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Simulation of statistical mean and variance of normally distributed random values, transformed by nonlinear functions $\sqrt{|X|}$ and $\sqrt{X}$

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This paper presents theoretical studies of formation regularities for the statistical mean and variance of normally distributed random values with the unlimited argument values subjected to nonlinear transformations of functions $\sqrt{|X|}$ and $\sqrt{X}$. It is shown that for nonlinear square root transformation of a normally distributed random variable, the integrals of higher order mean $n > 1$ satisfy the inequality $\overline{(y - \overline{Y})^n} \neq 0$. On the basis of the theoretical research, the correct boundaries $m, \sigma \to \infty$ of error transfer formulas are suggested.

Keywords: *statistical mean, variance, transformation, normal distribution, random variable.*

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1. Introduction

Among the methods of analyzing physical measurement data, averaging algorithms rank high, including the most commonly known Cauchy mean values, such as arithmetic mean and mean square deviation (MSD). If the arithmetic mean reflects the center of the statistical probability distribution (the expected value), MSD is its length.

Standard algorithms for statistical data analysis are designed primarily for normal distribution, so random experimental data would be first checked for normal distribution. In physical statistical models, there are widely used such distributions as the Cauchy (Lorentz) ones, which have no mathematical expectation since the integral of averaging diverges [1]. In addition, if the data sample contains extreme values, then estimating the center of a distribution by the arithmetic mean may also be incorrect. In this case, weighted estimates are used [2].

On the other hand, in the process of statistical processing, experimental data are often transformed by nonlinear functions, which, as we know [3, 4], is accompanied by a change in the law of probability distribution and by the emergence of constraints on the set of allowed values of a random variable (RV), as in the case of transformations by square radicals. Such a problem is particularly relevant for transformations of normally $N_X(m_X, \sigma_X)$ distributed RV since for them the averaging integrals are not always expressed through elementary functions in the form of tabular integrals, and one has to use the Taylor series [2].

The approximate formulas for calculating the mean and variance of transformations of normally $N_X(m_X, \sigma_X)$ distributed RV using the Taylor algorithm are know [5]. In the recent paper [6], this approach was developed for the following type of transformations

$$\sqrt{Y}, \quad Y = \begin{cases} |X|, & \text{(a)} \\ X, & \text{(b)} \end{cases} \tag{1}$$

however, without taking into account the fact that transformations of the type (1,a) of normally $N_X(m_X, \sigma_X)$ distributed RV with the set of values $X \in (-\infty, +\infty)$ are accompanied by a change in the law of distribution [7, 8], and by the restriction of the set of values to $X \in [0, +\infty)$ for (1,b).

These shortcomings of the statistical model are eliminated in the given work. Moving closer to the agreement on the level "$3\sigma_{Y_{\mathrm{FND}}}$" with confidence probability $P\big(|X-\overline{Y_{\mathrm{FND}}}| \leqslant \sigma_{\mathrm{FND}}\big) = 2\,\mathrm{erf}(3) = 0.998$ between functions (1) and their representation in the form of Taylor series, the formulas of statistical mean of normally $N_X(m_X,\sigma_X)$ distributed data are substantiated and the comparative analysis with corresponding error transfer formulas (7) is carried out [9]. It should be noted that accurate statistical analysis of data is important for maximum objectivity in their visualization in modern advanced technologies, for example in probe microspectroscopy and nanospectroscopy [10], in the development of physical tests for pseudorandom number generators [11], in the processing of physical measurements data in medicine [12], etc.

2. Results and discussion

Transformation (1,a). According to [7, 8], a normally distributed RV transformed by the law (1,a) acquires a folded normal distribution (ND) or distribution of a module of a normally distributed RV, which in coordinates $(|m_X|,\sigma_X)$ is described by the probability density function

$$f_{Y=|X|}(y=|x|) = \begin{cases} \sqrt{\dfrac{p}{\pi}}\exp\left(-\frac{(y-m_X)^2}{2\sigma_X^2}\right) + \sqrt{\dfrac{p}{\pi}}\exp\left(-\frac{(y+m_X)^2}{2\sigma_X^2}\right), & \text{if} \quad y \geqslant 0, \quad p = \dfrac{1}{2\sigma_X^2}, \\ 0, & \text{if } y < 0, \end{cases} \tag{2}$$

with the mean $\overline{Y_{\mathrm{FND}}}$ and the variance $D_{Y_{\mathrm{FND}}}$:

$$\overline{Y_{\mathrm{FND}}} = \sqrt{\frac{2}{\pi}}\,\sigma_X \exp\left(-\frac{m_X^2}{2\sigma_X^2}\right) + m_X\,\mathrm{erf}\left(\frac{m_X}{\sqrt{2}\sigma_X}\right), \tag{3}$$

$$D_{Y_{\mathrm{FND}}} = \overline{(Y_{\mathrm{FND}})^2} - \big(\overline{Y_{\mathrm{FND}}}\big)^2 = \sigma_X^2 + m_X^2 - \big(\overline{Y_{\mathrm{FND}}}\big)^2. \tag{4}$$

