MATHEMATICAL MODELING AND COMPUTING, Vol. 8, No. 4, pp. 705–715 (2021)

Nonlinear elliptic equations with variable exponents involving singular nonlinearity

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(Received 23 May 2021; Accepted 7 June 2021)

In this paper, we prove the existence and regularity of weak positive solutions for a class of nonlinear elliptic equations with a singular nonlinearity, lower order terms and L^1 datum in the setting of Sobolev spaces with variable exponents. We will prove that the lower order term has some regularizing effects on the solutions. This work generalizes some results given in [1–3].

Keywords: *Sobolev spaces with variable exponents, singular nonlinearity, elliptic equation.*

2010 MSC: 35J70, 35J60, 35B65 **DOI:** 10.23939/mmc2021.04.705

1. Introduction

1.1. Introduction of our problem

Consider the nonlinear elliptic problem

$$\begin{cases} -\operatorname{div}\left(a(x)|\nabla u|^{p(x)-2}\nabla u\right) + b(x)u|u|^{r(x)-1} = \dfrac{f}{u^{\gamma(x)}} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where Ω is a bounded open subset of $\mathbb{R}^N$ $(N \geqslant 2)$ with Lipschitz boundary $\partial\Omega$, f is a positive (that is $f(x) \geqslant 0$ and not zero a.e.) function in $L^1(\Omega)$, and $p, r\colon \overline{\Omega} \to (0,+\infty)$, $\gamma\colon \overline{\Omega} \to (0,1)$ are continuous functions and satisfying

$$1 < p^- := \inf_{x\in\overline{\Omega}} p(x) \leqslant p^+ := \sup_{x\in\overline{\Omega}} p(x) < N, \tag{2}$$

$$p(x) - 1 < r(x), \tag{3}$$

$$0 < \gamma^- := \inf_{x\in\overline{\Omega}} \gamma(x) \leqslant \gamma^+ := \sup_{x\in\overline{\Omega}} \gamma(x) < 1, \quad \text{and} \quad |\nabla\gamma| \in L^\infty(\Omega) \tag{4}$$

where $a(x)$, $b(x)$ are measurable functions verifying for some positive numbers α, β, μ, ν the next conditions

$$0 < \alpha \leqslant a(x) \leqslant \beta, \quad 0 < \mu \leqslant b(x) \leqslant \nu. \tag{5}$$

Equations with variable exponents appear in various mathematical models. In some cases, they provide realistic models for the study of natural phenomena in electro-rheological fluids and important applications are related to image processing. We refer the reader to [4–6] and the references therein.

For constant-exponent cases (i.e., $p(x) = p$, $r(x) = r$ and $\gamma(x) = \gamma$), the existence and regularity of solutions to problem (1) are studied in [1, 3, 7, 8]. They proved that the solution is in $W_0^{1,q}(\Omega)$ and $u^{r+\gamma}$ belongs to $L^1(\Omega)$, where $q = \frac{pr}{p+1-\gamma}$. The problem was also considered in [9], when $b(x) = 0$ and γ, p was a constants with $0 \leqslant \gamma < 1$, $f \in L^m(\Omega)$ $(m \geqslant 1)$. The authors in [9] prove the existence and uniqueness results. If $p(x) = 2$ and γ, r were constants, the problem (1) has been treated in [10].

In case without the lower-order term in (1) (i.e., $b(x) = 0$) and the exponent $p(x) \equiv p$, the problem (1), have been treated in [11], under the hypothesis $f \in L^m(\Omega)$ $(m \geqslant 1)$. If $m = 1$ and $0 < \gamma^- \leqslant \gamma(x) \leqslant \gamma^+ < 1$ the authors proved that the solution belongs to $W_0^{1,q}(\Omega)$, where $q = \frac{N(p+\gamma^- -1)}{N+\gamma^- -1}$.

1.2. Preliminary work

For some preliminary results on Lebesgue and Sobolev spaces with variable exponent, we give the definition of $L^{p(\cdot)}(\Omega)$ only, for more details, see [12, 13] or monographs [14, 15]. For an open $\Omega \subset \mathbb{R}^N$, let $p\colon \Omega \to [1, +\infty)$ be a measurable function such that

$$1 < p^- = \operatorname{ess\,inf} p, \quad p^+ = \operatorname{ess\,sup} p < +\infty.$$

Let define Lebesgue space with variable exponent $L^{p(\cdot)}(\Omega)$ to consist of all measurable functions $u\colon \Omega \to \mathbb{R}$ for which the convex modular

$$\rho_{p(\cdot)}(u) = \int_\Omega |u|^{p(x)} dx,$$

is finite. The expression

$$\|u\|_{p(\cdot)} := \|u\|_{L^{p(\cdot)}(\Omega)} = \inf\left\{\lambda > 0,\ \rho_{p(\cdot)}\left(\frac{u}{\lambda}\right) \leqslant 1\right\}$$

defines a norm in $L^{p(\cdot)}(\Omega)$, called the Luxemburg norm, and $\left(L^{p(\cdot)}(\Omega),\ \|u\|_{p(\cdot)}\right)$ is uniformly convex Banach space. Its dual space is isomorphic to $L^{p'(\cdot)}(\Omega)$, where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$. For all $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}(\Omega)$, the Hölder type inequality

$$\left|\int_\Omega u\, v\, dx\right| \leqslant \left(\frac{1}{p^-} + \frac{1}{p'^-}\right) \|u\|_{p(\cdot)} \|v\|_{p'(\cdot)} \leqslant 2\|u\|_{p(\cdot)} \|v\|_{p'(\cdot)},$$

holds true. Sobolev space is defined with variable exponent

$$W^{1,p(\cdot)}(\Omega) = \left\{u \in L^{p(\cdot)}(\Omega) \text{ and } |\nabla u| \in L^{p(\cdot)}(\Omega)\right\},$$

endowed with the norm

$$\|u\|_{1,p(\cdot)} = \|u\|_{W^{1,p(\cdot)}(\Omega)} = \|u\|_{p(\cdot)} + \|\nabla u\|_{p(\cdot)}.$$

