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Nonlinear the first kind Fredholm integro-differential first-order equation with degenerate kernel and nonlinear maxima

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In this note, the problems of solvability and construction of solutions for a nonlinear Fredholm one-order integro-differential equation with degenerate kernel and nonlinear maxima are considered. Using the method of degenerate kernel combined with the method of regularization, we obtain an implicit the first-order functional-differential equation with the nonlinear maxima. Initial boundary conditions are used to ensure the solution uniqueness. In order to use the method of a successive approximations and prove the one value solvability, the obtained implicit functional-differential equation is transformed to the nonlinear Volterra type integro-differential equation with the nonlinear maxima.

Keywords: *integro-differential equation, nonlinear functional-differential equation, degenerate kernel, nonlinear maxima, regularization, one value solvability.*

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1. Formulation of the problem

Integro-differential equations are the mathematical models to describe many physical phenomena and the operation in technical systems. Analytical and iterative methods are important in application of integro-differential equations [1–8].

In this paper, we study the initial value problem of one value solvability and construction of solutions of a nonlinear the first-order Fredholm integro-differential equation with the degenerate kernel and the nonlinear maxima. It is easy to replace the given equation by the implicit differential equation in case, when a kernel of integral is degenerate one. This equation is convenient to transform into Volterra integro-differential equation for solving by the method of successive approximations. The integral and integro-differential equations with degenerate kernels were considered by many authors (see, for example [9–20]). So, using the method of degenerate kernel combined with the regularization method, we obtain an implicit functional-differential equation with the nonlinear maxima. It is known fact that Fredholm functional integro-differential equation of the first kind is ill-posed. So, we use the initial boundary conditions to ensure the uniqueness of the solution. In order to use the successive approximations method, we transform the implicit functional-differential equation to the nonlinear Volterra type functional integro-differential equation, which is ill-posed, too. The one value solvability of this problem we have proved by given initial boundary conditions.

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On the segment $[0;T]$ the following nonlinear Fredholm integro-differential equation of first kind and first-order is considered

$$\lambda \int_0^T K(t,s)\, F\big(s, u(s), \max\{u(\tau)|\, \tau \in [h_1(s,u(s)); h_2(s,u(s))]\}, \dot{u}(s)\big)\, ds = f(t) \tag{1}$$

under the following conditions

$$\left[\begin{array}{l} u(0) = \varphi_{01} = \text{const}, \\ \dot{u}(0) = \varphi_{02} = \text{const}, \\ u(t) = \varphi_1(t), \quad t \in [-h_{01}; 0], \\ u(t) = \varphi_2(t), \quad t \in [T; T + h_{02}], \end{array}\right. \tag{2}$$

where $0 < T$ is given real number, λ is nonzero parameter of marching, $F(t,u,v,\vartheta) \in C([0;T] \times X \times X \times X)$, $h_i(t,u) \in C([0;T] \times X)$, $-h_{01} < h_1(t,u) < h_2(t,u) < T + h_{02}$, $0 < h_{0i} = \text{const}$, $i = 1,2$, $\varphi_1(t) \in C[-h_{01};0]$, $\varphi_2(t) \in C[T;T+h_{02}]$, $K(t,s) = \sum_{i=1}^k a_i(t) b_i(s)$, $0 \neq a_i(t), b_i(s) \in C[0;T]$, X is closed set on real number set. Here it is assumed that each system of functions $a_i(t)$, $i = \overline{1,k}$, and $b_i(s)$, $i = \overline{1,k}$, is linearly independent, $\varphi_1(0) = \varphi_{01}$, $\varphi_2(T) = u(T)$.

2. Method of degenerate kernel

Taking into account the degeneracy of the kernel, equation (1) is written in the following form

$$\lambda \int_0^T \sum_{i=1}^k a_i(t)\, b_i(s)\, F\big(s, u(s), \max\{u(\tau)|\tau \in [h_1(s,u(s)); h_2(s,u(s))]\}, \dot{u}(s)\big)\, ds = f(t). \tag{3}$$

