\sqrt{\varepsilon}\tau)^k}{k!}\frac{\partial^k b_0(\varphi,t)}{\partial x^k}\right.$$
$$\left.+\varepsilon\left(b_1(x,t)-\sum_{k=0}^{2N-1}\frac{(\sqrt{\varepsilon}\tau)^k}{k!}\frac{\partial^k b_1(\varphi,t)}{\partial x^k}\right)+\ldots+\varepsilon^N\left(b_N(x,t)-b_N(\varphi,t)\right)\right]$$
$$\times\left(\sum_{j=0}^{2N}\varepsilon^{j/2}V_j\right)\times\left(\sum_{j=0}^{2N}\varepsilon^{j/2}\frac{\partial V_j}{\partial x}+\frac{1}{\sqrt{\varepsilon}}\sum_{j=0}^{2N}\varepsilon^{j/2}\frac{\partial V_j}{\partial\tau}\right)+\mathrm{O}\big(\varepsilon^{N+1/2}\big), \quad (26)$$

where $\varphi=\varphi(t)$.

Since the functions $a_j(x,t)$, $j=\overline{0,N}$, are infinitely differentiable, then for all $(x,t)\in\Omega_\mu(\Gamma)$ the following inequality is true

$$\left|a_j(x,t)-\sum_{k=0}^{2N-j}\frac{(\sqrt{\varepsilon}\tau)^k}{k!}\frac{\partial^k a_j(\varphi,t)}{\partial x^k}\right|\leqslant C_j\left|\left(\sqrt{\varepsilon}\tau\right)^{2N+1-j}\right|,\quad \varphi=\varphi(t),$$

where the constant C_j, $j=\overline{0,N}$, depends on a compact $K\subset\mathbf{R}\times[0;T]$.

Similar asymptotic estimations hold for the functions $b_k(x,t)$, $k=\overline{0,N}$, $V_j(x,t,\tau)$, $u_j(x,t)$, $j=\overline{0,2N}$. Thus, we draw a conclusion that the residual function $g_N(x,t,\varepsilon)$ satisfies asymptotic relation $g_N(x,t,\varepsilon)=O(\varepsilon^N)$ as $\varepsilon\to0$ for all $(x,t)\in\Omega_\mu(\Gamma)$.

Now let us consider the set $D^+\backslash\Omega_\mu(\Gamma)$. Since for all $k\in\mathbf{N}$ any function $f(x,t,\tau)\in G_1$ satisfies inequality

$$|f(x,t,\tau)|\leqslant\frac{C_k(f)}{(\tau^2+1)^k},\quad \tau\geqslant0,$$

with some constant $C_k(f)$, $k\in\mathbf{N}$, depending on $K\subset\mathbf{R}\times[0;T]$, then relation

$$\left|\left(\sqrt{\varepsilon}\tau\right)^{2k}f(x,t,\tau)\right|\leqslant\varepsilon^k C_k(f),\quad \tau\geqslant0,\quad k\in\mathbf{N},$$

holds. It implies statement of Theorem 1 for all $(x,t)\in D^+\backslash\Omega_\mu(\Gamma)$.

Finally we consider the set $D^-\backslash\Omega_\mu(\Gamma)$. Using the representation of the function $V_j(x,t,\tau)$, $j=\overline{1,2N}$, in the form (20) we conclude

$$\left|V_j(x,t,\tau)-u_j^-(x,t)\right|\leqslant\frac{C_{1k}}{(|\tau|^2+1)^k},\qquad \left|\frac{\partial}{\partial\tau}V_j(x,t,\tau)\right|\leqslant\frac{C_k(V_j)}{(|\tau|^2+1)^k},$$

for all $\tau\leqslant0$ and any $k\in\mathbf{N}$, where C_{1k}, $C_k(V_j)$ are constants depending on a compact $K\subset\mathbf{R}\times[0;T]$.

Recall now that we are considering the asymptotic accuracy of the constructed solution as $\varepsilon\to0$. This means that the variable τ is unbounded, i.e. $|\tau|\to+\infty$. Since $V_j(x,t,\tau)-u_j^-(x,t)\to0$ sufficiently quickly as $\tau\to-\infty$, where $u_j^-(x,t)$ is a solution of the Cauchy problem (20), we come to statement of the theorem in the set $D^-\backslash\Omega_\mu(\Gamma)$.

Putting together all the properties proved above, we arrive at asymptotic estimation for the residual function of the form $g_N(x,t,\varepsilon)=\mathrm{O}\big(\varepsilon^N\big)$ as $\varepsilon\to0$ for all $(x,t)\in K$, where $K\subset\mathbf{R}\times[0;T]$ is an arbitrary compact.