The space $\left(W^{1,p(\cdot)}(\Omega), \|u\|_{1,p(\cdot)}\right)$ is reflexive Banach space. Next, we define also

$$W_0^{1,p(\cdot)}(\Omega) = \left\{u \in W^{1,p(\cdot)}(\Omega),\ u = 0 \text{ on } \partial\Omega\right\},$$

endowed with the norm $\|.\|_{1,p(\cdot)}$.

The space $W_0^{1,p(\cdot)}(\Omega)$ is separable and reflexive provided that with $1 < p^- \leqslant p^+ < \infty$.

Proposition 3 (Ref. [16, Poincaré inequality]). There exists a constant $C > 0$, such that

$$\|u\|_{p(\cdot)} \leqslant C\|\nabla u\|_{p(\cdot)}, \quad \forall u \in W_0^{1,p\cdot\cdot)}(\Omega).$$

An important role in manipulating the generalized Lebesgue and Sobolev spaces is played by the modular $\varrho_{p(\cdot)}(\Omega)$ of the space $L^{p(\cdot)}(\Omega)$. We have the following result

Proposition 4 (Ref. [14]). If (u_n), $u \in L^{p(\cdot)}(\Omega)$ and $p^+ < +\infty$, then the following properties hold true:

(i) $\min\left(\rho_{p(\cdot)}(u)^{\frac{1}{p^+}}, \rho_{p(\cdot)}(u)^{\frac{1}{p^-}}\right) \leqslant \|u\|_{p(\cdot)} \leqslant \max\left(\rho_{p(\cdot)}(u)^{\frac{1}{p^+}}, \rho_{p(\cdot)}(u)^{\frac{1}{p^-}}\right)$,

(ii) $\min\left(\|u\|^{p^-}_{p(\cdot)}, \|u\|^{p^+}_{p(\cdot)}\right) \leqslant \rho_{p(\cdot)}(u) \leqslant \max\left(\|u\|^{p^-}_{p(\cdot)}, \|u\|^{p^+}_{p(\cdot)}\right)$,

(iii) $\|u\|_{p(\cdot)} \leqslant \rho_{p(\cdot)}(u) + 1$,

Next, we recall some embedding results regarding variable exponent Lebesgue–Sobolev spaces. If $p, \theta \colon \Omega \to (1, +\infty)$ are Lipschitz continuous function satisfying (2) and $p(x) \leqslant \theta(x) \leqslant p^*(x)$ for any $x \in \Omega$, where $p^*(x) = \frac{Np(x)}{N-p(x)}$, then there exists a compact embedding

$$W^{1,p(\cdot)}(\Omega) \hookrightarrow\hookrightarrow L^{\theta(\cdot)}(\Omega) \hookrightarrow L^{\theta^-}(\Omega), \tag{6}$$

where $\theta^- = \inf\limits_{x\in\overline{\Omega}} \theta(x)$.

1.3. Statement of main result

Definition 1. *Let $f \in L^1(\Omega)$. A function $u \in W_0^{1,1}(\Omega)$ is a weak solution to problem (1), if*

$$\forall \omega \subset\subset \Omega,\ \exists c_\omega > 0 \quad \text{such that} \quad u \geqslant c_\omega \ \text{a.e. in}\ \omega, \quad u^{r(x)} \in L^1(\Omega),$$

and

$$\int_\Omega a(x)|\nabla u|^{p(x)-2}\nabla u \cdot \nabla\varphi\, dx + \int_\Omega b(x)u^{r(x)}\varphi\, dx = \int_\Omega \frac{f\varphi}{u^{\gamma(x)}}\, dx, \tag{7}$$

for every $\varphi \in C_0^1(\Omega)$.

In this paper we will show the following result.

Theorem 1. *Suppose that assumptions (2)–(4) hold. Let $f \in L^1(\Omega)$, $f \geqslant 0$ in Ω and that $f \not\equiv 0$ in Ω i.e. f is a function which is strictly positive on every compactly contained subset of Ω. Assume that*

$$p(x) > 1 + \frac{1-\gamma(x)}{r(x)}. \tag{8}$$

Then, the problem (1) has at least one weak solution $u \in W_0^{1,q(\cdot)}(\Omega)$, with

$$q(x) = \frac{p(x)}{1 + \frac{1-\gamma(x)}{r(x)}}. \tag{9}$$

Moreover $u^{r(x)+\gamma(x)}$ belongs to $L^1(\Omega)$.

Remark 1.

- The assumption (4) implies $1 < q(\cdot) < p(\cdot)$.
- The assumption (3) implies $q(\cdot) > p(\cdot) - 1$.

In order to prove this result, we will work by approximation, "truncating" the singular term $\frac{1}{u^{\gamma(x)}}$ so that it becomes not singular at the origin. We will get some a priori estimates on the solutions u_n of the approximating problems, which will allow us to pass to the limit and find a solution to problem (1).