Using the notation

$$\vartheta(t) = F\big(t, u(t), \max\{u(\tau)|\tau \in [h_1(t,u(t)); h_2(t,u(t))]\}, \dot{u}(t)\big) \tag{4}$$

and introducing new unknown function $\vartheta_\varepsilon(t)$, we obtain from (3) approximation Fredholm second kind integral equation with the small parameter

$$\varepsilon\, \vartheta_\varepsilon(t) = f(t) - \lambda \int_0^T \sum_{i=1}^k a_i(t)\, b_i(s)\, \vartheta_\varepsilon(s)\, ds, \tag{5}$$

where $0 < \varepsilon$ is the small parameter and

$$\lim_{\varepsilon \to 0} \vartheta_\varepsilon(t) = \vartheta(t). \tag{6}$$

Using new notation

$$\alpha_i = \int_0^T b_i(s)\, \vartheta_\varepsilon(s)\, ds, \tag{7}$$

the integral equation (5) can be rewritten as follows

$$\vartheta_\varepsilon(t) = \frac{1}{\varepsilon} \left[f(t) - \lambda \sum_{i=1}^k a_i(t)\, \alpha_i \right]. \tag{8}$$

Substituting (8) into (7), we obtain the system of the linear equations (SLE)

$$\alpha_i + \lambda \sum_{j=1}^k \alpha_j\, A_{ij} = B_i, \quad i = \overline{1,k}, \tag{9}$$

where

$$A_{ij} = \frac{1}{\varepsilon} \int_0^T b_i(s)\, a_j(s)\, ds, \quad B_i = \frac{1}{\varepsilon} \int_0^T b_i(s)\, f(s)\, ds. \tag{10}$$

Consider the following determinants:

$$\Delta(\lambda) = \begin{vmatrix} 1+\lambda A_{11} & A_{12} & \dots & A_{1k} \\ A_{21} & 1+\lambda A_{22} & \dots & A_{2k} \\ \dots & \dots & \dots & \dots \\ A_{k1}^m & A_{k2} & \dots & 1+\lambda A_{kk} \end{vmatrix} \neq 0, \tag{11}$$

$$\Delta_i(\lambda) = \begin{vmatrix} 1+\lambda A_{11} & \dots & A_{1(i-1)} & B_1 & A_{1(i+1)} & \dots & A_{1k} \\ A_{21} & \dots & A_{2(i-1)} & B_2 & A_{2(i+1)} & \dots & A_{1k} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ A_{k1} & \dots & A_{k(i-1)} & B_k & A_{k(i+1)} & \dots & 1+\lambda A_{kk} \end{vmatrix}, \quad i = \overline{1,k}.$$

SLE (9) is uniquely soluble for any finite right-hand sides, if the nondegeneracy condition (11) of the Fredholm determinant is satisfied. The determinant $\Delta(\lambda)$ in (11) is a polynomial with respect to λ of degree not greater than k. The equation $\Delta(\lambda) = 0$ has at most k different real roots. We denote them by μ_l ($l = \overline{1,p}$, $1 \leqslant p \leqslant k$). Then $\lambda = \mu_l$ are called irregular values of the spectral parameter λ. Other values of the spectral parameter $\lambda \neq \mu_l$ are called regular ones. The solutions of SLE (9) for the regular values of parameter λ are written as

$$\alpha_i = \frac{\Delta_i(\lambda)}{\Delta(\lambda)}, \quad i = \overline{1,k}. \tag{12}$$

Substituting (12) into (8),

$$\vartheta_\varepsilon(t) = \frac{1}{\varepsilon} \left[f(t) - \lambda \sum_{i=1}^{k} a_i(t) \frac{\Delta_i(\lambda)}{\Delta(\lambda)} \right]. \tag{13}$$

By virtue of formula (10), we suppose that

$$f(t) = \lambda \sum_{i=1}^{k} a_i(t)\, c_i, \quad c_i - \lambda \frac{\Delta_i(\lambda)}{\Delta(\lambda)} = \varepsilon C_i, \tag{14}$$

where c_i, $C_i = \text{const}$, $i = \overline{1,k}$.

The parameter λ is marching parameter between free term function $f(t)$ and kernel of integral equation (1). So, we choose one of the regular λ values satisfying the first of condition (14). Then, taking into account limit passing formula (6), from (13) we obtain

$$\vartheta(t) = \lambda \sum_{i=1}^{k} C_i\, a_i(t). \tag{15}$$

Now the function $\vartheta(t)$ is known and defined by the formula (15). We rewrite the implicit equation (4) as

$$G\big(t, u(t), \max\{u(\tau) | \tau \in [h_1(t, u(t)); h_2(t, u(t))]\}, \dot{u}(t)\big) = 0 \tag{16}$$

with given conditions (2), where $G = F - \vartheta$.

