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Mathematical modeling and analysis of Phytoplankton–Zooplankton–Nanoparticle dynamics

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In this paper, we investigate the population dynamics of phytoplankton–zooplankton–nanoparticle model with diffusion and density dependent death rate of predator. The functional response of predator in this model is considered as Beddington–DeAngelis type. The stability analysis of the equilibrium points is observed by applying the Routh–Hurwitz criterion. Numerical simulations are given to illustrate the theoretical results.

1. Introduction

The prey–predator interaction model was first introduced by Lotka (1924) and Volterra (1926) for species relations by arriving at an expression in terms of nonlinear differential equations. In population dynamics, a well-known fundamental structure is the prey–predator interaction. In accordance with this, various models are formulated within prey and predator species to analyse and understand the interaction and dynamics. The population and epidemic dynamics give rise to modeling techniques that plays a vital role in various research fields. Introducing a spatially heterogeneous environment to the Lotka–Volterra predator–prey model reveals that spatial diffusion leads to ample dynamics of ecological populations. To understand the basic features of spatially distributed interaction, many investigators have used reaction–diffusion related equations.

In recent decade, the impact of presence of nanoparticles (NPs) in real life applications such as medicine, construction, automotive, agriculture etc. has attracted researcher's interest. Sometimes, nanoparticles may be hazardous to living organisms, marine organisms and plankton. The nanoparticle reduces the growth rate of phytoplankton population and hence the system leads to limit cycle oscillation [1]. A wide variety of biological and geological processes generate natural NPs and as evidence, some natural NPs may be hazardous to algae, fish species and invertebrates as well as mammals [2]. Interaction of NPs with phytoplankton cells results in growth suppression in phytoplankton species [3–7]. A mathematical model is formulated based on the problem of viral infection on phytoplankton–zooplankton system [8].

The global dynamics of the prey–predator model with density dependent death rate of predator has been investigated [9]. The dynamics of predator–prey system with disease in prey is studied [10]. The stability analysis of solution of reaction–diffusion system has been carried out in which their concentrations are spatially modulated with the wavenumber [11]. The dynamics of predator-prey model with Beddington–DeAngelis functional response has been analysed [12–14]. This type of functional response differs from ratio-dependent form as it keeps away from some of the behaviours of ratio-dependent form at low densities. The effect of environmental toxins affecting plankton dynamics has been studied [15,16].

The existence and uniqueness theorems for boundary value problems and properties of the solutions have been discussed [17]. The boundedness and existence theory for nonlinear systems can be formu-

lated in number of ways [18, 19]. The dynamical complexity of a prey–predator model with nonlinear predator harvesting has been studied [20]. The dynamics of system with ratio-dependent functional response has been investigated [21, 22]. Mathematical models dealing with diffusive prey–predator models are formulated and its dynamics are analysed which helped to understand the concepts [23–26]. Analytical approximation of predator-prey models has been derived [27–30].

To the best of our knowledge, no researcher has formulated the diffusive phytoplankton–zooplankton–nanoparticle model with density dependent death rate of predator. Hence we construct the following model and analyse its local stability.

2. Mathematical formulation

The simple prey–predator model with Beddington–DeAngelis functional response where prey and predator represents phytoplankton–zooplankton is [1]

$$\frac{dU}{dt} = rU\left(1 - \frac{U}{k}\right) - \frac{cUV}{b + \gamma U + \alpha V},$$

$$\frac{dV}{dt} = \frac{\phi cUV}{b + \gamma U + \alpha V} - \mu V.$$

Here U denotes density of phytoplankton and V denotes density of zooplankton at time t. The phytoplankton population follows the logistic growth with carrying capacity k. The growth rate of phytoplankton is denoted by r, predation rate is denoted by c, b is the saturation constant, α is the effect of predator interference, γ denotes the food weighting factor, μ is death rate of zooplankton and ϕ measures the efficiency of converting prey intake into new predator.