2. Approximating problems

Hereafter, let denote by T_k the truncation function at the level $k > 0$, defined by $T_k(s) = \max\{-k, \min\{s, k\}\}$ for every $s \in \mathbb{R}$.

Let (f_n) $(f_n > 0)$ be a sequence of bounded functions defined in Ω which converges to $f > 0$ in $L^1(\Omega)$, and verifies the inequalities $f_n \leqslant n$ and $f_n \leqslant f$ for every $n \geqslant 1$ (for example $f_n = T_n(f)$). Consider the following approximate equation

$$\begin{cases} -\operatorname{div}\left(a(x)|\nabla u_n|^{p(x)-2}\nabla u_n\right) + b(x)u_n|u_n|^{r(x)-1} = \dfrac{f_n}{\left(u_n + \frac{1}{n}\right)^{\gamma(x)}} & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{10}$$

Theorem 2. *Let $f \in L^1(\Omega)$, and let $r, p\colon \overline{\Omega} \to (1, +\infty)$, $\gamma\colon \overline{\Omega} \to (0, 1)$ are continuous functions. Assume that (2) and (5) holds true. Then the problem (10) has a nonnegative solution $u_n \in W_0^{1,p(\cdot)}(\Omega)$.*

Lemma 1 (Ref. [17]). *Suppose that the hypotheses of Theorem 2 are satisfied. Then there exists at least one solution $u_n \in W_0^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$ to the problem (10) in the sense that*[1]

$$\int_\Omega a(x)|\nabla u_n|^{p(x)-2}\nabla u_n \cdot \nabla\varphi + \int_\Omega b(x)u_n|u_n|^{r(x)-1}\varphi = \int_\Omega \frac{f_n}{\left(u_n + \frac{1}{n}\right)^{\gamma(x)}}\varphi, \tag{11}$$

for every $\varphi \in W_0^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$.

Proof. This proof derived from Schauder–Tychonov fixed point Theorem (see, for example, [18, p. 581], [19, p. 298]). Let n in $\mathbb{N}$ be fixed, let v be a function in $L^{p(\cdot)}(\Omega)$, we know that the following non-singular problem

$$\begin{cases} -\operatorname{div}\left(a(x)|\nabla w|^{p(x)-2}\nabla w\right) + b(x)|w|^{r(x)-1}w = \dfrac{f_n}{\left(|v| + \frac{1}{n}\right)^{\gamma(x)}} & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases} \tag{12}$$

Therefore, the Minty–Browder Theorem (see, e.g. [20]) implies that problem (12) has a unique solution $w \in W_0^{1,p(x)}(\Omega)$. Let us define a map

$$G\colon L^{p(\cdot)}(\Omega) \to L^{p(\cdot)}(\Omega)$$

and define $w = G(v)$ to be the unique solution of (12). Taking w as test function,

$$\alpha\int_\Omega |\nabla w|^{p(x)} \leqslant \int_\Omega a(x)|\nabla w|^{p(x)-2}\nabla w \cdot \nabla w = \int_\Omega \frac{f_n w}{\left(|v| + \frac{1}{n}\right)^{\gamma(x)}} \leqslant n^{\gamma_+ + 1}\int_\Omega |w|.$$

Using Young's inequality for all $\varepsilon > 0$, Poincaré inequality, (6), and Proposition 2, we get

$$\begin{aligned} \int_\Omega |\nabla w|^{p(x)}dx &\leqslant \frac{C(\varepsilon)n^{\gamma^- + 1}}{\alpha} + \varepsilon\int_\Omega |w|^{p^-}dx \\ &\leqslant \frac{C(\varepsilon)n^{\gamma^- + 1}}{\alpha} + \varepsilon\int_\Omega |\nabla w|^{p^-}dx \\ &\leqslant \frac{C(\varepsilon)n^{\gamma^- + 1}}{\alpha} + \varepsilon\int_\Omega |\nabla w|^{p(x)}dx. \end{aligned}$$

Let choose $\varepsilon = \frac{1}{2}$, then by Proposition 2, we obtain

$$\|\nabla w\|_{p(.)}^{\rho} \leqslant \frac{Cn^{\gamma^- + 1}}{\alpha},$$

where

$$\rho = \begin{cases} p^+ & \text{if } \|\nabla w\|_{p(.)} \geqslant 1, \\ p^- & \text{if } \|\nabla w\|_{p(.)} \leqslant 1. \end{cases}$$

Using the Poincaré inequality on the left hand side, we have

[1] For the sake of simplicity we will use when referring to the integrals the following notation $\int_\Omega f = \int_\Omega f dx$.

$$\|w\|_{p(.)} \leqslant \left(\frac{Cn^{\gamma^{-}+1}}{\alpha}\right)^{\frac{1}{\rho}} = C_n,$$

where C_n is a positive constant independent form v and w, thus, we have that the ball of $L^{p(\cdot)}(\Omega)$ of radius C_n is invariant for G. It is easy to prove, using the Sobolev embedding, that G is both continuous and compact on $L^{p(\cdot)}(\Omega)$, so that by Schauder's fixed point Theorem there exists u_n in $W_0^{1,p(x)}(\Omega)$, for every fixed n such that $u_n = S(u_n)$, i.e., u_n solves

$$\begin{cases} -\operatorname{div}\big(a(x)\,|\nabla u_n|^{p(x)-2}\,\nabla u_n\big) + b(x)\,|u_n|^{r(x)-1}\,u_n = \dfrac{f_n}{\big(|u_n| + \frac{1}{n}\big)^{\gamma(x)}} & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{13}$$