3. Transform into nonlinear Volterra type integro-differential equation

Studying the solvability of implicit functional-differential equation (16) we use the method of successive approximations combined with the method of compressing mapping. However, it is impossible to apply the method of successive approximations to the equation (16) with the nonlinear maxima directly. Therefore, the following method is proposed.

On the segment $[0;T]$ the arbitrary positive defined and continuous function $K_0(t)$ is considered. We introduce the notation

$$\psi(t,s)=\int_s^t K_0(\theta)\,d\theta,\quad \psi(t,0)=\psi(t),\quad t\in[0;T].$$

It is obvious that $\psi(t,s)=\psi(t)-\psi(s)$. By the solution of equation (1) we mean a continuous function $u(t)$ on the segment $[0;T]$ that satisfies equation (1) with the given conditions (2) and the Lipschitz condition:

$$\max\big\{\|u(t)-u(s)\|;\|\dot u(t)-\dot u(s)\|\big\}\leqslant L_0|t-s|, \tag{17}$$

where $0<L_0=\text{const}$, $\|u(t)\|=\max\limits_{0\leqslant t\leqslant T}|u(t)|$.

We write the implicit equation (16) as

$$\dot u(t)+\int_0^t K_0(s)\,\dot u(s)\,ds=\dot u(t)+\int_0^t K_0(s)\,\dot u(s)\,ds \\ +G\big(t,u(t),\max\{u(\tau)|\tau\in[h_1(t,u(t));h_2(t,u(t))]\},\dot u(t)\big),\quad t\in[0;T].$$

Hence, using resolvent of the kernel $[-K_0(s)]$,

$$\dot u(t)=\dot u(t)+\int_0^t K_0(s)\,\dot u(s)\,ds+G\big(t,u(t),\max\{u(\tau)|\tau\in[h_1(t,u(t));h_2(t,u(t))]\},\dot u(t)\big) \\ +\int_0^t K_0(s)\,\exp\{-\psi(t,s)\}\bigg\{-\dot u(s)+\int_0^s K_0(\theta)\,\dot u(\theta)\,d\theta \\ -G\left(s,u(s),\max\left\{u(\tau)|\tau\in[h_1(s,u(s));h_2(s,u(s))]\right\},\dot u(s)\right)\bigg\}\,ds,\quad t\in[0;T]. \tag{18}$$

Appling Dirichlet's formula to (18) (see [21]), we derive the following Volterra type nonlinear functional integro-differential equation

$$\dot u(t)=\mathrm{Im}_1(t;\dot u)\equiv\int_0^t H(t,s)\,\dot u(s)\,ds \\ +\Big[\dot u(t)+G\left(t,u(t),\max\left\{u(\tau)|\tau\in[h_1(t,u(t));h_2(t,u(t))]\right\},\dot u(t)\right)\Big]\exp\{-\psi(t)\} \\ +\int_0^t K_0(s)\exp\{-\psi(t,s)\}\Big\{\dot u(t)-\dot u(s)+G\left(t,u(t),\max\left\{u(\tau)|\tau\in[h_1(t,u(t));h_2(t,u(t))]\right\},\dot u(t)\right) \\ -G\left(s,u(s),\max\left\{u(\tau)|\tau\in[h_1(s,u(s));h_2(s,u(s))]\right\},\dot u(s)\right)\Big\}\,ds,\quad t\in[0;T], \tag{19}$$

where

$$H(t,s)=K_0(s)\exp\{-\psi(t,s)\}-\int_s^t K_0(\theta)\exp\{-\psi(t,\theta)\}\,d\theta. \tag{20}$$

Integrating functional integro-differential equation (19) on the interval $(0;t)$ with the initial condition $u(0)=\varphi_{01}$, we obtain the following functional integro-differential equation

$$u(t) = \mathrm{Im}_2(t;u) \equiv \varphi_{01} + \int_0^t (t-s)H(t,s)\,\dot{u}(s)\,ds$$
$$+ \int_0^t \Big[\dot{u}(s) + G\left(s, u(s), \max\left\{u(\tau)|\tau \in [h_1(s,u(s)); h_2(s,u(s))]\right\}, \dot{u}(s)\right)\Big] \exp\{-\psi(s)\}\,ds$$
$$+ \int_0^t (t-s)K_0(s)\exp\{-\psi(t,s)\}\Big\{\dot{u}(t) - \dot{u}(s) + G\left(t, u(t), \max\left\{u(\tau)|\tau \in [h_1(t,u(t)); h_2(t,u(t))]\right\}, \dot{u}(t)\right)$$
$$- G\left(s, u(s), \max\left\{u(\tau)|\tau \in [h_1(s,u(s)); h_2(s,u(s))]\right\}, \dot{u}(s)\right)\Big\}\,ds, \quad t \in [0;T], \tag{21}$$