In the presence of NPs, the growth rate of phytoplankton decreases and it is inversely proportional to number of contacts between phytoplankton and NPs [1, 3–7]. This phenomenon is included in the above model with an assumed modified growth rate. The assumed functional form is $\frac{r}{1 + \beta\beta_1 NU}$. Hence the growth rate of prey in the absence of predator takes the form $\frac{rU}{1 + \beta\beta_1 NU}\left(1 - \frac{U}{k}\right)$.

The mathematical model of internalization of NPs into phytoplankton is given by [1]

$$\frac{dU}{dt} = \frac{rU}{1 + \beta\beta_1 NU}\left(1 - \frac{U}{k}\right) - \frac{cUV}{b + \gamma U + \alpha V}, \tag{1}$$

$$\frac{dV}{dt} = \frac{\phi cUV}{b + \gamma U + \alpha V} - \mu V, \tag{2}$$

$$\frac{dN}{dt} = A - \beta NU - eN. \tag{3}$$

Here N denotes the density of nanoparticle, β is the contact rate of NP with phytoplankton, β_1 is the proportionality constant. It is assumed that nanoparticles are added at rate A into the aquatic environment and e denotes the natural depletion rate.

In mathematical ecology, the classical Lotka–Volterra model reveals only the population dynamics due to predation in certain circumstances where the densities of predator and prey are spatially independent. In this model, the population is homogeneously distributed and the species does not naturally develop strategies for existence. By including diffusion process, it could be quite complicated since different densities of prey and predator cause different population movements. These movements can be found by the concentration of same species (diffusion). Also, we add the density dependent death rate of predator in the model [9].

In the model [1], we consider phytoplankton to be microalgae which has movement. By considering the inhomogeneous spatial distribution of phytoplankton-zooplankton within a fixed boundary at any given time and the tendency of diffusion of each species, we arrive at a following system of reaction–diffusion type,

$$\frac{\partial U}{\partial t} = D_1\frac{\partial^2 U}{\partial x^2} + \frac{rU}{1 + \beta\beta_1 NU}\left(1 - \frac{U}{k}\right) - \frac{cUV}{b + \gamma U + \alpha V}, \tag{4}$$

$$\frac{\partial V}{\partial t} = D_2 \frac{\partial^2 V}{\partial x^2} + \frac{\phi c U V}{b + \gamma U + \alpha V} - \delta V^2 - \mu V, \tag{5}$$

$$\frac{\partial N}{\partial t} = D_3 \frac{\partial^2 N}{\partial x^2} + A - \beta N U - e N. \tag{6}$$

Boundary condition:

$$U(x,0) \geqslant 0, \quad V(x,0) \geqslant 0, \quad N(x,0) \geqslant 0, \quad x \in \Delta, \tag{7}$$

$$\frac{\partial U}{\partial x} = \frac{\partial V}{\partial x} = \frac{\partial N}{\partial x} = 0, \quad x \in \partial\Delta, \quad t > 0, \tag{8}$$

where Δ is a bounded domain, $\frac{\partial}{\partial x}$ denotes the outward normal derivative on $\partial\Delta$, δ is the predator intraspecific competition rate and D_1, D_2, D_3 are diffusion coefficients of phytoplankton, zooplankton and nanoparticle respectively. We have considered zero-flux Neumann boundary conditions [10]. This condition denotes that there is no emigration or immigration across the boundary i.e., the phytoplankton and zooplankton cannot leave the domain Δ and the species from outer of the domain cannot enter.

Theorem 1. *All the solutions of system (4)–(8) are uniformly bounded.*

Proof. Using comparison theorem and from [1, 10], the equation (6) becomes

$$\frac{dN}{dt} \leqslant A - eN.$$

On integration, we get,

$$0 < N(t) \leqslant e^{-et}\left(N(0) - \frac{A}{e}\right) + \frac{A}{e}.$$

As $t \to \infty$, $N(t) \leqslant \frac{A}{e}$, since $\sup\limits_{t\to\infty} N(t) = \frac{A}{e}$.