Using as a test function $u_n^- = \min\{u_n, 0\}$, one has $u_n \geqslant 0$. Since the right hand side of (10) belongs to $L^\infty(\Omega)$ and we proceed in the same way as [21] and obtain u_n belongs to $L^\infty(\Omega)$ (although its norm in $L^\infty(\Omega)$ may depend on n). ■

Lemma 2. *Suppose that the hypotheses of Theorem 2 are satisfied. Then the sequence u_n is increasing with respect to n, $u_n > 0$ in Ω, and for every $\omega \subset\subset \Omega$ there exists $c_\omega > 0$ (independent on n) such that*

$$u_n(x) \geqslant c_\omega > 0, \quad \forall x \in \Omega, \quad \forall n \in \mathbb{N}. \tag{14}$$

Moreover there exists the pointwise limit $u \geqslant c_\omega$ of the sequence u_n.

Proof. [Proof of the Lemma 2] Due to $0 \leqslant f_n \leqslant f_{n+1}$ and $\gamma(x) > 0$,

$$-\operatorname{div}\big(a(x)|\nabla u_n|^{p(x)-2}\nabla u_n\big) + b(x)u_n^{r(x)} = \frac{f_n}{\big(u_n + \frac{1}{n}\big)^{\gamma(x)}} \quad \leqslant \frac{f_{n+1}}{\Big(u_n + \frac{1}{n+1}\Big)^{\gamma(x)}}.$$

So that

$$\begin{aligned} &-\operatorname{div}\big(a(x)|\nabla u_n|^{p(x)-2}\nabla u_n\big) + \operatorname{div}\big(a(x)|\nabla u_{n+1}|^{p(x)-2}\nabla u_{n+1}\big) + b(x)u_n^{r(x)} - b(x)u_{n+1}^{r(x)} \\ &\qquad \leqslant f_{n+1}\left[\frac{\Big(u_{n+1} + \frac{1}{n+1}\Big)^{\gamma(x)} - \Big(u_n + \frac{1}{n+1}\Big)^{\gamma(x)}}{\Big(u_n + \frac{1}{n+1}\Big)^{\gamma(x)}\Big(u_{n+1} + \frac{1}{n+1}\Big)^{\gamma(x)}}\right]. \end{aligned} \tag{15}$$

Let choose $(u_n - u_{n+1})_+ = \max\{u_n - u_{n+1}, 0\}$ as test function in (15). In the left hand side we use (5) and the monotonicity of the $p(x)$-laplacian operator as well as the monotonicity of the function $t \to |t|^{r(x)-1}t$. For the right hand, using the fact that $\gamma(x) \geqslant 0$ and $f_{n+1} \geqslant 0$, it follows

$$\left[\left(u_{n+1} + \frac{1}{n+1}\right)^{\gamma(x)} - \left(u_n + \frac{1}{n+1}\right)^{\gamma(x)}\right](u_n - u_{n+1})_+ \leqslant 0. \tag{16}$$

By (16), one can get

$$\alpha \int_\Omega |\nabla(u_n - u_{n+1})_+|^{p(x)} \leqslant 0,$$

which implies that $(u_n - u_{n+1})_+ = 0$ a.e. in Ω, that is, $u_n \leqslant u_{n+1}$ for every $n \in \mathbb{N}$. Since the sequence (u_n) is increasing with respect to n, we only need to prove that (14) holds for u_1. Due to Lemma 1, $u_1 \in L^\infty(\Omega)$, i.e., there exists a constant c_0 (depending only on Ω and N) such that $\|u_1\|_{L^\infty(\Omega)} \leqslant c\|f_1\|_{L^\infty}(\Omega) \leqslant c_0$, then

$$-\operatorname{div}\big(a(x)|\nabla u_1|^{p(x)-2}\nabla u_1\big) + b(x)u_1^{r(x)} = \frac{f_1}{(u_1+1)^{\gamma(x)}} \geqslant \frac{f_1}{(c_0+1)^{\gamma(x)}} \geqslant 0.$$

Since $\frac{f_1}{(c_0+1)^{\gamma(x)}}$ is not identically zero, the strong maximum principle implies that $u_1 > 0$ in Ω (see [22]). Since $u_n \geqslant u_1$ for every $n \in \mathbb{N}$, (14) holds for u_n (with the same constant c_ω which is then independent on n). ■

Proof. [Proof of the Theorem 2] In virtue of the Lemma 1 and Lemma 2, there exists at least one nonnegative weak solution $u_n \in W_0^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$ of problem (10). ■

3. A priori estimates

In the remainder of this section, we denote by C_i $i = 1, 2, 3, \dots$ various positive constants depending only on the data of the problem, but not on n.