This means that the first part of the statement is proven.

Since the property $V_j(x,t,\tau)\in G_1$, $j=\overline{0,2N}$, from (25) we deduce that asymptotic solution (21) satisfies equation (1) with accuracy $\mathrm{O}(\varepsilon^{N+1/2})$ as $\tau\to\pm\infty$ on the set $\mathbf{R}\times[0;T]$.

The theorem is completely proven. ■

4. Example

Let us demonstrate the technique of constructing the asymptotic step-like solutions for example. We set

$$a(x,t,\varepsilon)=-(x^2+1)^{3/2},\quad b(x,t,\varepsilon)=1,\tag{27}$$

and consider the variable coefficient Korteweg–de Vries equation with a small parameter of the following form

$$\varepsilon\,u_{xxx}=-(x^2+1)^{3/2}u_t+u\,u_x.\tag{28}$$

Let proceed to determining terms of the asymptotic expansion. To simplify the calculations we assume that the regular part of the asymptotic is trivial.

Firstly it is necessary to search the phase function $\varphi = \varphi(t)$. Under given formulation of the problem it satisfies equation (15) which is reduced to the following relation

$$(\varphi^2+1)^{5/2}\,\frac{d\varphi}{dt} = \gamma, \quad \gamma \in \mathbf{R}. \tag{29}$$

Lemma 3. *For any positive γ the Cauchy problem for differential equation (29) under initial condition $\varphi(0) = 0$ has solution defined for all $t \in \mathbf{R}$.*

Proof. Taking into account the initial condition $\varphi(0) = 0$ we reduce the Cauchy problem to relation of the form

$$\varphi\sqrt{\varphi^2+1}\,\left(8\varphi^4+26\varphi^2+33\right) + 15\ln\left|\varphi+\sqrt{\varphi^2+1}\right| = 48\gamma t. \tag{30}$$

Thus, the function $\varphi = \varphi(t)$ is given implicitly.

Let us verify conditions of the implicit function theorem [10]. The functions on the left and the right sides of (30) are defined for all values of φ, t and are infinitely differentiable with respect to their arguments. In particular, the first derivative of the left side function in variable φ is positive. The applying the implicit function theorem ensures the existence of a function $\varphi = \varphi(t)$ defined in some neighborhood of the initial point $(0,0)$.

Moreover, the function $\varphi = \varphi(t)$ is determined for all $t \in \mathbf{R}$ [11]. The last property follows from its monotonicity as well as its finiteness as $|t| \to +\infty$ accordingly relation (30) and inequality $0 < \varphi'(t) < \gamma$.

This completes the proof of Lemma 3. ■

Easy to see that condition (16) takes place in the case $\gamma > 0$ because of inequality $A(\varphi,\varphi',t) = (\varphi^2+1)^{3/2}\,\varphi' > 0$.

Formulas (17) provide us with the main singular term in the following form

$$V_0(x,t,\tau) = v_0(t,\tau) = -\frac{3\cosh^{-2}\vartheta(t,\tau)}{\varphi^2+1}, \tag{31}$$

where

$$\vartheta(t,\tau) = \frac{\sqrt{\gamma}\,\tau}{2\sqrt{\varphi^2+1}}, \quad \tau = \frac{x-\varphi}{\sqrt{\varepsilon}}, \quad \varphi = \varphi(t), \quad (t,\tau) \in \mathbf{R}^2.$$

Now let set $\gamma = 1$. By calculating $\mathcal{F}_1(t,\tau)$ accordingly (9) we get

$$\Phi_1(t,\tau) = \frac{12\varphi}{(\varphi^2+1)^3}\left[\sqrt{\varphi^2+1}\,(\tanh\vartheta(t,\tau)-1) - \tau\cosh^{-2}\vartheta(t,\tau)\right].$$