Remark 1. The nonlinear functional integro-differential equations (19) and (21) are ill-posed [22], so we will study them with given conditions (2). In addition, we consider the conditions (2) as $u(t-0) = u(t+0)$ at the points $t=0$ and $t=T$.

Let the conditions (11) and (14) are satisfied. Then, instead of Fredholm functional integro-differential equation of the first kind (1) we will study Volterra type functional integro-differential equations (19) and (21) with conditions (2).

Theorem 1. *Let the conditions* (17) *are satisfied and*

1. $\|G(t,u(t),v(t),\vartheta(t))\| \leqslant M_0$, $0 < M_0 = \mathrm{const}$;
2. $|G(t,u_1(t),v_1(t),\vartheta_1(t)) - G(t,u_2(t),v_2(t),\vartheta_2(t))|$ $\leqslant L_1(t)\left(|u_1(t)-u_2(t)| + |v_1(t)-v_2(t)| + |\vartheta_1(t)-\vartheta_2(t)|\right)$;
3. $|h_i(t,u_1(t)) - h_i(t,u_2(t))| \leqslant L_{2i}(t)\,|u_1(t)-u_2(t)|$, $0 < L_{2i}(t) \in C[0;T]$, $i=1,2$;
4. $\rho < 1$, *where* $\rho = \frac{1}{2}\max\limits_{0\leqslant t\leqslant T}[P_1(t)+P_2(t)+V_1(t)+V_2(t)]$, *with*

$$V_1(t) = L_1(t)\left[2 + L_0(L_{21}(t)+L_{22}(t))\right]\tilde{Q}(t), \quad V_2(t) = \int_0^t (t-s)\,Q(t,s)\,ds + (1+L_1(t))\tilde{Q}(t),$$
$$P_1(t) = L_1(t)\left[2 + L_0(L_{21}(t)+L_{22}(t))\right]Q(t,0), \quad P_2(t) = \int_0^t Q(t,s)\,ds + (1+L_1(t))Q(t,0),$$
$$Q(t,s) = \exp\{-\psi(t)\} + 2\int_s^t K_0(\theta)\exp\{-\psi(t,\theta)\}\,d\theta.$$

Then the nonlinear functional integro-differential equation (21) *with conditions* (2) *has a unique solution on the segment* $[0;T]$.

Proof. We suppose that Picard iteration processes for the functional integro-differential equations (19) and (21) are given by

$$\dot{u}_0(t) = \varphi_{02}, \quad \dot{u}_{n+1}(t) = \mathrm{Im}_1(t;\dot{u}_n), \quad n \in \mathbb{N}, \quad t \in [0;T], \tag{22}$$

$$\left[\begin{array}{ll} u_0(t) = \varphi_1(t), & t \in [-h_1;0], \\ u_0(t) = \varphi_{01}, & t \in [0;T], \\ u_0(t) = \varphi_2(t), & t \in [T;T+h_2], \end{array}\right. \qquad \left[\begin{array}{ll} u_{n+1}(t) = \varphi_1(t), & t \in [-h_1;0], \\ u_{n+1}(t) = \mathrm{Im}_2(t;u_n), \quad n \in \mathbb{N}, & t \in [0;T], \\ u_{n+1}(t) = \varphi_2(t), & t \in [T;T+h_2], \end{array}\right. \tag{23}$$

Firstly, we estimate the function $H(t,s)$, given by formula (20):

$$|H(t,s)| \leqslant K_0(s)\exp\{-\psi(t,s)\} + 2\int_s^t K_0(\theta)\exp\{-\psi(t,\theta)\}\,d\theta = Q(t,s). \tag{24}$$