Lemma 3. *Let $k > 0$ be fixed. The sequence $(T_k(u_n))$, where u_n is a solution to (13), is bounded in $W_0^{1,p(.)}(\Omega)$.*

Proof. Taking $T_k(u_n)$ as a test function in (13), one can obtain

$$\int_\Omega a(x)|\nabla u_n|^{p(x)-2}\nabla u_n \cdot \nabla T_k(u_n) + \int_\Omega b(x)u_n^{r(x)}T_k(u_n) = \int_\Omega \frac{f_n}{\left(|u_n| + \frac{1}{n}\right)^{\gamma(x)}}T_k(u_n).$$

Using (5), $f_n \leqslant f$, $T_k(u_n) \neq 0$, and dropping the nonegative order term,

$$\int_\Omega |\nabla T_k(u_n)|^{p(x)}dx \leqslant \frac{k}{\alpha}\|f\|_{L^1(\Omega)}. \tag{17}$$

As a consequence of Proposition 4 and (17), $T_k(u_n)$ is bounded in $W_0^{1,p(.)}(\Omega)$. ■

Lemma 4. *Suppose that the hypotheses of Theorem 1 are satisfied. Then, the sequence u_n is bounded in $W_0^{1,q(.)}(\Omega)$, where $q(\cdot)$ is given by (9). Moreover $(u_n^{r(x)+\gamma(x)})$ belongs to $L^1(\Omega)$.*

Proof. Taking $\varphi(x,u) = (u_n+1)^{\gamma(x)} - 1$, as test function in (13), by (4), (5), and the fact that for a.e. $x \in \Omega$

$$\nabla\varphi(x,u) = \nabla\gamma(x)(u_n+1)^{\gamma(x)}\ln(u_n+1) + \gamma(x)\frac{\nabla u_n}{(u_n+1)^{\gamma(x)}},$$

we obtain

$$\gamma^-\alpha\int_\Omega \frac{|\nabla u_n|^{p(x)}}{(1+u_n)^{1-\gamma(x)}} + \mu\int_\Omega u_n^{r(x)}\left[(u_n+1)^{\gamma(x)} - 1\right]$$
$$\leqslant C_1\int_\Omega |\nabla u_n|^{p(x)-1}(u_n+1)^{\gamma(x)}\ln(u_n+1) + \int_\Omega f\frac{\left[(u_n+1)^{\gamma(x)} - 1\right]}{\left(u_n + \frac{1}{n}\right)^{\gamma(x)}}.$$

Using the fact that $|u_n|^{\theta(x)} \geqslant 2^{1-\theta^+}(1+u_n)^{\theta(x)} - 1$ (here $\theta(x) = r(x)$ and $\theta(x) = \gamma(x)$),

$$\gamma^-\alpha\int_\Omega \frac{|\nabla u_n|^{p(x)}}{(1+u_n)^{1-\gamma(x)}} + 2^{1-r^+}\mu\int_\Omega (u_n+1)^{r(x)+\gamma(x)}$$
$$\leqslant C_2 + \frac{1}{2^{1-\gamma^+}}\int_\Omega f + C_1\int_\Omega |\nabla u_n|^{p(x)-1}(u_n+1)^{\gamma(x)}\ln(u_n+1). \tag{18}$$

The last term in (18) can be estimated by application of Young's inequality

$$(1+u_n)^{\gamma(x)}\ln(1+u_n)|u_n|^{p(x)-1} = (1+u_n)^{1-\frac{1-\gamma(x)}{p(x)}}\ln(1+u_n)|u_n|^{p(x)-1}(1+u_n)^{-\frac{(1-\gamma(x))(p(x)-1)}{p(x)}}$$
$$\leqslant C_3(1+u_n)^{p(x)-(1-\gamma(x))}(\ln(1+u_n))^{p(x)} + \varepsilon\frac{|\nabla u_n|^{p(x)}}{(u_n+1)^{1-\gamma(x)}}. \tag{19}$$

Let choose $\varepsilon = \frac{\gamma^-\alpha}{2C_1}$, then by (18) and (19) one can obtain

$$\frac{1}{2}\int_\Omega \frac{|\nabla u_n|^{p(x)}}{(1+u_n)^{1-\gamma(x)}} + 2^{1-r^+}\mu\int_\Omega (u_n+1)^{r(x)+\gamma(x)}$$
$$\leqslant C_4 + C_5\int_\Omega (u_n+1)^{p(x)-(1-\gamma(x))}(\ln(u_n+1))^{p(x)}. \tag{20}$$

The hypothesis (3) implies $(1+t)^{p(x)-1-r(x)-c}(\ln(1+t))^{p(x)}$ is bounded for all $x \in \overline{\Omega}$ and $t \in \mathbb{R}^+$. By another application of Youngs inequality, the next is true

$$(u_n+1)^{p(x)-(1-\gamma(x))}(\ln(u_n+1))^{p(x)} = (u_n+1)^{r(x)+\gamma(x)+c}(u_n+1)^{p(x)-1-r(x)-c}(\ln(u_n+1))^{p(x)} \leqslant \varepsilon(u_n+1)^{r(x)+\gamma(x)} + C_6. \tag{21}$$

Therefore, by (20), (21),

$$\int_\Omega \frac{|\nabla u_n|^{p(x)}}{(1+u_n)^{1-\gamma(x)}} + \int_\Omega (u_n+1)^{r(x)+\gamma(x)} \leqslant C_7. \tag{22}$$

Since $r(x) \geqslant 0$ and $\gamma(x) \geqslant 0$, then

$$\int_\Omega u_n^{r(x)} \leqslant \int_\Omega (u_n+1)^{r(x)} \leqslant \int_\Omega (u_n+1)^{r(x)+\gamma(x)} \leqslant C_7. \tag{23}$$

The inequality (23) implies that $(u_n^{r(x)+\gamma(x)})$ is bounded in $L^1(\Omega)$. Let $q(x) < p(x)$, using Young's inequality and (22), it follows

$$\begin{aligned}\int_\Omega |\nabla u_n|^{q(x)} &= \int_\Omega \frac{|\nabla u_n|^{q(x)}}{(u_n+1)^{(1-\gamma(x))\frac{q(x)}{p(x)}}} \\ &\leqslant C_8 \int_\Omega \frac{|\nabla u_n|^{p(x)}}{(u_n+1)^{1-\gamma(x)}} + C_9 \int_\Omega (u_n+1)^{(1-\gamma(x))\frac{q(x)}{p(x)-q(x)}} \\ &\leqslant C_{10} + C_9 \int_\Omega (u_n+1)^{(1-\gamma(x))\frac{q(x)}{p(x)-q(x)}}.\end{aligned} \tag{24}$$