As a result of formula (14) as $j = 1$, one receives that the first singular term on the discontinuity curve is given with expression

$$\begin{aligned} v_1(t,\tau) = \frac{3\varphi}{(\varphi^2+1)^2}\Bigg[\Bigg[&\left(36-10\sqrt{\varphi^2+1}\right)\tau + \left((20\tau+12)\sqrt{\varphi^2+1}-10\tau\right)\cosh^{-2}\vartheta(t,\tau) \\ &- \left(30+10\sqrt{\varphi^2+1}\right)\tau\cosh^{-4}\vartheta(t,\tau) \\ - \Bigg(5\sqrt{\varphi^2+1} + \frac{1}{\sqrt{\varphi^2+1}} &- 35\sqrt{\varphi^2+1}\cosh^{-2}\vartheta(t,\tau) + \frac{105\,\tau^2}{4\sqrt{\varphi^2+1}}\cosh^{-4}\vartheta(t,\tau) \\ + 140\sqrt{\varphi^2+1}\ln\cosh\vartheta(t,\tau) - 3\tau\Bigg)&\tanh\vartheta(t,\tau)\Bigg]\cosh^{-2}\vartheta(t,\tau) - 4\sqrt{\varphi^2+1}\,(\tanh\vartheta(t,\tau)-1)\Bigg]. \end{aligned} \tag{32}$$

Taking into account (8), we have

$$\nu_1(t) = -(\varphi^2+1)\lim_{\tau\to-\infty}\Phi_1(t,\tau) = 24\frac{\varphi\sqrt{\varphi^2+1}}{(\varphi^2+1)^2}, \tag{33}$$

$$\eta_1(t,\tau) = -\frac{1}{2}\tanh\vartheta(t,\tau) + \frac{1}{2}, \quad \psi_1(t,\tau) = v_1(t,\tau) - \nu_1(t)\eta_1(t,\tau). \tag{34}$$

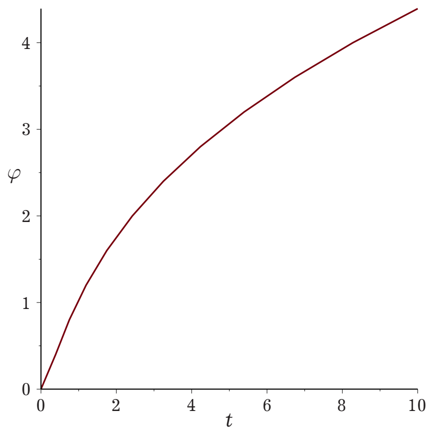

Fig. 1. The discontinuity curve $x = \varphi(t)$, $t \geqslant 0$, $\gamma = 1$.

To prolong the function $v_1(t,\tau)$ from the discontinuity curve Γ (see Fig. 1) we solve the Cauchy problem of form (18), (19)

$$-(x^2+1)^{-3/2}\frac{\partial}{\partial t}u_1^-(x,t) = 0, \quad u_1^-(x,t)\big|_\Gamma = \nu_1(t) \tag{35}$$

and obtain its solution as follows

$$u_1^-(x,t) = 24x(x^2+1)^{-3/2}. \tag{36}$$

Representation (20) implies formula for the first singular term as

$$V_1(x,t,\tau) = u_1^-(x,t)\eta_1(t,\tau) + \psi_1(t,\tau), \tag{37}$$

where the functions $u_1^-(x,t)$, $\eta_1(t,\tau)$, $\psi_1(t,\tau)$ are found in exact form above.

Higher terms of the asymptotic step-like solution can be found by means of formula (14).

Theorem 2. *The function*

$$Y_1(x,t,\varepsilon) = V_0(x,t,\tau) + \sqrt{\varepsilon}V_1(x,t,\tau), \quad \tau = \frac{x-\varphi(t)}{\sqrt{\varepsilon}}, \tag{38}$$

is the asymptotic step-like solution to the singular perturbed Korteweg–de Vries equation (28) and satisfies the equation with accuracy $O(\sqrt{\varepsilon})$ *as* $\varepsilon \to 0$ *for all* $(x,t) \in \mathbf{R}^2$. *Moreover, it satisfies equation (1) with accuracy* $O(\varepsilon)$ *as* $\tau \to \pm\infty$ *on the set* $\mathbf{R} \times [0;T]$.

In (38) the functions $V_0(x,t,\tau)$, $V_1(x,t,\tau)$ *are defined by formulas (31), (37) according to expressions (36), (33), (32).*

Proof. The statement of the theorem follows from proving Theorem 1. ■

Remark 5. It should be mentioned that asymptotic solution (38) is defined globally, i.e. for all $(x,t) \in \mathbf{R}^2$, in contrast to Theorem 1.

The plots of the main and the first terms, as well as the plots of the first approximation of the asymptotic step-like solution of problem (28) for $\varepsilon = 0.95$ and $\varepsilon = 0.25$ are given in Figs. 2–4 respectively.