It is obvious that the following estimates are true

$$\|\dot{u}_0(t)\| \leqslant |\varphi_{02}| < \infty, \tag{25}$$

$$\|u_0(t)\| \leqslant \max\left\{|\varphi_{01}|;\ \max_{-h_1\leqslant t\leqslant 0}|\varphi_1(t)|\,;\ \max_{T\leqslant t\leqslant T+h_2}|\varphi_2(t)|\right\} = \Delta_0 < \infty. \tag{26}$$

By virtue of conditions of theorem and Picard processes (22) and (23), by using estimates (25) and (26), for the first approximations we obtain the next estimates

$$\begin{aligned}
|\dot{u}_1(t)| \leqslant &\int_0^t \|H(t,s)\| \cdot \|\dot{u}_0(s)\|\, ds + \Big[\|\dot{u}_0(t)\| \\
&+ \|G\left(t, u_0(t), \max\left\{u_0(\tau)|\tau \in [h_1(t,u_0(t)); h_2(t,u_0(t))]\right\}, \dot{u}_0(t)\right)\|\Big] \exp\{-\psi(t)\} \\
&+ \int_0^t K_0(s) \exp\{-\psi(t,s)\}\Big[\|\dot{u}_0(t) - \dot{u}_0(s)\| \\
&+ 2\,\|G(t, u_0(t)), \max\left\{u_0(\tau)|\tau \in [h_1(t,u_0(t)); h_2(t,u_0(t))]\right\}, \dot{u}_0(t)\|\Big]\, ds \\
\leqslant &\, |\varphi_{02}| \int_0^t Q(t,s)\, ds + (|\varphi_{02}| + M_0)\exp\{-\psi(t)\} \\
&+ \int_0^t K_0(s)\exp\{-\psi(t,s)\}(L_0|t-s| + 2M_0)\, ds \\
\leqslant &\, |\varphi_{02}| \int_0^t Q(t,s)\, ds + \Delta_{11} Q(t,0),
\end{aligned} \tag{27}$$

where

$$\Delta_{11} = \max\left\{|\varphi_{02}| + M_0; L_0 T + 2M_0\right\};$$

$$\begin{aligned}
|u_1(t)| \leqslant \Delta_0 &+ \int_0^t \|(t-s)H(t,s)\| \cdot \|\dot{u}_0(s)\|\ ds + \int_0^t \exp\{-\psi(s)\}\Big[\ \|\dot{u}_0(s)\| \\
&+ \Big\|G\left(s, u_0(s), \max\{u_0(\tau)|\tau \in [h_1(s,u_0(s)); h_2(s,u_0(s))]\}, \dot{u}_0(s)\right)\Big\|\Big]\, ds \\
&+ \int_0^t (t-s) K_0(s)\exp\{-\psi(t,s)\}\Big[\ \|\dot{u}_0(t) - \dot{u}_0(s)\| \\
&+ 2\Big\|G\left(t, u_0(t), \max\left\{u_0(\tau)|\tau \in [h_1(t,u_0(t)); h_2(t,u_0(t))]\right\}, \dot{u}_0(t)\right)\Big\|\Big]\, ds \\
\leqslant \Delta_0 &+ |\varphi_{02}| \int_0^t (t-s)Q(t,s)\, ds + (|\varphi_{02}| + M_0)\int_0^t \exp\{-\psi(s)\}\, ds \\
&+ \int_0^t (t-s)K_0(s)\exp\left\{-\psi(t,s)\right\}\left(L_0|t-s| + 2M_0\right)\ ds \\
\leqslant \Delta_0 &+ |\varphi_{02}| \int_0^t (t-s)Q(t,s)\, ds + \Delta_{11}\tilde{Q}(t),
\end{aligned} \tag{28}$$

where

$$\tilde{Q}(t) = \int_0^t \exp\{-\psi(s)\}\, ds + 2\int_0^t (t-s)K_0(s)\exp\{-\psi(t,s)\}\, ds.$$