Set

$$(1-\gamma(x))\frac{q(x)}{p(x)-q(x)} = r(x).$$

Then this equality and (23)–(24) yield

$$\int_\Omega |\nabla u_n|^{q(x)} \leqslant C_{11}. \tag{25}$$

∎

Lemma 5. *Let u_n be a solution to problem* (13)*. Then*

$$\int_{\{u_n>k\}} u_n^{r(x)} \leqslant \frac{1}{\mu k^{\gamma^+}} \int_{\{u_n>k\}} f, \quad \forall k>0, \quad \lim_{|E|\to 0} \int_E u_n^{r(x)} = 0,$$

uniformly with respect to n, for every measurable subset E in Ω.

Proof. Let $k > 0$ and ψ_j be a sequence of increasing, positive, uniformly bounded $C^\infty(\Omega)$ functions, such that $\psi_j(s) \to \chi_{\{s>k\}}$, as $j \to +\infty$. Choosing $\psi_j(u_n)$ in (13), using (5),

$$\mu \int_\Omega u_n^{r(x)} \psi_j(u_n) \leqslant \int_\Omega \frac{f_n}{\left(u_n+\frac{1}{n}\right)^{\gamma(x)}} \psi_j(u_n).$$

Therefore, as j tends to infinity and that $k^{\gamma^-} \leqslant \left(k+\frac{1}{n}\right)^{\gamma^-} \leqslant \left(u_n+\frac{1}{n}\right)^{\gamma(x)}$ in the set $\{u_n > k\}$,

$$\int_{\{u_n>k\}} u_n^{r(x)} \leqslant \frac{1}{\mu k^{\gamma^-}} \int_{\{u_n>k\}} f. \tag{26}$$

By (26), for any measurable subset E in Ω, we have

$$\int_E u_n^{r(x)} = \int_{E\cap\{u_n\leqslant k\}} u_n^{r(x)} + \int_{E\cap\{u_n>k\}} u_n^{r(x)} \leqslant k^{r^+}|E| + \frac{1}{\mu k^{\gamma^-}} \int_{\{u_n>k\}} f. \tag{27}$$

Since $f \in L^1(\Omega)$, we may choose $k = k_\varepsilon$ large enough such that

$$\int_{\{u_n>k\}} f \leqslant \varepsilon. \tag{28}$$

Therefore, the estimates (27)–(28) imply that

$$\int_E u_n^{r(x)} \leqslant k_\varepsilon^{r^+}|E| + \frac{\varepsilon}{\mu k_\varepsilon^{\gamma^-}},$$

and lemma is thus proved. ■

Lemma 6. *Let u_n be a solution to problem* (13)*. Then*

$$\lim_{|E|\to 0}\int_E |\nabla u_n|^{q(x)} = 0, \quad \textit{uniformly with respect to } n, \tag{29}$$

for every measurable subset E in Ω and $q(\cdot)$ given by (9)*.*

Proof. Let $\varepsilon > 0$, by Lemma 4, we may choose $k = k_\varepsilon$ large enough such that

$$\int_{E\cap\{u_n>k\}} |\nabla u_n|^{q(x)} \leqslant \varepsilon. \tag{30}$$

From the estimate (17) and that $q(x) < p(x)$, it comes

$$\int_{E\cap\{u_n\leqslant k\}} |\nabla T_k(u_n)|^{q(x)} \leqslant \varepsilon. \tag{31}$$

By (30) and (31), for any measurable subset E in Ω, we have

$$\int_E |\nabla u_n|^{q(x)} = \int_{E\cap\{u_n\leqslant k\}} |\nabla u_n|^{q(x)} + \int_{E\cap\{u_n>k\}} |\nabla u_n|^{q(x)} \leqslant 2\varepsilon.$$

As a result $|\nabla u_n|^{q(x)}$ is equiintegrable in $L^1(\Omega)$. Thus (29) is proved. ■

4. Proof of the main theorem

By Lemma 3, the sequence $(u_n)_n$ is bounded in $W_0^{1,q(\cdot)}(\Omega)$. Therefore, there exists a function $u \in W_0^{1,q(\cdot)}(\Omega)$ such that (up to a subsequence)

$$\begin{cases} u_n \rightharpoonup u & \text{in } W_0^{1,q(\cdot)}(\Omega), \\ u_n \to u & \text{a.e. in } \Omega. \end{cases} \tag{32}$$

Proposition 5. If the sequence $T_k(u_n)$ of the truncates of the solutions u_n of (13) is bounded in $W_0^{1,p(\cdot)}(\Omega)$. Then

$$T_k(u_n) \to T_k(u) \quad \text{strongly in } \; W_0^{1,p(\cdot)}(\Omega), \tag{33}$$

as $n \to \infty$, for every $k > 0$. In particular $\nabla u_n \to \nabla u$ a.e. in Ω.