Given $m_X = 0$ and

$$f_{\mathrm{FND}}(y) = \begin{cases} \sqrt{\dfrac{2}{\pi}}\,\sigma_X \exp\Big(-\dfrac{y^2}{2\sigma_X^2}\Big), & \text{if } y \geqslant 0, \\ 0, & \text{if } y < 0. \end{cases}$$

Therefore, according to (2), the folded ND reflects the transformed by the law $Y = |X|$ normally distributed data $X \in (-\infty,+\infty)$ in the range of argument values $y > 0$.

Considering the domain of function definition (2), in the following calculations we will take into account the known table integral (No. 7 (2.3.15) [13])

$$\int_0^{\infty} y^n e^{-py^2-qy}\mathrm{d}y = \frac{(-1)^n}{2}\sqrt{\frac{\pi}{p}}\frac{\partial^n}{\partial q^n}\left[\exp\left(\frac{q^2}{4p}\right)\mathrm{erfc}\left(\frac{q}{2\sqrt{p}}\right)\right], \tag{5}$$

where $\mathrm{erfc}(\xi) = 1-\mathrm{erf}(\xi) = \frac{2}{\sqrt{\pi}}\int_0^x \exp(-\xi^2)\mathrm{d}\xi$ is a special nonanalytic error function [14], and the distribution function (2) is transformed to the form:

$$f_{\mathrm{FND}}(y) = \sqrt{\frac{p}{\pi}}\exp\left(-\frac{m_X^2}{2\sigma_X^2}\right)\big(\exp(-py^2-\theta y) + \exp(-py^2-qy)\big), \quad \theta = -\frac{m_X}{\sigma_X^2}, \quad q = \frac{m_X}{\sigma_X^2}, \tag{6}$$

and calculate its normalizing constant C_{FND}:

$$C_{\mathrm{FND}} = \frac{\exp\Big(-\frac{m_X^2}{2\sigma_y^2}\Big)}{\int_0^{\infty}\exp(-py^2-qy)\mathrm{d}y + \int_0^{\infty}\exp(-py^2-\theta y)\mathrm{d}y} = \frac{2}{1-\mathrm{erf}\Big(\frac{m_X}{\sqrt{2}\,\sigma_X}\Big) + 1 + \mathrm{erf}\Big(\frac{m_X}{\sqrt{2}\,\sigma_X}\Big)} = 1. \tag{7}$$

Thus, the application of the transformation $|X|$ to the input normally $N_X(m_X, \sigma_X)$ distributed RV with a set of values $X \in (-\infty, +\infty)$ does not change the normalizing constant (7) of function (2) and the variance (4), but only the mean (3) is changed.

To prove that the substitution $\theta = -\frac{m_X}{2\sigma_X^2}$ in (6) in order to apply (5) was correct, we calculate formulas (3) and (4):

$$\begin{aligned}\overline{Y_{\mathrm{FND}}} &= C_{\mathrm{FND}}\sqrt{\frac{p}{\pi}}\exp\left(-\frac{m_y^2}{2\sigma_y^2}\right)\left[\int_0^\infty y\exp(-px^2-qx)\mathrm{d}x + \int_0^\infty y\exp(-px^2-\theta x)\mathrm{d}x\right] \\ &= m_X\operatorname{erf}\left(\frac{m_X}{\sqrt{2}\sigma_X}\right) + \sqrt{\frac{2}{\pi}}\,\sigma_X\exp\left(-\frac{m_y^2}{2\sigma_y^2}\right). \qquad (8)\end{aligned}$$

Given RV, Y and $\overline{Y_{FND}}$ are statistically independent, the variance D_{FND} is calculated by the formula:

$$D_{\mathrm{FND}} = \overline{(Y_{\mathrm{FND}})^2} - \left(\overline{Y_{\mathrm{FND}}}\right)^2,$$

where the mean $\overline{(Y_{\mathrm{FND}})^2}$ is:

$$\begin{aligned}\overline{(Y_{\mathrm{FND}})^2} &= C_{\mathrm{FND}}\sqrt{\frac{p}{\pi}}\exp\left(-\frac{m_y^2}{2\sigma_y^2}\right)\left[\int_0^\infty y^2\exp\left(-px^2-qx\right)\mathrm{d}x + \int_0^\infty y^2\exp\left(-px^2-\theta x\right)\mathrm{d}x\right] \\ &= \frac{1}{2p} + \left(\frac{q}{2p}\right)^2 = \sigma_X^2 + m_X^2 \quad \Longrightarrow \quad D_{\mathrm{FND}} = \sigma_X^2 + m_X^2 - \left(\overline{Y_{\mathrm{FND}}}\right)^2,\end{aligned}$$

which agrees with (4).