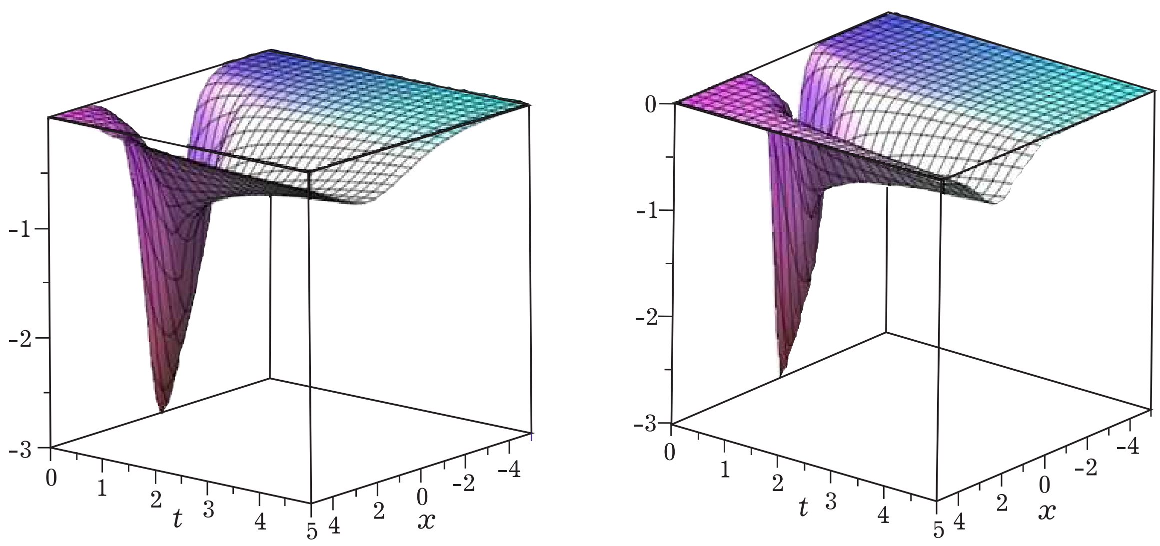

Fig. 2. The main term $V_0(x,t,\varepsilon)$ as $\varepsilon = 0.95$ (at the left) and $\varepsilon = 0.25$ (at the right) for $\gamma = 1$.

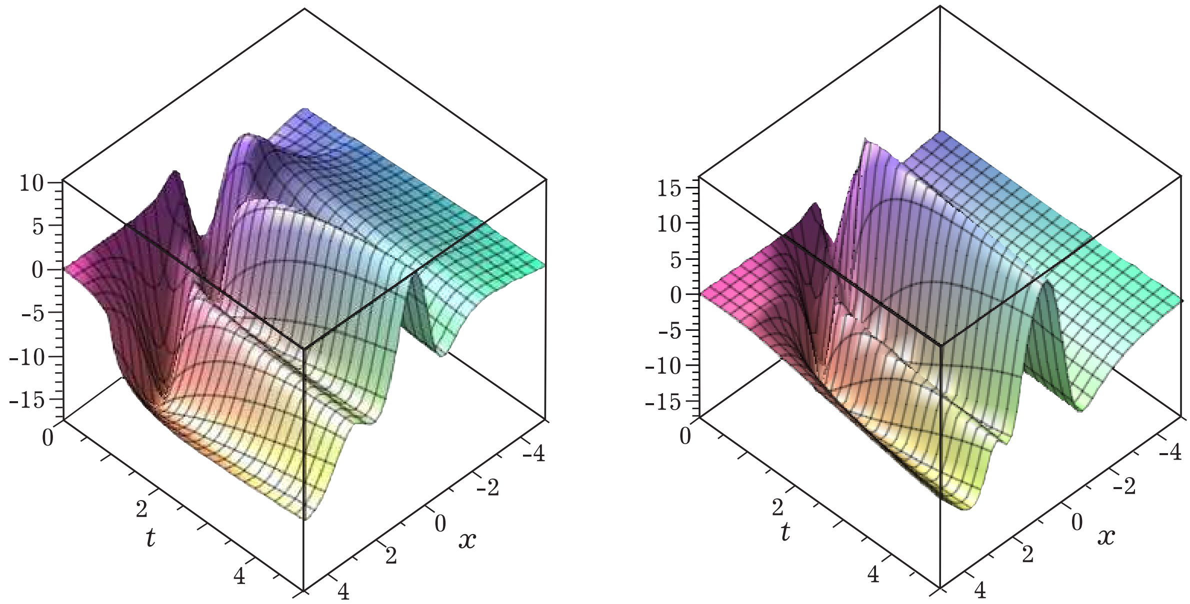

Fig. 3. The first term $V_1(x, t, \tau)$ as $\varepsilon = 0.95$ (at the left) and $\varepsilon = 0.25$ (at the right).