We define the function H as $H = \phi U + V$. The derivative of H with respect to time is given by

$$\frac{dH}{dt} \leqslant \phi r U \left(1 - \frac{U}{k}\right) - \mu V,$$

$$\frac{dH}{dt} + \xi H \leqslant U\left(\phi r + \xi\phi - \phi r \frac{U}{k}\right) + (\xi - \mu)V,$$

$$\frac{dH}{dt} + \eta H \leqslant m.$$

Here, $\xi < \mu$, $m = \frac{\phi k(r+\xi)^2}{4r}$, hence

$$0 < H(t) \leqslant e^{-\xi t}\left(H(0) - \frac{m}{\xi}\right) + \frac{m}{\xi}.$$

As $t \to \infty$, $H(t) \leqslant \frac{m}{\xi}$, since $\sup\limits_{t\to\infty} H(t) = \frac{m}{\xi}$.

Hence the system is bounded above. Since the system is bounded above, it is understood that the species under consideration cannot grow exponentially or infinitely. Due to limited resource, the density of species is bounded. ∎

3. Stability Analysis

To observe the dynamics of the species population, it requires stability analysis. In this section, we carry out the linear stability analysis for the model (4)–(6) with boundary condition (7)–(8). It helps in understanding the qualitative behaviour of the nonlinear dynamical systems. The local behaviour of nonlinear system is studied by means of linearization technique. Generally, the system is linearized around each equilibrium point and it is perturbed with a very small quantity. This procedure is used to check whether the system comes back to its corresponding equilibrium point or meets some other equilibrium point/attractor.

Equilibrium points. The equilibrium points of the system are:
- Axial Equilibrium E_1: $\left(0, 0, \frac{A}{e}\right)$,
- Planar Equilibrium E_2: $\left(k, 0, \frac{A}{\beta k + e}\right)$,
- Interior Equilibrium E^*: (U^*, V^*, N^*),

where we consider the positive root of the equation (9) as U^* and $U^* > k$

$$\phi c U^* - \delta b \left[\frac{r(k - U^*)(b + \gamma U^*)}{ck(1 + \beta\beta_1 N^* U^*) - r\alpha(k - U^*)}\right] - \delta\gamma U^* \left[\frac{r(k - U^*)(b + \gamma U^*)}{ck(1 + \beta\beta_1 N^* U^*) - r\alpha(k - U^*)}\right] - \mu b$$

$$-\mu\gamma U^* - \alpha\delta\left[\frac{r(k - U^*)(b + \gamma U^*)}{ck(1 + \beta\beta_1 N^* U^*) - r\alpha(k - U^*)}\right]^2 - \mu\alpha\left[\frac{r(k - U^*)(b + \gamma U^*)}{ck(1 + \beta\beta_1 N^* U^*) - r\alpha(k - U^*)}\right] = 0, \quad (9)$$

$$V^* = \left[\frac{r(k - U^*)(b + \gamma U^*)}{ck(1 + \beta\beta_1 N^* U^*) - r\alpha(k - U^*)}\right],$$

$$N^* = \frac{A}{\beta U^* + e}.$$

The Jacobian matrix $J(E)$ of the model (4)–(8) is

$$J(E) = \begin{bmatrix} \frac{r}{(1+\beta\beta_1 N^* U^*)^2} \\ -\left[\frac{k(1+\beta\beta_1 N^* U^*)2rU^* - rU^{*2}(k\beta\beta_1 N^*)}{(k(1+\beta\beta_1 N^* U^*))^2}\right] \\ -\left[\frac{(b+\alpha V^*)cV^*}{(b+\gamma U^*+\alpha V^*)^2}\right] & -\frac{(b+\gamma U^*)cU^*}{(b+\gamma U^*+\alpha V^*)^2} & -\frac{rU^*\left(1-\frac{U^*}{k}\right)\beta\beta_1 U^*}{(1+\beta\beta_1 N^* U^*)^2} \\ \frac{(b+\alpha V^*)\phi cV^*}{(b+\gamma U^*+\alpha V^*)^2} & \frac{(b+\gamma V^*)\phi cV^*}{(b+\gamma U^*+\alpha V^*)^2} - 2\delta V^* - \mu & 0 \\ -\beta U^* & 0 & -\beta U^* - e \end{bmatrix},$$