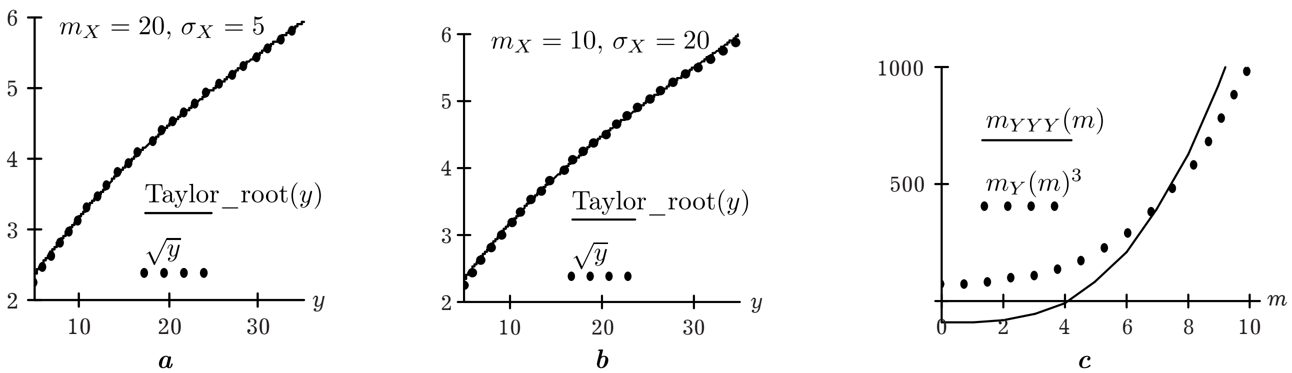

Fig. 1. The illustration of the consistency rule "$3\sigma_X$" of the function $\sqrt{Y}$ expansion in a Taylor series in approximation considering four terms for two values of the ratio $\frac{m_X}{\sigma_X} = 0.25$ (***a***) and $\frac{m_X}{\sigma_X} = 2$ (***b***), ***c***: illustration of the implementation of inequality $\overline{(Y_{\mathrm{FND}})^3} \neq \left(\overline{Y_{\mathrm{FND}}}\right)^3$, where $\overline{(Y_{\mathrm{FND}})^3} = m_{YYY}(m)$, $\left(\overline{Y_{\mathrm{FND}}}\right)^3 = m_Y(m)^3$.

The integral (5) with the fractional value of the exponent $n = \frac{1}{2}$ is not tabular, so we expand the radical $\sqrt{Y}$ into the Taylor series in terms of functions Y_{FND}. As follows from Fig. 1, when changing the parameters m_X and σ_X of input RV, "$3\sigma_{Y_{FND}}$", the level of agreement with the confidence probability $P(|Y - \overline{Y_{\mathrm{FND}}}| \leqslant \sigma_{\mathrm{FND}}) = 2\operatorname{erf}(3) = 0.998$ with the function $\sqrt{Y}$ is achieved taking into account the first four members of the expansion:

$$\begin{aligned}\sqrt{Y_{\mathrm{FND}}} &\cong \sqrt{\overline{Y_{\mathrm{FND}}}} + \frac{\left(y - \overline{Y_{\mathrm{FND}}}\right)}{2\sqrt{\overline{Y_{\mathrm{FND}}}}} - \frac{\left(y - \overline{Y_{\mathrm{FND}}}\right)^2}{8\overline{Y_{\mathrm{FND}}}\sqrt{\overline{Y_{\mathrm{FND}}}}} + \frac{\left(y - \overline{Y_{\mathrm{FND}}}\right)^3}{16\left(\overline{Y_{FND}}\right)^2\sqrt{\overline{Y_{FND}}}} + \ldots \quad \Longrightarrow \\ \overline{\sqrt{Y_{\mathrm{FND}}}} &\cong \sqrt{\overline{Y_{\mathrm{FND}}}} + \frac{1}{2\sqrt{\overline{Y_{\mathrm{FND}}}}}\overline{\left(y - \overline{Y_{\mathrm{FND}}}\right)} - \frac{1}{8\overline{Y_{\mathrm{FND}}}\sqrt{\overline{Y_{\mathrm{FND}}}}}\overline{\left(y - \overline{Y_{\mathrm{FND}}}\right)^2} \\ &\quad + \frac{1}{16\left(\overline{Y_{\mathrm{FND}}}\right)^2\sqrt{\overline{Y_{\mathrm{FND}}}}}\overline{\left(y - \overline{Y_{\mathrm{FND}}}\right)^3} + \ldots, \qquad (9)\end{aligned}$$