Proof. By Lemma 3 $T_k(u_n)$ is bounded in $W_{loc}^{1,p(\cdot)}(\Omega)$, it weakly converges in this space to its pointwise limit $T_k(u)$. Moreover, since $f_n \geqslant 0$ and $u_n \geqslant 0$ a.e., we have that

$$-\mathrm{div}\big(a(x)|\nabla u_n|^{p(x)-2}\nabla u_n\big) + b(x)u_n^{r(x)} \geqslant 0,$$

for all $n \in \mathbb{N}$ and $k > 0$.

Now we fix $\phi \in C_0^1(\Omega)$ such that $0 \leqslant \phi \leqslant 1$ on Ω and such that $\phi \equiv 1$ on a fixed subset ω of Ω. Then, thanks to the monotonicity of the $p(x)$-laplacian operator, (5), and that $T_k(u_n) \geqslant T_k(u)$ (since $u_n \to u \leqslant u_n$), we can conclude that the following holds

$$\begin{aligned}
0 < \beta & \int_\omega \Big(|\nabla T_k(u_n)|^{p(x)-2}\nabla T_k(u_n) - |\nabla T_k(u)|^{p(x)-2}\nabla T_k(u) \Big) \cdot \nabla (T_k(u_n) - T_k(u)) \\
& + \nu \int_\omega u_n^{r(x)} (T_k(u_n) - T_k(u)) \\
& = \beta \int_\Omega \Big(|\nabla T_k(u_n)|^{p(x)-2}\nabla T_k(u_n) - |\nabla T_k(u)|^{p(x)-2}\nabla T_k(u) \Big) \cdot \nabla (T_k(u_n) - T_k(u))\phi \\
& + \nu \int_\Omega u_n^{r(x)} (T_k(u_n) - T_k(u))\phi \\
& = \beta \int_\Omega |\nabla T_k(u_n) \nabla T_k(u_n)|^{p(x)-2} \nabla [(T_k(u_n) - T_k(u))\phi] \\
& - \beta \int_\Omega |\nabla T_k(u_n)|^{p(x)-2}\nabla T_k(u_n) \cdot \nabla \phi [T_k(u_n) - T_k(u)] \\
& - \beta \int_\Omega |\nabla T_k(u)|^{p(x)-2}\nabla T_k(u) \cdot \nabla (T_k(u_n) - T_k(u))\phi \\
& + \nu \int_\Omega u_n^{r(x)} (T_k(u_n) - T_k(u))\phi
\end{aligned} \tag{34}$$

By Lemma 5, we obtain

$$u_n^{r(x)} \to u^{r(x)} \quad \text{strongly in} \;\; L^1(\Omega).$$

Therefore, since $T_k(u_n)$ strongly converges to $T_k(u)$ in $L^{p(\cdot)}(\Omega)$ (Lemma 3),

$$\int_\Omega u_n^{r(x)} (T_k(u_n) - T_k(u))\phi \to 0, \quad \text{as} \;\; n \to \infty. \tag{35}$$

It's well known that $|\nabla T_k(u)|^{p(x)-2}\nabla T_k(u) \in L_{loc}^{p'(\cdot)}(\Omega)$, and $\nabla (T_k(u_n) - T_k(u))\phi$ tends to zero weakly in $L^p(\Omega)$, therefore one can get

$$\int_\Omega |\nabla T_k(u)|^{p(x)-2}\nabla T_k(u) \cdot \nabla (T_k(u_n) - T_k(u))\phi \to 0, \quad \text{as} \;\; n \to \infty. \tag{36}$$

$\nabla \phi [T_k(u_n) - T_k(u)]$ strongly converges to zero in $L^{p(\cdot)}(\Omega)$. Thus

$$\int_\Omega |\nabla T_k(u_n)|^{p(x)-2}\nabla T_k(u_n) \cdot \nabla \phi [T_k(u_n) - T_k(u)] \to 0, \quad \text{as} \;\; n \to \infty. \tag{37}$$

From (34)–(37),

$$\int_\omega \Big(|\nabla T_k(u_n)|^{p(x)-2}\nabla T_k(u_n) - |\nabla T_k(u)|^{p(x)-2}\nabla T_k(u) \Big) \cdot \nabla (T_k(u_n) - T_k(u)) \to 0,$$

then $T_k(u_n)$ strongly converges to $T_k(u)$ in $W_0^{1,p(\cdot)}(\omega)$ for all $k > 0$, i.e., since ω is arbitrary, that $T_k(u_n)$ strongly converges to $T_k(u)$ in$W_{loc}^{1,p(\cdot)}(\Omega)$.

Choosing $\phi \equiv 1$ and repeating the same proof, we obtain that $T_k(u_n)$ strongly converges to $T_k(u)$ in $W_0^{1,p(\cdot)}(\Omega)$, then $\nabla u_n \to \nabla u$ a.e. in Ω. ■

Proof. [Proof of the Theorem 1] It is easy to pass to the limit in the right hand side of problems (13). On the other hand, using Lemma 2,

$$0 \leqslant \left| \frac{f_n \varphi}{\left(u_n + \frac{1}{n}\right)^{\gamma(x)}} \right| \leqslant \frac{\|\varphi\|_\infty}{c_\omega^{\gamma^-}} f,$$

for every $\varphi \in C_0^1(\Omega)$, using Lebesgue Theorem and (32), it follows that

$$\lim_{n\to\infty} \int_\Omega \frac{f_n \varphi}{\left(u_n + \frac{1}{n}\right)^{\gamma(x)}} = \int_\Omega \frac{f\varphi}{u^{\gamma(x)}}. \tag{38}$$

By the same argument, we get

$$\lim_{n\to\infty}\int_\Omega b(x)u_n^{r(x)}\varphi=\int_\Omega u^{r(x)}\varphi. \tag{39}$$