λ is the eigenvalue of $I\Delta + J(E)$ iff λ is an eigenvalue of matrix $-\sigma_i I + J(E)$ [10]. Then the matrix is

$$|\lambda I + \sigma_i I - J(E)| = \begin{vmatrix} \lambda + \sigma_1 - M_{11}^* & -M_{12}^* & -M_{13}^* \\ -M_{21}^* & \lambda + \sigma_2 - M_{22}^* & -M_{23}^* \\ -M_{31}^* & -M_{32}^* & \lambda + \sigma_3 - M_{33}^* \end{vmatrix},$$

where

$$M_{11} = \frac{r}{(1 + \beta\beta_1 N^* U^*)^2} - \left[\frac{k(1 + \beta\beta_1 N^* U^*)2rU^* - rU^{*2}(k\beta\beta_1 N^*)}{(k(1 + \beta\beta_1 N^* U^*))^2}\right] - \left[\frac{(b + \alpha V^*)cV^*}{(b + \gamma U^* + \alpha V^*)^2}\right],$$

$$M_{12} = -\frac{(b + \gamma U^*)cU^*}{(b + \gamma U^* + \alpha V^*)^2},$$

$$M_{13} = -\frac{rU^*\left(1 - \frac{U^*}{k}\right)\beta\beta_1 U^*}{(1 + \beta\beta_1 N^* U^*)^2},$$

$$M_{21} = \frac{(b + \alpha V^*)\phi cV^*}{(b + \gamma U^* + \alpha V^*)^2},$$

$$M_{22} = \frac{(b + \gamma V^*)\phi cV^*}{(b + \gamma U^* + \alpha V^*)^2} - 2\delta V^* - \mu,$$

$$M_{23} = 0,$$

$$M_{31} = -\beta U^*,$$

$$M_{32} = 0,$$

$$M_{33} = -\beta U^* - e.$$

Theorem 2. *The system* (4)–(8) *around* E_1: $\left(0, 0, \frac{A}{e}\right)$ *is locally asymptotically stable (LAS) if* $r < \sigma_1$.

Proof. The Jacobian matrix of corresponding axial equilibrium E_1 is given by

$$J(E_1) = \begin{bmatrix} r & 0 & 0 \\ 0 & -\mu & 0 \\ 0 & 0 & -e \end{bmatrix}.$$

The eigenvalues of the matrix are given by roots of the characteristic equation

$$|\lambda I + \sigma_i I - J(E_1)| = \begin{vmatrix} \lambda + \sigma_1 - r & 0 & 0 \\ 0 & \lambda + \sigma_2 + \mu & 0 \\ 0 & 0 & \lambda + \sigma_3 + e \end{vmatrix},$$

$$(\lambda + \sigma_1 - r)\left[(\lambda + \sigma_2 + \mu)(\lambda + \sigma_3 + e)\right] = 0.$$

The roots are $\lambda = -\sigma_1 + r$, $\lambda = -\sigma_2 - \mu$, $\lambda = -\sigma_3 - e$.

Hence E_1 is locally asymptotically stable if $r < \sigma_1$.   ■

Theorem 3. *The system* (4)–(8) *around* E_2: $\left(k, 0, \frac{A}{\beta k + e}\right)$ *is Locally Asymptotically Stable if the root of the characteristic equation* $\rho_0 \lambda^3 + \rho_1 \lambda^2 + \rho_2 \lambda + \rho_3 = 0$ *of the Jacobian matrix satisfies Routh–Hurwitz criterion i.e.* $\rho_0 > 0$, $\rho_1 > 0$, $\rho_1 \rho_2 - \rho_0 \rho_3 > 0$, $\rho_3 > 0$.