where the statistical averaging of higher order expansions is calculated as integrals

$$\overline{\left(y-\overline{Y_{\rm FND}}\right)^n} = \frac{C_{\rm FND}\exp\left(-\frac{m_X^2}{2\sigma_X^2}\right)}{\sqrt{2\pi}\sigma_X} \times \left[\int_0^\infty \left(y-\overline{Y_{\rm FND}}\right)^n \exp(-py^2-qy){\rm d}y + \int_0^\infty \left(y-\overline{Y_{\rm FND}}\right)^n \exp(-py^2-\theta y){\rm d}y\right]. \quad (10)$$

The first order mean

$$\overline{(y-\overline{Y_{FND}})} = \overline{Y_{\rm FND}} - \overline{Y_{\rm FND}} = 0$$

is equal to zero and the second order mean is $\overline{(y-\overline{Y_{\rm FND}})^2} = D_{Y_{\rm FND}}$, therefore

$$\overline{\sqrt{Y_{\rm FND}}} \cong \sqrt{\overline{Y_{\rm FND}}} - \frac{D_{Y_{\rm FND}}}{8\overline{Y_{\rm FND}}\sqrt{\overline{Y_{\rm FND}}}} + \frac{1}{16\left(\overline{Y_{\rm FND}}\right)^2\sqrt{\overline{Y_{\rm FND}}}}\overline{\left(y-\overline{Y_{\rm FND}}\right)^3} + \dots. \quad (11)$$

The series (11) differs from the corresponding one in [6] by the formula (8) due to the fact that the known regularity is used for (8) [6] that the integrals in even limits from odd functions (10) are zero. However, for transformations (1), this pattern does not hold, i.e., the third-order mean $\overline{(y-\overline{Y_{\rm FND}})^3} \neq 0$ and the higher order mean values are not zero. In fact,

$$\begin{aligned}\overline{\left(y-\overline{Y_{\rm FND}}\right)^3} &= \overline{(Y_{\rm FND})^3} - 3\overline{(Y_{\rm FND})^2}\,\overline{Y_{\rm FND}} + 3\overline{Y_{\rm FND}}\left(\overline{Y_{\rm FND}}\right)^2 - \left(\overline{Y_{\rm FND}}\right)^3 \\ &= \overline{(Y_{FND})^3} - 3\overline{(Y_{\rm FND})^2}\,\overline{Y_{\rm FND}} + 2\left(\overline{Y_{\rm FND}}\right)^3.\end{aligned}$$

However, in the systems with the variance of statistically independent RV $D_Y = \overline{Y^2} - (\overline{Y})^2 > 0$ therefore $\overline{Y^2} \neq (\overline{Y})^2$. Let us prove that $\overline{Y^3} \neq (\overline{Y})^3$ and calculate the mean $\overline{Y^3}$:

$$\begin{aligned}\overline{(Y_{\rm FND})^3} &= \sqrt{\frac{p}{\pi}}\,C_{\rm FND}\exp\left(-\frac{m_X^2}{2\,\sigma_X^2}\right)\left[\int_0^\infty y^3\exp(-py^2-qy){\rm d}y + \int_0^\infty y^3\exp(-py^2-\theta y){\rm d}y\right] \\ &= \left(2\sigma_X^2 m_X + m_X^3\right){\rm erf}\left(\frac{m_X}{\sqrt{2}\sigma_X}\right) - \sqrt{\frac{2}{\pi}}\,\sigma_X\left(\sigma_X^2+2m_X^2\right)\exp\left(-\frac{m_X^2}{2\sigma_X^2}\right).\end{aligned}$$