By virtue of the first and the second conditions of theorem, analogously to estimates (27) and (28) for the arbitrary difference of approximations

$$\begin{aligned}
|\dot{u}_{n+1}(t) - \dot{u}_n(t)| \leqslant &\int_0^t \|H(t,s)\| \cdot \|\dot{u}_n(s) - \dot{u}_{n-1}(s)\| \\
&+ \exp\{-\psi(t)\}\Big[\|\dot{u}_n(t) - \dot{u}_{n-1}(t)\| + L_1(t)\|u_n(t) - u_{n-1}(t)\| \\
&+ \Big\|\max\left\{u_n(\tau)|\tau \in [h_1(t,u_n(t)); h_2(t,u_n(t))]\right\} \\
&- \max\left\{u_{n-1}(\tau)|\tau \in [h_1(t,u_{n-1}(t)); h_2(t,u_{n-1}(t))]\right\}\| + \|\dot{u}_n(t) - \dot{u}_{n-1}(t)\|)\Big]\Big] \\
&+ 2\int_0^t K_0(s)\exp\left\{-\psi(t,s)\right\}\Big[\ \|\dot{u}_n(s) - \dot{u}_{n-1}(s)\| + L_1(s)\|u_n(s) - u_{n-1}(s)\| \\
&+ \Big\|\max\left\{u_n(\tau)|\tau \in [h_1(s,u_n(s)); h_2(s,u_n(s))]\right\}
\end{aligned}$$

$$- \max\{u_{n-1}(\tau)|\tau \in [h_1(s, u_{n-1}(s)); h_2(s, u_{n-1}(s))]\}\big\|$$
$$+ \|\dot{u}_n(s) - \dot{u}_{n-1}(s)\|\Big] ds. \tag{29}$$

To continue estimate the norm in (29) we use condition (17) and the third condition of the theorem. Then

$$\begin{aligned}
&\|\max\{u_n(\tau)|\tau \in [h_1(s, u_n(s)); h_2(s, u_n(s))]\} - \max\{u_{n-1}(\tau)|\tau \in [h_1(s, u_{n-1}(s)); h_2(s, u_{n-1}(s))]\}\| \\
&\leqslant \|\max\{u_n(\tau)|\tau \in [h_1(s, u_n(s)); h_2(s, u_n(s))]\} - \max\{u_{n-1}(\tau)|\tau \in [h_1(s, u_n(s)); h_2(s, u_n(s))]\}\| \\
&\quad + \|\max\{u_{n-1}(\tau)|\tau \in [h_1(s, u_n(s)); h_2(s, u_n(s))]\} \\
&\quad - \max\{u_{n-1}(\tau)|\tau \in [h_1(s, u_{n-1}(s)); h_2(s, u_{n-1}(s))]\}\| \\
&\leqslant \|\max\{|u_n(\tau) - u_{n-1}(\tau)| : \tau \in [h_1(s, u_n(s)) : h_2(s, u_n(s))]\}\| \\
&+ L_0 \sum_{i=1}^{2} \|h_i(s,\, u_n(s)) - h_i(s, u_{n-1}(s))\| \leqslant [1 + L_0(L_{21}(s) + L_{22}(s))]\, \|u_n(s) - u_{n-1}(s)\|\,.
\end{aligned} \tag{30}$$

Substituting (30) into (29),

$$\begin{aligned}
|\dot{u}_{n+1}(t) - \dot{u}_n(t)| &\leqslant \int_0^t Q(t, s)\|\dot{u}_n(s) - \dot{u}_{n-1}(s)\|\, ds + \exp\{-\psi(t)\} \\
&\times \big[L_1(t)[2 + L_0(L_{21}(t) + L_{22}(t))]\, \|u_n(t) - u_{n-1}(t)\| + (1 + L_1(t))\, \|\dot{u}_n(t) - \dot{u}_{n-1}(t)\|\big] \\
&+ 2\int_0^t K_0(s) \exp\{-\psi(t, s)\} \big[L_1(s)[2 + L_0(L_{21}(s) + L_{22}(s))]\, \|u_n(s) - u_{n-1}(s)\| \\
&+ (1 + L_1(s))\, \|\dot{u}_n(s) - \dot{u}_{n-1}(s)\|\,\big]\, ds \\
&\leqslant P_1(t)\, \|u_n(t) - u_{n-1}(t)\| + P_2(t)\, \|\dot{u}_n(t) - \dot{u}_{n-1}(t)\|\,,
\end{aligned} \tag{31}$$

where

$$P_1(t) = L_1(t)\, [2 + L_0(L_{21}(t) + L_{22}(t))]\, Q(t, 0),$$
$$P_2(t) = \int_0^t Q(t, s)\, ds + (1 + L_1(t)) Q(t, 0),$$
$$Q(t, s) = \exp\{-\psi(t)\} + 2\int_0^t K_0(s) \exp\{-\psi(t, s)\}\, ds;$$