Proof. The Jacobian matrix of corresponding planar equilibrium E_2 is given by

$$J(E_2) = \begin{bmatrix} -\dfrac{r}{\left(1 + \beta\beta_1 k\left(\frac{A}{\beta k + e}\right)\right)^2} - \dfrac{r\left(\frac{k\beta\beta_1 A}{\beta k + e}\right)}{\left(1 + \beta\beta_1 k\left(\frac{A}{\beta k + e}\right)\right)^2} & -\dfrac{ck}{b + \gamma k} & 0 \\ 0 & \dfrac{\phi ck}{b + \gamma k} - \mu & 0 \\ \beta k & 0 & -\beta k - e \end{bmatrix}$$

The eigenvalues of the matrix are given by roots of the characteristic equation

$$|\lambda I + \sigma_i I - J(E_2)| = \begin{vmatrix} \lambda + \sigma_1 + \dfrac{r}{\left(1 + \beta\beta_1 k\left(\frac{A}{\beta k + e}\right)\right)^2} & & \\ + \dfrac{r\left(\frac{k\beta\beta_1 A}{\beta k + e}\right)}{\left(1 + \beta\beta_1 k\left(\frac{A}{\beta k + e}\right)\right)^2} & \dfrac{ck}{b + \gamma k} & 0 \\ 0 & \lambda + \sigma_2 - \dfrac{\phi ck}{b + \gamma k} + \mu & 0 \\ \beta k & 0 & \lambda + \sigma_3 + \beta k + e \end{vmatrix},$$

$$\left[\lambda + \sigma_1 + \dfrac{r}{\left(1 + \beta\beta_1 k\left(\frac{A}{\beta k + e}\right)\right)^2} + \dfrac{r\left(\frac{k\beta\beta_1 A}{\beta k + e}\right)}{\left(1 + \beta\beta_1 k\left(\frac{A}{\beta k + e}\right)\right)^2}\right]$$

$$\times \left[\left(\lambda + \sigma_2 - \dfrac{\phi ck}{b + \gamma k} + \mu\right)(\lambda + \sigma_3 + \beta k + e)\right] = 0,$$

$$\lambda^3 + \lambda^2 \left[\sigma_3 + \beta k + e + \sigma_2 - \frac{\phi c k}{b + \gamma k} + \mu + \sigma_1 + \frac{r + r\left(\frac{\beta \beta_1 k A}{\beta k + e}\right)}{\left(1 + \frac{\beta \beta_1 k A}{\beta k + e}\right)^2} \right]$$

$$+ \lambda \left[\sigma_2 \sigma_3 + \sigma_2 \beta k + \sigma_2 e - \frac{\phi c k}{b + \gamma k} \sigma_3 - \frac{\phi c k^2}{b + \gamma k} \beta - \frac{\phi c k}{b + \gamma k} e + \mu \sigma_3 + \mu \beta k \right.$$

$$+ \mu e + \sigma_1 \sigma_3 + \sigma_1 \beta k + \sigma_1 e + \sigma_1 \sigma_2 - \sigma_1 \frac{\phi c k}{b + \gamma k} + \sigma_1 \mu$$

$$+ \left(\sigma_3 + \beta k + e + \sigma_2 - \frac{\phi c k}{b + \gamma k} + \mu \right) \frac{r + r\left(\frac{\beta \beta_1 k A}{\beta k + e}\right)}{\left(1 + \frac{\beta \beta_1 k A}{\beta k + e}\right)^2} \right]$$

$$+ \left[\left(\sigma_2 \sigma_3 + \sigma_2 \beta k + \sigma_2 e - \frac{\phi c k}{b + \gamma k} \sigma_3 - \frac{\phi c k^2}{b + \gamma k} \beta - \frac{\phi c k}{b + \gamma k} e + \mu \sigma_3 + \mu \beta k + \mu e \right) \right.$$

$$\left. \times \left(\sigma_1 + \frac{r + r\left(\frac{\beta \beta_1 k A}{\beta k + e}\right)}{\left(1 + \frac{\beta \beta_1 k A}{\beta k + e}\right)^2} \right) \right] = 0,$$

which is of the form

$$\rho_0 \lambda^3 + \rho_1 \lambda^2 + \rho_2 \lambda + \rho_3 = 0.$$

By Routh–Hurwitz criterion, the system is locally asymptotically stable when

$$\rho_0 > 0, \quad \rho_1 > 0, \quad \rho_1 \rho_2 - \rho_0 \rho_3 > 0, \quad \rho_3 > 0.$$