The inequality $\overline{Y^3} \neq \left(\overline{Y}\right)^3$ is confirmed in Fig. 1c, where the notation $\overline{(Y_{\rm FND})^3} = m_{YYY}(m)$, $\left(\overline{Y_{\rm FND}}\right)^3 = m_Y(m)^3$ is introduced. Therefore,

$$\begin{aligned}\left(\overline{(y-\overline{Y_{\rm FND}})^3}\right)^3 = &-\sqrt{\frac{2}{\pi}}\,\sigma_X\left(\sigma_X^2+2m_X^2\right)\exp\left(-\frac{m_X^2}{2\sigma_X^2}\right) - 3\left(\sigma_X^2+m_X^2\right)m_X\,{\rm erf}\left(\frac{m_X}{\sqrt{2}\sigma_X}\right) \\ &+ 2\left(m_X\,{\rm erf}\left(\frac{m_X}{\sqrt{2}\sigma_X}\right) + \sqrt{\frac{2}{\pi}}\sigma_X\exp\left(-\frac{m_X^2}{2\sigma_X^2}\right)\right)^3.\end{aligned}$$

Since equality $\overline{(\sqrt{Y_{\rm FND}})^2} = \overline{Y_{\rm FND}}$ is satisfied, the variance is calculated by the formula:

$$D_{\sqrt{Y_{\rm FND}}} = \overline{Y_{\rm FND}} - \left(\overline{\sqrt{Y_{\rm FND}}}\right)^2. \quad (12)$$

Transformation (1,b). Although the function $\sqrt{X}$ limits the set of allowed values of RV X by a positive half-bounded interval $X \in [0,+\infty)$ the distribution of the transformed RV is described by the truncated (left) ND:

$$f_{N(m_X,\sigma_X)}(x) = \sqrt{\frac{p}{\pi}}\exp\left(-\frac{(x-m_X)^2}{2\sigma_X^2}\right) = \sqrt{\frac{p}{\pi}}\exp\left(-\frac{m_X^2}{2\,\sigma_X^2}\right)\exp(-px^2-\theta x), \quad x>0. \quad (13)$$

According to the table integral $\sqrt{\frac{p}{\pi}}\int\exp\left(-\frac{(x-m_X)^2}{2\sigma_X^2}\right){\rm d}x = -\frac{1}{2}\,{\rm erf}\left(\frac{x-m_X}{\sqrt{2}\sigma_X}\right)$ [15] or (5), the normalizing constant C_X of function (13) is equal to:

$$C_{[0,\infty)} = \frac{1}{\sqrt{\frac{p}{\pi}}\exp\left(-\frac{m_X^2}{2\sigma_X^2}\right)\int_0^\infty x^0\exp(-px^2-\theta x)\mathrm{d}x} = \frac{2}{1+\operatorname{erf}\left(\frac{m_X}{\sqrt{2}\sigma_X}\right)}.$$

Then the statistical mean $\overline{X_{[0,\infty)}}$ is:

$$\begin{aligned}\overline{X_{[0,\infty)}} &= \sqrt{\frac{p}{\pi}}\,C_{[0,\infty)}\int_0^{+\infty} x\exp\left(-\frac{(x-m_X)^2}{2\sigma_X^2}\right)\mathrm{d}x \\ &= \sqrt{\frac{p}{\pi}}\,C_{[0,\infty)}\exp\left(-\frac{m_X^2}{2\sigma_X^2}\right)\int_0^\infty x\exp(-py^2-\theta y)\mathrm{d}x = m_X + \sqrt{\frac{2}{\pi}}\,\sigma_X\,K(m_X,\sigma_X),\end{aligned}$$

where the input function is

$$K(m_X,\sigma_X) = \frac{\exp\left(-\frac{m_X^2}{2\sigma_X^2}\right)}{1+\operatorname{erf}\left(\frac{m_X}{\sqrt{2}\sigma_X}\right)}, \tag{14}$$

the graphs of which are shown in Fig. 2 and the standard RV representation $X = m_X + \sigma_X Z \Rightarrow \begin{cases}\text{if } x=0 \text{ then } z=-\frac{m_X}{\sigma_X}\\ \text{if } x=\infty \text{ then } z=\infty\end{cases}$ is used and the change of integration boundaries is taken into account.