and

$$\begin{aligned}
&|u_{n+1}(t) - u_n(t)| \leqslant \int_0^t (t - s)\, Q(t, s)\, \|\dot{u}_n(s) - \dot{u}_{n-1}(s)\|\; ds + \int_0^t \exp\{-\psi(s)\} \\
&\times \big[L_1(s)\, [2 + L_0(L_{21}(s) + L_{22}(s))]\, \|u_n(s) - u_{n-1}(s)\| + (1 + L_1(s))\, \|\dot{u}_n(s) - \dot{u}_{n-1}(s)\|\,\big]\, ds \\
&+ 2\int_0^t (t - s) K_0(s) \exp\{-\psi(t, s)\} \\
&\times \big[L_1(s)\, [2 + L_0(L_{21}(s) + L_{22}(s))]\, \|u_n(s) - u_{n-1}(s)\| + (1 + L_1(s))\, \|\dot{u}_n(s) - \dot{u}_{n-1}(s)\|\,\big]\, ds \\
&\leqslant V_1(t)\, \|u_n(t) - u_{n-1}(t)\| + V_2(t)\, \|\dot{u}_n(t) - \dot{u}_{n-1}(t)\|\,,
\end{aligned} \tag{32}$$

where

$$V_1(t) = L_1(t)\, [2 + L_0(L_{21}(t) + L_{22}(t))]\, \tilde{Q}(t), \tag{33}$$
$$V_2(t) = \int_0^t (t - s)\, Q(t, s)\, ds + (1 + L_1(t))\, \tilde{Q}(t), \tag{34}$$
$$\tilde{Q}(t) = \int_0^t \exp\{-\psi(s)\}\, ds + 2\int_0^t (t - s)\, K_0(s) \exp\{-\psi(t, s)\}\, ds. \tag{35}$$

From the estimates (31) and (32) it follows that

$$\|U_{n+1}(t) - U_n(t)\| \leqslant \rho \cdot \|U_n(t) - U_{n-1}(t)\|\,, \tag{36}$$

where

$$\|U_{n+1}(t) - U_n(t)\| \leqslant \max\left\{\|u_{n+1}(t) - u_n(t)\|; \|\dot{u}_{n+1}(t) - \dot{u}_n(t)\|\right\},$$

$$\rho = \frac{1}{2}\max_{0\leqslant t\leqslant T}\left[P_1(t) + P_2(t) + V_1(t) + V_2(t)\right].$$

Choosing the function $K_0(t)$, let take into account that

$$\psi(t,s) = \int_s^t K_0(\theta)\,d\theta \gg 1, \quad t \in [0;T].$$

Hence, we obtain that $\exp\{-\psi(t)\} \ll 1$. So, the functions $H(t,s)$ and $Q(t,s)$ are small. Then the functions $L_1(t)$, $L_{2i}(t)$, $i = 1,2$ we can choose such that $\rho < 1$ and the last condition of the theorem is satisfied. We consider the solution of the integro-differential equations (19) and (21) in the space of the continuous functions $C[0;T]$, satisfying condition (17). Since $\|u_{n+1}(t) - u_n(t)\| \leqslant \|U_{n+1}(t) - U_n(t)\|$, it follows from the estimate (36) that the integral operator on the right-hand side of (21) with conditions (2) is compressing mapping. So, from the estimates (25)–(28) and (36) implies that the integro-differential equation (21) with conditions (2) has a unique solution on the segment $[0;T]$. The theorem is proved. ■

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Нелінійне інтегро-диференціальне рівняння Фредгольма першого порядку з виродженим ядром і нелінійними максимумами

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У цій статті розглянуто проблеми розв'язності та побудови розв'язків нелінійного інтегро-диференціального рівняння Фредгольма першого порядку з виродженим ядром та нелінійними максимумами. Використовуючи метод виродженого ядра у поєднанні з методом регуляризації, отримано неявне функціонально-диференціальне рівняння першого порядку з нелінійними максимумами. Використовуємо початкові граничні умови, щоб забезпечити єдиність розв'язку. Для застосування методу послідовного наближення та доведення однозначного розв'язування, перетворено отримане неявне функціонально-диференціальне рівняння до нелінійного інтегро-диференціального рівняння Вольтерра з нелінійними максимумами.

Ключові слова: *інтегро-диференціальне рівняння, нелінійне функціонально-диференціальне рівняння, вироджене ядро, нелінійні максимуми, регуляризація, однозначне розв'язування.*