Hence E_2 is locally asymptotically stable. ∎

Theorem 4. *The system (4)–(8) around E^*: (U^*, V^*, N^*) is LAS if the roots of characteristic equation $\omega_0 \lambda^3 + \omega_1 \lambda^2 + \omega_2 \lambda + \omega_3 = 0$ of the Jacobian matrix satisfy Routh–Hurwitz criterion i.e. $\omega_0 > 0$, $\omega_1 > 0$, $\omega_1 \omega_2 - \omega_0 \omega_3 > 0$, $\omega_3 > 0$.*

Proof. The variational matrix of corresponding interior equilibrium E^* is given by

$$J(E^*) = \begin{bmatrix} M_{11}^* & M_{12}^* & M_{13}^* \\ M_{21}^* & M_{22}^* & M_{23}^* \\ M_{31}^* & M_{32}^* & M_{33}^* \end{bmatrix}.$$

The eigenvalues of the matrix are given by roots of the characteristic equation which is of the form

$$\omega_0 \lambda^3 + \omega_1 \lambda^2 + \omega_2 \lambda + \omega_3 = 0.$$

By Routh–Hurwitz criterion, the system is locally asymptotically stable when

$$\omega_0 > 0, \quad \omega_1 > 0, \quad \omega_1 \omega_2 - \omega_0 \omega_3 > 0, \quad \omega_3 > 0.$$

Hence the point E^* is locally asymptotically stable. ∎

4. Comparative study

Comparison of stability conditions for equilibrium points of the system (4)–(8) with system in absence of diffusion and density dependent death rate of predator (1)–(3). The trivial equilibrium is unstable in system (1)–(3) and it is LAS if $r < \sigma_1$ in (4)–(8) i.e., the population density cannot increase or decrease indefinitely.

Equilibriums	System (1)–(3)	System (4)–(8)
E_1: $\left(0, 0, \frac{A}{e}\right)$	Unstable	LAS if $r < \sigma_1$
E_2: $\left(k, 0, \frac{A}{\beta k + e}\right)$	$Eck < \mu(b+k)$	$\rho_1 > 0$, $\rho_3 > 0$, $\rho_1\rho_2 > \rho_3$, ρ_* are defined in theorem 2.
E^*: (U^*, V^*, N^*)	$\sigma_1 > 0$, $\sigma_3 > 0$, $\sigma_1\sigma_2 > \sigma_3$, σ_* are defined in [1]	$\omega_1 > 0$, $\omega_3 > 0$, $\omega_1\omega_2 > \omega_3$, ω_* are defined in theorem 3.

5. Numerical simulation

(**a**) The temporal solution of U (**b**) The temporal solution of V (**c**) The temporal solution of N
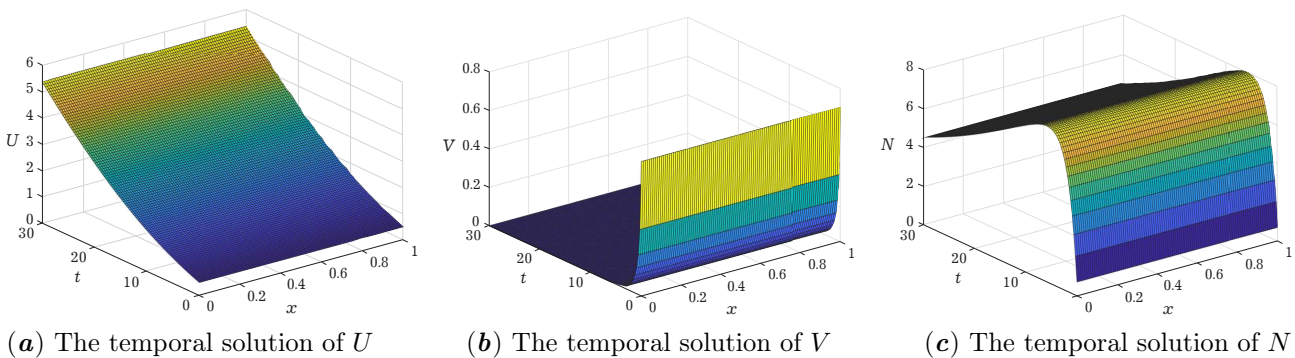
Fig. 1. Plot of numerical simulation of the system (4)–(8). The initial conditions are $U(x,0) = 0.5$, $V(x,0) = 0.7$, $N(x,0) = 0.7$ and parameter values are $r = 0.2$, $k = 20$, $c = 0.8$, $b = 8$, $\gamma = 1$, $\alpha = 0.1$, $E = 0.33$, $\mu = 0.8$, $A = 5$, $e = 0.5$, $\beta = 0.12$, $\beta_1 = 0.8$, $\sigma_1 = 1$, $\sigma_2 = 0.5$, $\sigma_3 = 0.1$, $\delta = 3.2$.