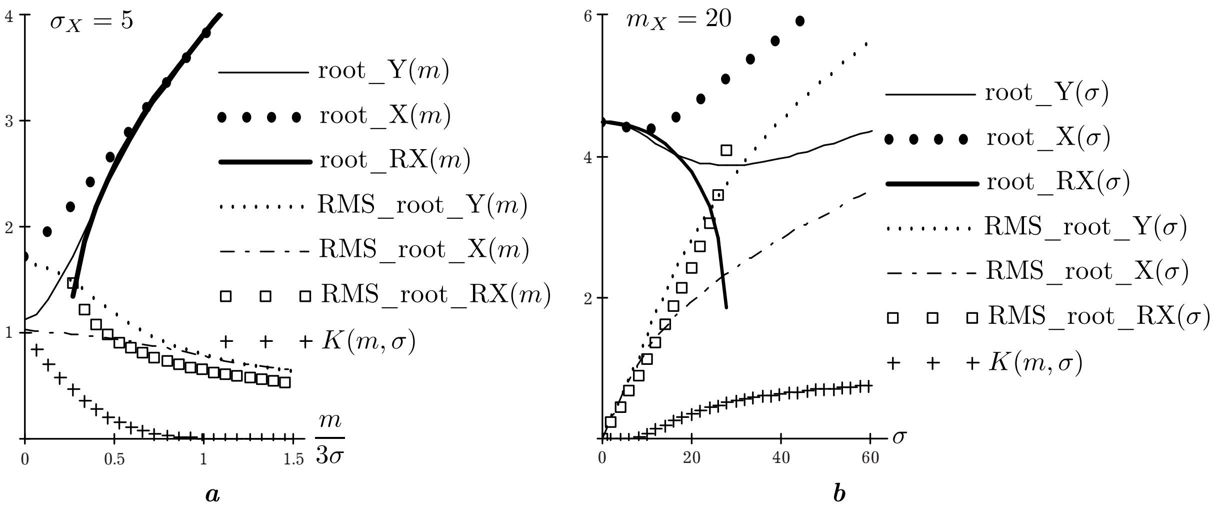

Fig. 2. The dependence graphs for $\sigma_X = \text{const}$ (a) and $m_X = \text{const}$ (b) mean and MSD transformations $\sqrt{|X|}$, $\sqrt{X}$, built by formulas root_Y(m) (9), RMS_root_Y(m) (12), root_X(m) (17), RMS_root_X(m) (19) and root_RX(m), RMS_root_RX(m) (7) [9], and the graphs of function $K(m,\sigma)$ (14).

Now let us calculate the mean $\overline{\left(X_{[0,\infty)}\right)^2}$:

$$\begin{aligned}\overline{\left(X_{[0,\infty)}\right)^2} &= \sqrt{\frac{p}{\pi}}\,C_{[0,\infty)}\int_0^{+\infty} x^2\exp\left(-\frac{(x-m_X)^2}{2\sigma_X^2}\right)\mathrm{d}x \\ &= \sigma_X^2 + m_X^2 + \sqrt{\frac{2}{\pi}}m_X\sigma_X\frac{\exp\left(-\frac{m_y^2}{2\sigma_X^2}\right)}{1+\operatorname{erf}\left(\frac{m_X}{\sqrt{2}\,\sigma_X}\right)}.\end{aligned}$$

Then, the variance is:

$$\begin{aligned}D_{X_{[0,\infty)}} &= \overline{\left(x-\overline{X_{[0,\infty)}}\right)^2} = \overline{\left(X_{[0,\infty)}\right)^2} - \left(\overline{X_{[0,\infty)}}\right)^2 \\ &= m_X^2 + \sigma_X^2 + \sqrt{\frac{2}{\pi}}m_X\sigma_X K(m_X,\sigma_X) - \left(m_X + \sqrt{\frac{2}{\pi}}\sigma_X K(m_X,\sigma_X)\right)^2.\end{aligned} \tag{15}$$

Let us check the result (15):

$$D_{X_{[0,\infty)}} = \sqrt{\frac{p}{\pi}}\, C_{[0,\infty)} \int_0^{+\infty} \left(x - \overline{X_{[0,\infty)}}\right)^2 \exp\left(-\frac{(x-m_X)^2}{2\sigma_X^2}\right) \mathrm{d}x$$
$$= \sqrt{\frac{p}{\pi}}\, C_{[0,\infty)} \exp\left(-\frac{m_Y^2}{2\sigma_X^2}\right) \int_0^{+\infty} \left(x - \overline{X_{[0,\infty)}}\right)^2 \exp(-px^2 - \theta x)\, \mathrm{d}x$$
$$= \sigma_X^2 + m_X^2 - 2\,\overline{X_{[0,\infty)}}\, m_X + \left(\overline{X_{[0,\infty)}}\right)^2 - \left[-m_X + 2\,\overline{X_{[0,\infty)}}\right] \sqrt{\frac{2}{\pi}}\, \sigma_X\, K(m_X, \sigma_X).$$