For the first term, by Proposition 5 we have that

$$a(x)|\nabla u_n|^{p(x)-2}\nabla u_n\to a(x)|\nabla u|^{p(x)-2}\nabla u \quad \text{a.e. in } \Omega,$$

furthermore $a(x)|\nabla u_n|^{p(x)-2}\nabla u_n$ is majorette by $\beta|\nabla u_n|^{p(x)-1}$. Observe that $p(x)-1<q(x)$, by Lemma 6 and Vitali's Theorem, we have

$$\lim_{n\to\infty} a(x)|\nabla u_n|^{p(x)-2}\nabla u_n\cdot\nabla\varphi=\int_\Omega a(x)|\nabla u|^{p(x)-2}\nabla u\cdot\nabla\varphi. \tag{40}$$

Hence from (38)–(39) we can deduce (7). ■

[1] Boccardo L., Croce G. The impact of a lower order term in a Dirichlet problem with a singular nonlinearity. Portugaliae Mathematica. **76** (3–4), 4075 (2019).

[2] Oliva F. Regularizing effect of absorption terms in singular problems. Journal of Mathematical Analysis and Applications. **472** (1), (2019).

[3] Sbai A., El Hadfi Y. Regularizing effect of absorption terms in singular and degenerate elliptic problems. arXiv preprint arXiv:2008.03597 (2020).

[4] Callegari A., Nashman A. A nonlinear singular boundary-value problem in the theory of pseudoplastic fluids. SIAM J. Appl. Math. **38**, 275–281 (1980).

[5] Keller H. B., Chohen D. S. Some positive problems suggested by nonlinear heat generators. Indiana Univ. Math. J. **16** (12), 1361–1376 (1967).

[6] Ambrosetti A., Brézis H., Cerami G. Combined effects of concave and convex nonlinearities in some elliptic problems. J. Funct. Anal. **122** (2), 519–543 (1994).

[7] El Ouardy M., El Hadfi Y., Ifzarne A. Existence and regularity results for a singular parabolic equations with degenerate coercivity. Discrete & Continuous Dynamical Systems – S. 1–25 (2021).

[8] Sbai A., El Hadfi Y. Degenerate elliptic problem with a singular nonlinearity. arXiv e-prints arXiv:2005.08383 (2020).

[9] Canino A., Sciunzi B., Trombetta A. Existence and uniqueness for p-Laplace equations involving singular nonlinearities. Nonlinear Differential Equations and Applications. **23**, Article number: 8 (2016).

[10] De Cave L M., Oliva F. On the regularizing effect of some absorption and singular lower order terms in classical Dirichlet problem with L^1 data. Journal of Elliptic and Parabolic Equations. **2**, 73–85 (2016).

[11] Chu Y., Gao R., Sun Y. Existence and regularity of solutions to a quasilinear elliptic problem involving variable sources. Boundary Value Problems. **2017**, Article number: 1 (2017).

[12] Fan X., Zhao D. On the spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$. Journal of Mathematical Analysis and Applications. **263** (2), 424–446 (2001).

[13] Kováĉik O., Ràkosník J. On spaces $L^{p(x)}(\Omega)$ and $W^{k,p(x)}(\Omega)$. Czechoslovak Mathematical Journal. **41** (4), 592–618 (1991).

[14] Diening L., Harjulehto P., Hästö P., Ruzicka M. Lebesgue and Sobolev spaces with variable exponents. Lecture notes in mathematics. Heidelberg, Springer (2017).

[15] Rŭžička M. Electrorheological fluids: modeling and mathematical theory. Lecture notes in mathematics. Berlin, Springer-Verlag (2000).

[16] Carmona J., Martiínez-Aparicio Pedro J., Rossi Julio D. A singular elliptic equation with natural growth in the gradient and a variable exponent. Nonlinear Differential Equations and Applications. **22** (6), 1935–1948 (2015).

[17] Fan X. Positive solution to $p(x)$-Laplacian Dirichlet problems with sign-changing non-linearities. Glasgow Mathematical Journal. **52** (3), 505–516 (2010).

[18] Papageorgiou N. S., Winkert P. Applied Nonlinear Functional Analysis. De Gruyter, Berlin (2018).

[19] Papageorgiou N. S., Rădulescu V. D., Repovš D. D. Nonlinear Analysis – Theory and Methods. Springer, Cham (2019).

[20] Brezis H. Analyse fonctionnelle theorie et applications. New York, Springer (2010).

[21] Stampacchia G. Le problème de Dirichlet pour les équations elliptiques du second ordre à coefficients discontinus. Annales de l'Institut Fourier. **15** (1), 189–257 (1965).

[22] Zhang Q. A strong maximum principle for differential equations with nonstandard $p(x)$-growth conditions. Journal of Mathematical Analysis and Applications. **312** (1), 24–32 (2005).

Нелінійні еліптичні рівняння зі змінними показниками, що включають сингулярну нелінійність

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У статті доводиться існування та регулярність слабких додатних розв'язків для класу нелінійних еліптичних рівнянь із нелінійною сингулярністю, членами нижчого порядку та L^1 в заданні просторів Соболєва зі змінними показниками. Доведено, що член нижчого порядку має деякий регуляризуючий вплив на розв'язок. Ця робота узагальнює деякі результати, наведені в [1–3] .

Ключові слова: *простори Соболєва зі змінними показниками, сингулярна нелінійність, еліптичне рівняння.*