In this section, the feasibility of our stability condition is illustrated by applying numerical values. The initial conditions for system (4)–(8) are assumed as $U(x,0) = 0.5$, $V(x,0) = 0.7$, $N(x,0) = 0.7$, $x \in [0,1]$.

Example 1. In system (4)–(8), we set the parametric values as $r = 0.2$, $k = 20$, $c = 0.8$, $b = 8$, $\gamma = 1$, $\alpha = 0.1$, $\phi = 0.33$, $\mu = 0.8$, $A = 5$, $e = 0.5$, $\beta = 0.12$, $\beta_1 = 0.8$, $\sigma_1 = 1$, $\sigma_2 = 0.5$, $\sigma_3 = 0.1$, $\delta = 3.2$.

By Theorem 3, we see that the equilibrium point is E_2: $(20, 0, 1.72413793)$ and $\rho_1 = 5.188637689 > 0$, $\rho_2 = 7.763154058 > 0$, $\rho_3 = 4.238048444 > 0$, $\rho_1\rho_2 - \rho_3 = 36.04214529 > 0$ satisfies Routh–Hurwitz criterion which ensure the system is locally asymptotically stable.

Example 2. In system (4)–(8), we set the parametric values as $r = 0.2$, $k = 20$, $c = 1$, $b = 8$, $\gamma = 1$, $\alpha = 0.1$, $\phi = 1.2$, $\mu = 0.8$, $A = 5$, $e = 0.5$, $\beta = 0.12$, $\beta_1 = 0.8$, $\sigma_1 = 1$, $\sigma_2 = 0.5$, $\sigma_3 = 0.1$, $\delta = 3.2$.

By Theorem 4, we see that the equilibrium point is E^*: $(19.98525550, 0.0009573951429, 1.725190500)$ and $\omega_1 = 4.489944219 > 0$, $\omega_2 = 4.937772632 > 0$, $\omega_3 = 1.262456853 > 0$, $\omega_1\omega_2 - \omega_3 = 20.90786683 > 0$ satisfies Routh–Hurwitz criterion which ensure the system is locally asymptotically stable.

Figure 1 indicates the temporal solution of the system with initial condition and Neumann boundary condition (4)–(8) is locally asymptotically stable.

6. Conclusion

In this paper, we have considered the phytoplankton–zooplankton–nanoparticle model and it is incorporated with diffusion and density dependent death rate of predator. The functional response of the system is considered in the form of Beddington–DeAngelis type. The interacting species can move inside a closed domain whereas it cannot leave or enter the domain. Thus our proposed model is a reaction–diffusion system with Neumann boundary conditions. The equilibrium points and the condition to be locally asymptotically stable have been analysed. Numerical simulation supports our theoretical results and illustrates the LAS of equilibrium points. The current study can be extended to find global stability of the system as future scope of our work.

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Математичне моделювання та аналіз динаміка фітопланктон–зоопланктон–наночастинка

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У цій роботі досліджуємо популяційну динаміку моделі фітопланктон–зоопланктон–наночастинка із дифузійною залежністю швидкості загибелі хижака. Функціональна реакція хижака в цій моделі розглядається як реакція Беддінгтона–ДеАнджеліса. Аналіз стійкості точок рівноваги проводиться за допомогою критерію Рауса–Гурвіца. Для ілюстрації теоретичних результатів наведено чисельне моделювання.

Ключові слова: *модель "жертва–хижак", наночастинки, дифузія, функціональна реакція Беддінгтон–ДеАнджеліса, стійкість, чисельне моделювання.*