Let us apply expansion (8) to the function $\sqrt{X_{[0,\infty)}}$,

$$\sqrt{X_{[0,\infty)}} \cong \sqrt{\overline{X_{[0,\infty)}}} + \frac{\left(x - \overline{X_{[0,\infty)}}\right)}{2\sqrt{\overline{X_{[0,\infty)}}}} - \frac{\left(x - \overline{X_{[0,\infty)}}\right)^2}{8\,\overline{X_{[0,\infty)}}\sqrt{\overline{X_{[0,\infty)}}}} + \frac{\left(x - \overline{X_{[0,\infty)}}\right)^3}{16\left(\overline{X_{[0,\infty)}}\right)^2\sqrt{\overline{X_{[0,\infty)}}}} + \dots, \tag{16}$$

and carry out averaging (16):

$$\overline{\sqrt{X_{[0,\infty)}}} \cong \sqrt{\overline{X_{[0,\infty)}}} + \frac{\overline{\left(x - \overline{X_{[0,\infty)}}\right)}}{2\sqrt{\overline{X_{[0,\infty)}}}} - \frac{\overline{\left(x - \overline{X_{[0,\infty)}}\right)^2}}{8\,\overline{X_{[0,\infty)}}\sqrt{\overline{X_{[0,\infty)}}}} + \frac{\overline{\left(x - \overline{X_{[0,\infty)}}\right)^3}}{16\left(\overline{X_{[0,\infty)}}\right)^2\sqrt{\overline{X_{[0,\infty)}}}} + \dots$$
$$= \sqrt{\overline{X_{[0,\infty)}}} - \frac{D_{[0,\infty)}}{8\,\overline{X_{[0,\infty)}}\sqrt{\overline{X_{[0,\infty)}}}} + \frac{\overline{\left(x - \overline{X_{[0,\infty)}}\right)^3}}{16\left(\overline{X_{[0,\infty)}}\right)^2\sqrt{\overline{X_{[0,\infty)}}}} + \dots, \tag{17}$$

where $\overline{(x - \overline{X_{[0,\infty)}})} = 0$ and $\overline{(x - \overline{X_{[0,\infty)}})^2} = D_{[0,\infty)}$ are taken into account. Given $\overline{X^2} \neq \left(\overline{X}\right)^2$ and $\overline{X^3} \neq \left(\overline{X}\right)^3$,

$$\overline{\left(x - X_{[0,\infty)}\right)^3} = \overline{\left(X_{[0,\infty)}\right)^3} - 3\,\overline{\left(X_{[0,\infty)}\right)^2}\,\overline{X_{[0,\infty)}} + 2\left(\overline{X_{[0,\infty)}}\right)^3, \tag{18}$$

where

$$\overline{\left(X_{[0,\infty)}\right)^3} = \left[m_X^3 + 2\sigma_X^2 m_X\right] + \left[\sigma_X^2 + m_X^2\right]\sqrt{\frac{2}{\pi}}\sigma_X \frac{\exp\left(-\frac{m_X^2}{2\sigma_X^2}\right)}{1 + \operatorname{erf}\left(\frac{m_X}{\sqrt{2}\sigma_X}\right)}.$$

For statistically independent RV the variance of transformation (1,b) is:

$$D_{\sqrt{X_{[0,\infty)}}} = \overline{\left(\sqrt{X_{[0,\infty)}}\right)^2} - \left(\overline{\sqrt{X_{[0,\infty)}}}\right)^2 = \overline{X_{[0,\infty)}} - \left(\overline{\sqrt{X_{[0,\infty)}}}\right)^2. \tag{19}$$

Discussion. Figure 2 shows the graphs of the dependences of $m_X(\sigma_{\mathrm{X}} = \text{const})$ and $\sigma_{\mathrm{X}}(m_X = \text{const})$ on the mean root_Y(m) (9) and root_X(m) (17), MSD RMS_root_Y(m) (12) and RMS_root_X(m) (19) at intervals $3\sigma_X$, the approximation within which the agreement of Taylor series for the transformation functions (1) was achieved. We see that within the interval $3\sigma_X$, the individual regularities of the statistical mean values of the two transformations emerge (1). Beyond this uncertainty, the dependencies (9), (18), (12), (19) $m_X \to \infty$ ($\sigma_X = \text{const}$) and $\sigma_X \to 0$ ($m_X = \text{const}$) are very close to root_RX(m, σ) and RMS_root_RX(m, σ):

$$\sqrt{X} \cong \sqrt{\sqrt{m_X^2 - \frac{1}{2}\sigma_X^2}}, \quad \sigma_{\sqrt{X}} \cong \sqrt{m_X - \sqrt{m_X^2 + \frac{1}{2}\sigma_X^2}} \tag{20}$$

built by formulas (7) [9]. In addition, the graphs in Fig. 2 confirm the reservations made in [16, 17] regarding the existence of restrictions on the application of formulas (20). Given $\sigma_X = \text{const}$ (Fig. 2a), the restriction region is formed in the range of values $m_X > m_{X,\mathrm{lim}}$, with respect to some boundary $m_{X,\mathrm{lim}} = \frac{\sigma_X}{\sqrt{2}}$. Given $m_X = \text{const}$ (Fig. 2b, the boundary region is formed in the range of values