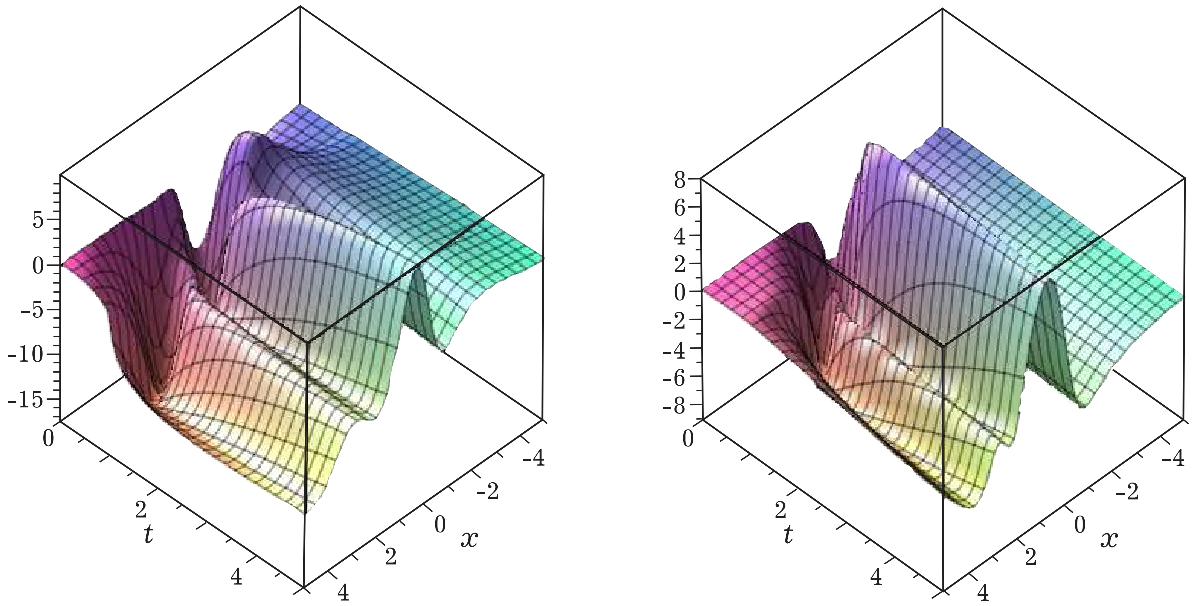

Fig. 4. The first asymptotic approximation $Y_1(x, t, \varepsilon)$ as $\varepsilon = 0.95$ (at the left) and $\varepsilon = 0.25$ (at the right).

A comparison of corresponding graphs for the main and the first terms demonstrates that for a suitable description of the qualitative properties of the asymptotic step-like solutions of the considered Korteweg–de Vries equation, it is necessary to construct at least the first asymptotic approximation.

5. Conclusions

We apply the nonlinear WKB-technique and present the algorithm of constructing the asymptotic step-like solution to the singularly perturbed Korteweg–de Vries equation with variable coefficients in detail. The theorems on the accuracy of the main and the higher terms of the asymptotic solution are proven. There is also presented example of applying the algorithm to the equation of the given type.

The soliton properties of solutions of singularly perturbed equations of integrable type are manifested in some neighborhoods of the so-called discontinuity curve, which in the general case is determined only for a finite interval. In this example, the corresponding discontinuity curve is defined for all values of the argument. As a result, the constructed asymptotic solution is global, i.e., defined for all values of arguments $(x,t) \in \mathbf{R}^2$, that is important.

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Асимптотичні розв'язки сходинкового типу для рівняння Кортевега–де Фріза із сингулярним збуренням та їх точність

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Дана стаття стосується побудови асимптотичних солітоноподібних розв'язків сходинкового типу для рівняння Кортевега–де Фріза зі змінними коефіцієнтами та малим параметром при старшій похідній. Асимптотичний сходинкового типу будується за допомогою нелінійного методу ВКБ. Представлено алгоритм побудови вищих асимптотичних наближень, доведено теорему про їх точність. Запропонований алгоритм продемонстровано на прикладі рівняння із конкретно заданими змінними коефіцієнтами. Знайдено основний доданок та перше асимптотичне наближення для даного прикладу, проведено їх аналіз та представлено твердження про їх асимптотичну точність.

Ключові слова: *рівняння Кортевега–де Фріза, асимптотичний розв'язок типу сходинки, солітон, сингулярне збурення, асимптотичні розв'язки.*