$\sigma > \sigma_{X,\lim}$ with respect to some boundary $\sigma > \sigma_{X,\lim} = \sqrt{2}m_X$. The differences between formulas (20) with the boundaries of the mean root_X(m) and MSD RMS_root_X(m) transformations $\sqrt{X}$ and $\sqrt{|X|}$ can be minimized and more optimal agreement can be achieved (Fig. 3) if formulas (20) are represented as follows:

$$\sqrt{Y} \cong \sqrt{\sqrt{m_X^2 + \frac{1}{2}\sigma_X^2}}, \quad \sigma_{\sqrt{Y}} \cong \sqrt{-m_X + \left(\sqrt{Y}\right)^2}. \tag{21}$$

In contrast to (20), the formulas (21) allow us to transfer approximate values of the mean and transfer errors (1) by the square root of a normally distributed RV over the whole range of values m, σ.

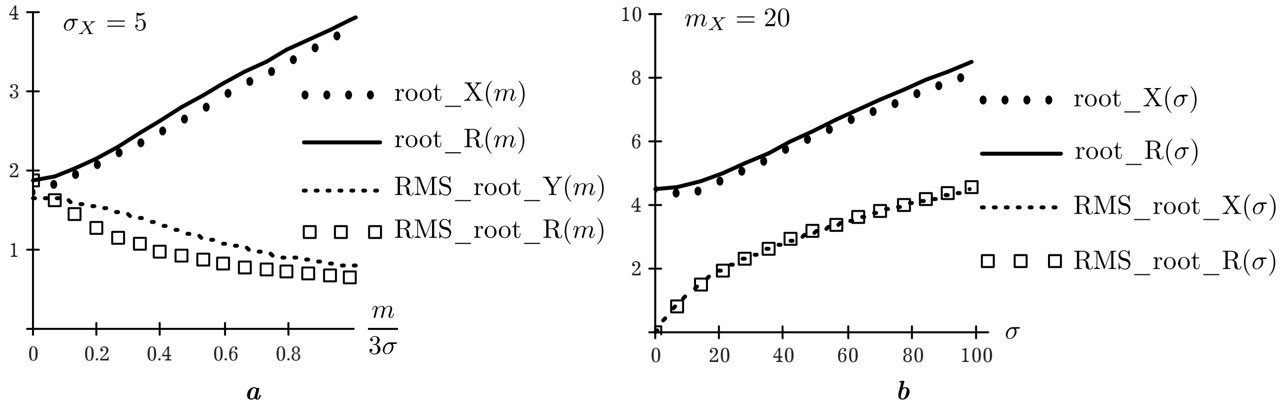

Fig. 3. The illustration of the consistency of formulas (21) with the dependencies (9), (18) and (12), (19) on the level "$3\sigma_X$".

3. Conclusions

The theoretical studies carried out in this work once again confirmed the importance of the correct application of the basic provisions of probability theory for the statistical averaging of random values of physical measurements subjected to nonlinear transformations. It is shown that for nonlinear transformation of a normally distributed RV with a square root, the integrals of higher order averaging $n > 1$ satisfy the inequality $\overline{\left(y - \overline{Y}\right)^n} \neq 0$. On the basis of the theoretical research, correct boundary $m, \sigma \to \infty$ formulas of error transfer are proposed.

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Моделювання статистичних середніх і дисперсії нормально розподілених випадкових величин, перетворених нелінійними функціями $\sqrt{|X|}$ та $\sqrt{X}$

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У роботі виконані теоретичні дослідження закономірностей формування статистично усереднених і дисперсії нормально розподілених випадкових значень із необмеженим інтервалом значень аргументу, які перетворені нелінійним перетворенням функціями $\sqrt{|X|}$ та $\sqrt{X}$. Показано, що для нелінійного перетворення нормально розподіленої випадкової змінної квадратним коренем, інтеграли статистичного усереднення вищих порядків $n > 1$ задовольняють нерівність $\overline{(y - \overline{Y})^n} \neq 0$. На основі проведених теоретичних досліджень запропоновано коректні граничні $m, \sigma \to \infty$.

Ключові слова: *статистичні середне, дисперсія, перетворення, нормальний розподіл, випадкова величина.*